PARTITIONING SUBGRAPHS OF PROFINITE ORDERED GRAPHS

STEFANIE HUBER, STEFAN GESCHKE, AND MENACHEM KOJMAN

Abstract. Let $\mathcal{K}$ be the class orderedf all inverse limits $G = \varprojlim_{n\in\mathbb{N}} G_n$ where each G_n is a finite ordered graph. $G \in \mathcal{K}$ is *universal* if every $B \in \mathcal{K}$ embeds continuously into G.

Theorem (1). *For every finite ordered graph A there exists a least natural number $k(A) \geq 1$ such that for every universal $G \in \mathcal{K}$, for every finite Baire measurable partition of the set $\binom{G}{A}$ of all copies of A in G, there is a closed copy $G' \subseteq G$ of G such that $\binom{G'}{A}$ meets at most $k(A)$ parts. In the arrow notation:*

$$G \to_{Baire} (G)^A_{<\infty|k(A)}.$$

Theorem (2). *The probability that $k(A) = 1$, for a finite ordered graph A, chosen randomly with uniform probability from all graphs on $\{0, 1, \dots, n-1\}$, tends to 1 as n grows to infinity, where $k(A)$ is the number given by Theorem (1).*

As a corollary:

Theorem (3). *The class $\mathcal{K}$ with Baire partitions satisfies with high probability the A-partition property for a finite ordered graph A, where the A-partition property is*

$$(\forall_{B\in\mathcal{K}})(\exists_{C\in\mathcal{K}})\, C \to_{Baire} (B)^A.$$

1. introduction

We consider open partitions of n-tuples from inverse limits of finite ordered graphs, and particularly from *universal* inverse limit graphs G, into which all inverse limit graphs embed as closed subgraphs. Such universal graphs can be defined with no mention of inverse systems as follows:

Definition 1.1. *A universal inverse limit of finite ordered graphs is a triple $G = \langle V, E, < \rangle$ where:*

- *V is a compact subset of $\mathbb{R} \setminus \mathbb{Q}$, $E \subseteq [V]^2$ and $<$ is the restriction of the standard order on $\mathbb{R}$ to V.*

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- *(Modular profiniteness) For any two distinct vertices $u, v \in V$ there is a partition of V to finitely many closed intervals such that*
  (a) *u, v belong to different intervals from the partition;*
  (b) *every interval I in the partition is a* module *in G, that is, for all $v \in V \setminus I$ and for all $x, y \in I$, $vEx \iff vEy$.*
- *(Universality) every nonempty open interval of V contains induced copies of all finite ordered graphs.*

The universality condition implies that V has no isolated points and also that every inverse limit graph G, or, equivalently, a graph that satisfies the first two conditions, embeds into G via a continuous order preserving map.

The standard question in infinite Ramsey Theory is whether for suitable partitions of n-tuples from a structure, there is a copy of the structure in itself on which the partition satisfies stricter conditions. The vertex set of every copy of a universal profinite ordered graph in itself is a topologically perfect subset of $\mathbb{R}$, so let us begin by recalling Blass' partition theorem for topologically perfect subsets of $\mathbb{R}$. Confirming a conjecture of Galvin, who also proved his conjecture for $n \leq 3$, Blass proved [2] that for every positive natural number n and a topologically perfect subset $P \subseteq \mathbb{R}$, for every finite Baire- or Lebesgue-measurable partition of $[P]^n$, the set of all n-tuples from P, there is a topologically perfect $P' \subseteq P$ such that $[P']^n$ meets at most $(n-1)!$ parts.

Blass actually identified an *open* partition of n-tuples from the Cantor space $2^{\mathbb{N}}$ (which embeds into every perfect set) to $(n-1)!$ parts which is (a) *persistent*: every topologically perfect $P \subseteq 2^{\mathbb{N}}$ has n-tuples in each of the $(n-1)!$ parts; and (b) *basic* for Baire and Lebesgue measurable finite partitions: for every finite Baire or Lebesgue measurable partition of $[2^{\mathbb{N}}]^k$ there is a closed copy $P \subseteq 2^{\mathbb{N}}$ of $2^{\mathbb{N}}$ on which the part of any n-tuple in the Blass open partition determines its part in the given partition. The Blass types are conveniently described in terms of the standard binary tree representation of $2^{\mathbb{N}}$ (see [20], where also a proof of Blass' theorem using Milliken's theorem is given).

Back to universal inverse limits, the mapping from an n-tuple to the isomorphism type of the ordered graph it induces is continuous, and in each copy of a universal inverse limit G in itself all isomorphism types of finite ordered graphs occur. Thus one may as well look separately at partitions of copies of a each ordered graph A. every copy G'?

For each finite ordered graph A we identify a finite open partition of the set $\binom{G}{A}$ of all copies of A in G to $k(A)$ parts, called *types*, which is persistent under taking copies of G in itself and basic for finite Baire-measurable partitions. (In a context like this, the numbers of types is called a *Ramsey degree*. See Section 10 of [12] for the introduction and a discussion of Ramsey degrees in a general setting.)

The number of types that occur in $\binom{G}{A}$ for a given ordered graph A depends strongly on the edge relation of A. For A with trivial graph structure, e.g. a complete graph or a graph with no edges of size n, the number $k(A)$ exceeds the number $(n-1)!$ of Blass' types. However — perhaps surprisingly — for a generic A, $k(A) = 1$: the probability that $k(A) = 1$ for a graph A chosen with uniform probability on $\{0, 1, \ldots, n-1\}$ tends to 1 as n grows to infinity. A random graph structure can almost certainly have only one type (see below).

Thus, the class of all inverse limits of finite ordered graphs satisfies the A-partition property with Baire-measurable partitions for a finite ordered graph A for "most" A, where the A partition property means, following [15]: for every inverse limit of finite ordered graphs B there exists and inverse limit of finite ordered graph C such that for every Baire measurable partition of $\binom{C}{A}$ to finitely many parts there is a closed copy of B in C such that $\binom{B}{A}$ meets exactly one of the parts. In the arrow notation:

$$C \to (B)^A.$$

Proving the theorem only in the generic case is easier and can be done without appealing to the Halpern-Läuchli partition theorem for trees. We do not separate the proof of the generic case from the proof of the general case below, but we do indicate after Lemma 3.3 below how to finish the proof for the generic case without Halpern-Läuchli.

Let us remark that, as $k(A) = 1$ for the complete graphs of size 2, universal ordered inverse limit graphs G satisfy the following relation, which Rado's countable homegeneous and universal graph does not:

$$G \to (G)^2.$$

Finally, a similar theorem about partitions of infinite closed subgraphs of universal inverse limits holds, but unlike the theorem presented here, we do not have for it at the moment a proof which does not use Forcing. It will be presented elsewhere.

2. Preliminaries and Notation

2.1. **Modular profinite graphs of countable weight.** Let X be a topological space. A graph G with the vertex set X is *clopen* if the edge-relation of G is a clopen subset of $X^2 \setminus \{(x, x) : x \in X\}$.

G is *[ordered] modular profinite* if it is the inverse limit of a system of [ordered] finite graphs whose bonding maps are [order-] modular. The name *modular profinite* is chosen in analogy to profinite groups, which are inverse limits of systems of finite groups. A thorough treatment of inverse limits of graphs can be found in [4]. Here we call a map f from the vertex set $V(G)$ of a graph G to the vertex set $V(H)$ of a graph H *[order-] modular* if for all $v, w \in V(G)$, either $f(v) = f(w)$ or $\{v, w\} \in E(G)$ iff $\{f(v), f(w)\} \in E(H)$ [and $v \leq w$ iff $f(v) \leq f(w)$]. Clearly, for each [order-] modular map $f : V(G) \to V(H)$ and each vertex $v \in V(H)$, the inverse image $f^{-1}(v)$ is a [order-] module of G, i.e., a set M of vertices of G such that for all

$u \in V(G) \setminus M$ either u is adjacent to all vertices in M or to no vertex in M [and $f^{-1}(v)$ is an interval]. If $f : G \to H$ is [order-] modular, then $\{f^{-1}(v) : v \in V(H)\}$ is a *[order-] modular partition* of G, i.e., a partition of the vertex set of G into [order-] modules, and the induced subgraph of H on the range of f is the *[ordered] modular quotient* of G by this modular partition.

In [4] modular profinite graphs have been studied in detail and it was shown that a graph G on a topological space X is modular profinite if and only if X is compact and zero-dimensional, G is a clopen graph on X, and the modular partitions of G into finitely many clopen sets separate the vertices of G.

For the purpose of this article it is unnecessary to go into more details of the definition of modular profinite graphs. It is enough to use the following description of modular profinite graphs of countable weight, using the Baire space of all infinite sequences of integers, ω^ω:

For distinct points $x, y \in \omega^\omega$ let $\Delta(x, y)$ denote the minimal $n \in \omega$ with $x(n) \neq y(n)$. The Baire space is a complete and separable metric space under the metric that assigns to two different sequences η, ν the distance $1/(\Delta(\eta, \nu) + 1)$. Let $S \subseteq \omega^\omega$ be a closed set with respect to the metric topology just defined on ω^ω. A coloring $c : [S]^2 \to 2$ is of *depth* 1 if for all $\{x, y\} \in [S]^2$ the color $c(x, y)$ only depends on $x \restriction (\Delta(x, y) + 1)$ and $y \restriction (\Delta(x, y) + 1)$. The coloring c corresponds to the graph $G_c = (S, c^{-1}(1))$. If the coloring $c : [S]^2 \to 2$ is of depth 1 and S is compact, then the graph G_c is modular profinite. Moreover, every modular profinite graph of countable weight is isomorphic to a graph of the form G_c [4, Theorem 3.9].

Talking about modular profinite graphs of the form G_c becomes easier if we can use the language of trees in the sense of set theory. Recall that a *tree* is a partially ordered set $(T, \leq)$ where for each $t \in T$ the set $\{s \in T : s \leq t\}$ is wellordered by $\leq$. A subtree S of a tree $(T, \leq)$ is a subset of S such that for all $s \in S$ and $t \in T$, if $t \leq s$, then $t \in S$. All the trees that we consider in this article are subtrees of $\omega^{<\omega}$, the tree of finite sequences of natural numbers ordered by inclusion. Note that for $s, t \in \omega^{<\omega}$, we have $s \subseteq t$ iff s is an initial segment of t. If T is a tree and $s, t \in T$, then s is an *immediate successor* of t iff s is a minimal element of T above t. The set of immediate successors of t in T is denoted by $\mathrm{succ}_T(t)$. An element t of T is a *splitting node* of T if t has at least two immediate successors. T is *perfect* if every $t \in T$ extends to a splitting node of T.

A modular ordered profinite graph G of countable weight is described by the following data: the set V of vertices, which is a compact subset of the *lexicographically ordered Baire space* ω^ω, and for each

$$t \in T = T(V) = \{x \restriction n : x \in V \wedge n \in \omega\}$$

a graph G_t on the set $\mathrm{succ}_T(t)$ such that distinct successors $s_0, s_1 \in \mathrm{succ}_T(t)$ form an edge in G_t iff all $x, y \in V$ with $s_0 \subseteq x$ and $s_1 \subseteq y$ form an edge in G. Since the coloring $c : [V]^2 \to 2$ corresponding to G is of depth 1, for t, s_0, s_1

as above, either all $x, y \in V$ with $s_0 \subseteq x$ and $s_1 \subseteq y$ form an edge in G or no such pair forms an edge. Thus, for every $t \in T$, the set $\{x \in V : t \subseteq x\}$ is an order-module in G.

In [5] it was proved that there is a universal modular profinite graph of countable weight, but using the language of continuous colorings. Such universal graphs are not unique up to isomorphism, but are unique with respect to bi-embeddability. A detailed description of such a graph is as follows:

Let R be the Rado graph, i.e., the unique countable universal and homogeneous graph. We assume that the set of vertices of R is just the set ω of natural numbers. For each $n \in \omega$ let R_n denote the induced subgraph of R on the set $\{0, \dots, n\}$. Now consider the subtree $T_{\max}$ of $\omega^{<\omega}$ that consists of all $t \in \omega^{<\omega}$ such that for all for all $i \in |t|$, $t(i) \leq i$, where $|t|$ denotes the length of the sequence t. For each t in $T_{\max}$ we choose G_t such that the map

$$\mathrm{succ}_{T_{\max}}(t) \to \{0, \dots, |t|\}; s \mapsto s(|t|)$$

is an isomorphism between G_t and $R_{|t|}$. This induces a graph structure on the space $[T_{\max}]$ of branches of $T_{\max}$. We call this graph $G_{\max}$.

Let us introduce some more notation for trees. For a subtree T of $\omega^{<\omega}$ and $t \in T$ let

$$T_t = \{s \in T : t \subseteq s \vee s \subseteq t\}.$$

For a subset Z of T let

$$T_Z = \{s \in T : s \text{ is comparable with some } t \in Z\}.$$

For $n \in \omega$ let $\mathrm{Lev}_T(n) = \{t \in T : |t| = n\}$. Observe that for all $t \in \omega^{<\omega}$ the natural number $|t|$ coincides with the domain of t. Also, $|t|$ is the height of t in any subtree T of $\omega^{<\omega}$ with $t \in T$. Finally, given $s, t \in \omega^{<\omega} \cup \omega^\omega$, let $s \wedge t$ denote the longest common initial segment of the two sequences s and t.

We shall be using the following fundamental induced Ramsey theorem:

Theorem 2.1 (Nešetřil, Rödl [15], also see [1])**.** *For every finite number r and finite ordered graphs B and L there is a finite ordered graph D such that for every coloring of the induced copies of L in D by r colors there is an induced copy of B in D in which all copies of L are colored by a single color. In symbols:*

$$D \rightarrowtail (B)^L_r.$$

Note that the Nešetřil-Rödl theorem implies that for any finite number N and all finite ordered graphs B and L there is a finite ordered graph D such that for any family $\mathcal{C}$ of at most N colorings of the induced copies of L in D by two colors, there is a copy B' of B in D such that each of the colorings in $\mathcal{C}$ is constant on the induced copies of L in B'. This is because we can code the N colorings in $\mathcal{C}$ by a single coloring c with 2^N colors and then apply the Nešetřil-Rödl theorem with $r = 2^N$ many colors, which, of course, follows from the version with 2 colors.

Lemma 2.2. *If H is any finite ordered graph and $t \in T_{\max}$ then there is an extension s of t in $T_{\max}$ such that H embeds (via an order preserving embedding) into G_s, where we consider G_s as an ordered graph with respect to the lexicographic order on the vertices of G_s.*

Proof. We consider the random graph with the usual ordering on ω. Since the graphs G_t are isomorphic to the initial segments R_n of R, it is enough to show that each finite ordered graph H embeds into the ordered random graph R.

Let $v_1, \dots, v_k$ be the increasing enumeration of the vertices of H. We define an order preserving embedding $e : H \to R$ as follows: Choose $e(v_1) \in V(R)$ arbitrarily. Now assume that we have chosen $e(v_1), \dots, e(v_i)$ for some $i \in \{1, \dots, k-1\}$ such that $e \restriction \{v_1, \dots, v_i\}$ is an order preserving embedding of the induced subgraph on $\{v_1, \dots, v_i\}$ into R. By the extension property of the random graph, there is a vertex w of R such that $(e \restriction \{v_1, \dots, v_i\}) \cup \{(v_{i+1}, w)\}$ is an embedding of the induced subgraph on the set $\{v_1, \dots, v_i, v_{i+1}\}$ into R as unordered graphs. But it is easy to see that in fact, there are infinitely many such vertices w. Hence we can find one that is larger than all the vertices $e(v_1), \dots, e(v_i)$ and call it $e(v_{i+1})$. This finishes the recursive construction of e. □

We are interested in induced subgraphs of $G_{\max}$ that contain copies of $G_{\max}$ itself. One way of getting such subgraphs is by constructing sufficiently large subtrees of $T_{\max}$. Given a subtree T of $T_{\max}$, for each $t \in T$ let G_t^T denote the induced subgraph of G_t on the set of immediate successors of t in T. Also, let $G(T)$ denote the induced subgraph of $G_{\max}$ on the set $[T]$ of infinite branches of T.

We call a tree $T \subseteq T_{\max}$ a $G_{\max}$-*tree* if for every finite ordered graph H and all $t \in T$ there is $s \in T$ such that $t \subseteq s$ and H embeds into G_s^T.

The main techniques of building a $G_{\max}$-subtree S of a $G_{\max}$-tree T is *fusion*: A sequence $(T_k)_{k\in\omega}$ is a *fusion sequence with witness* $(m_k)_{k\in\omega}$ if the following hold:

(1) $(m_k)_{k\in\omega}$ is a strictly increasing sequence of natural numbers.
(2) For all $k, \ell \in \omega$, if $k < \ell$, then T_ℓ is a $G_{\max}$-subtree of T_k such that $\mathrm{Lev}_{T_k}(m_k) = \mathrm{Lev}_{T_\ell}(m_k)$.
(3) For every finite ordered graph H, every $k \in \omega$, and every $t \in \mathrm{Lev}_{T_k}(m_k)$ there is $\ell > k$ such that t has an extension s in T_ℓ such that $|s| < m_\ell$ and H embeds into $G_s^{T_\ell}$.

It is easily checked that if $(T_k)_{k\in\omega}$ is a fusion sequence witnessed by $(m_k)_{k<\omega}$, then the *fusion* $\bigcap_{k\in\omega} T_k = \bigcup_{k\in\omega}(T_k \cap \omega^{\leq m_k})$ is a $G_{\max}$-tree. In practice, whenever we construct a fusion sequence $(T_k)_{k\in\omega}$ witnessed by $(m_k)_{k\in\omega}$, we will use some book-keeping that tells us that when we have already chosen T_k and m_k, we now have to find a splitting node s above a certain $t \in \mathrm{Lev}_{T_k}(m_k)$ such that a certain finite ordered graph H embeds into $G_s^{T_{k+1}}$. With the right book-keeping, which we will not specify precisely, this guarantees that $(T_k)_{k\in\omega}$ is a fusion sequence witnessed by $(m_k)_{k\in\omega}$.

For technical reasons we will sometimes assume that the $G_{\max}$-trees under consideration are *skew*, where a subtree T of $\omega^{<\omega}$ is *skew* if for all n T has at most one splitting node at level n.

In some specific situations less book-keeping is required to constuct $G_{\max}$ subtrees of a given $G_{\max}$-tree T. If S is a subtree of a subtree T of $\omega^{<\omega}$ we say that S is a *strong subtree* of T if every splitting node s of S satisfies $\mathrm{succ}_S(s) = \mathrm{succ}_T(s)$. A $G_{\max}$-tree T is *normal* if for all $t, s \in T$ with $t \subseteq s$ the following holds: if the finite ordered graph R_n embeds into G_t^T, then either G_s^T has only a single vertex or R_{n+1} embeds into G_s^T.

If T is a normal $G_{\max}$-tree, $t \in T$, and H is any finite ordered graph, then, in order to find an extension s of t in T such that G_s^T contains a copy of H all we have to do is to find any extension s of t that has enough restrictions that are splitting nodes. This will simplify the construction in the proof of Lemma .

Using fusion, we see that every $G_{\max}$-tree has a $G_{\max}$-subtree that is both skew and normal.

Lemma 2.3. *If T is a normal $G_{\max}$-tree and $S \subseteq T$ is a perfect strong subtree of T, then S is a $G_{\max}$-tree.*

Proof. Let $s \in S$ and let H be a finite ordered graph. Let $n \in \omega$ be such that H embeds into R_n. Since S is perfect, s has an extension t in S that is a splitting node and also has n restrictions that are splitting nodes. Since T is normal and t has at least n restrictions that are splitting nodes in T, R_n embeds into G_t^T. Since S is a strong subtree of T and t is a splitting node of S, $\mathrm{succ}_T(t) = \mathrm{succ}_S(t)$. It follows that R_n and hence H embeds into G_t^S. □

Definition 2.4. *For $n \geq 1$ we define a topology on $[G_{\max}]^n$ as follows: A set $O \subseteq [G_{\max}]^n$ is* open *if for all $H \in O$ there are open neighborhoods $U_1, \dots, U_n$ of the vertices of H such that all $H' \in [G_{\max}]^n$ that have exactly one vertex in each U_i are also in O. This topology is separable and induced by a complete metric. A coloring of n-tuples from $G_{\max}$ is* continuous *if it is continuous with respect to this topology.*

2.2. **Types.** Let T be a $G_{\max}$-tree and let H and H' be finite induced subgraphs of $G(T)$. We say that H and H' are *strongly isomorphic* if there is an isomorphism $\varphi : H \to H'$ of ordered graphs such that for all 2-element sets $\{x, y\}$ and $\{x', y'\}$ of vertices of H we have

$$\Delta(x, y) \leq \Delta(x', y') \Leftrightarrow \Delta(\varphi(x), \varphi(y)) \leq \Delta(\varphi(x'), \varphi(y')).$$

The *type* of a finite induced subgraph H of $G(T)$ is its strong isomorphism type. The map from n-tuples from $G_{\max}$ to their type τ is continuous. Let us call a type τ *skew* if for all $x \neq y$ and $x' \neq y'$, if $\Delta(x, y) = \Delta(x', y')$

then $x \wedge y = x' \wedge y'$, that is, there is at most one meet at each height in the subtree generated by the n-tuple.

Among all types of n-tuples let us single out the *equidistant type*, the type in which $\Delta(x, y) = k$ for some fixed k for all $x \neq y$ in the n-tuple, and thus also $x \wedge y$ is fixed for all $x \neq y$.

Claim 2.5. *If the type τ of an n-tuple $x_0 < x_1 \cdots < x_{n-1}$ from $G_{\max}$ is not the equidistant type then the ordered subgraph of $G_{\max}$ spanned by $\{x_0, \dots, x_{n-1}\}$ has a non-trivial order-module.*

Proof. Let t_0 be a node of maximal height which is a tree-meet of two distinct points from the n-tuple. Let $I = \{x_i : t_0 \subseteq x_i, i < n\}$. By the choice of t_0 it follows that $|I| \geq 2$, that I is an interval in $\langle x_0, x_1 \dots, x_{n-1}\rangle$ and $x_i \wedge x_j = t_0$ for all $x_i \neq x_j$ in I. Since τ is not the equidistant type, $|I| < n$. Let $x_i \notin I$ be arbitrary and fix some $x_j \in I$. As $\Delta(x_i, x_j) \neq |t_0|$, the maximality of $|t_0|$ shows that $\Delta(x_i, x_j) < |t_0|$ and consequently there is some $s_0 \subseteq t_0$ with $|s_0| < |t_0|$ such that $x_i \wedge x_j = s_0$ for all $x_j \in I$. Thus, for all $x_j, x_k \in I$ it holds that $x_j < x_i \iff x_k < x_i$ and $x_j E x_i \iff x_k E x_i$, affirming that I is an order-module. □

Given a type τ of a finite induced subgraph of $G(T)$, we denote by $\binom{G(T)}{\tau}$ the set of all induced subgraphs of $G(T)$ of type τ.

Corollary 2.6. *The probability for a graph on $\{0, 1, \dots n-1\}$, chosen randomly with uniform probability, to have a copy in a $G_{\max}$ with respect to any ordering with a type which is not the equidistant type, tends to* 0 *as n grows to infinity.*

Proof. It is enough to show that the probability of having a non-trivial module tends to zero as n grows, since a graph with no non-trivial module has no non-trivial order-module with respect to any ordering.

Suppose that $I \subseteq \{0, 1, \dots n-1\}$ is a nontrivial module of a graph on $\{0, 1, \dots n-1\}$. Then there are distinct $i, j \in I$ and $\ell < n$ outside I. Suppose there is $x < n$, distinct from i, j and ℓ, such that i and x form an edge iff j and x do not and moreover, ℓ and i form an edge iff ℓ and x do not. Then, since $i, j \in I$ and I is a module, we have $x \in I$ as well. By the same argument for x, i and ℓ instead of i, j and x, $\ell \in I$. But this contradicts the choice of ℓ.

It is a standard calculation that the probability that for every triple $\{i, j, \ell\} \subseteq \{0, 1, \dots, n-1\}$ there exists x as above tends to 1 as n tends to infinity. □

2.3. **The Halpern-Läuchli theorem.**

For the last step of the proof of the main theorem we shall be needing the classic partition theorem on level-products of trees by Halpern and Läuchli.

Definition 2.7. *Let T be a subtree of $\omega^{<\omega}$. For every $D \subseteq T$ and $n \in \omega$ let $\mathrm{Lev}_D(n) = D \cap \omega^n$. For subtrees $T_1, \dots, T_\ell$ of $\omega^{<\omega}$ let*

$$\bigotimes_{i=1}^{\ell} T_i = \bigcup_{n\in\omega} \prod_{i=1}^{\ell} \mathrm{Lev}_{T_i}(n).$$

For $D_1 \subseteq T_1, \dots, D_\ell \subseteq T_\ell$ let

$$\bigotimes_{i=1}^{\ell} D_i = \prod_{i=1}^{\ell} D_i \cap \bigotimes_{i=1}^{\ell} T_i.$$

For $n, m \in \omega$ with $n \leq m$, a sequence $(D_1, \dots, D_\ell)$ with $D_1 \subseteq T_1, \dots, D_\ell \subseteq T_\ell$ is (n,m)-dense *in $\bigotimes_{i=1}^{\ell} T_i$ if for all $(t_1, \dots, t_\ell) \in \prod_{i=1}^{\ell} \mathrm{Lev}_{T_i}(n)$ there is $(d_1, \dots, d_\ell) \in \prod_{i=1}^{\ell} \mathrm{Lev}_{D_i}(m)$ with $t_1 \subseteq d_1, \dots, t_\ell \subseteq d_\ell$.*

Theorem 2.8 (Halpern-Läuchli)**.** *Let $\ell, k > 0$ be natural numbers and let $T_1, \dots, T_\ell$ be finitely splitting subtrees of $\omega^{<\omega}$. For every coloring $c : \bigotimes_{i=1}^{\ell} T_i \to k$ there are $t_1 \in T_1, \dots, t_\ell \in T_\ell$ and $D_1 \subseteq (T_1)_{t_1}, \dots, D_\ell \subseteq (T_\ell)_{t_\ell}$ such that c is constant on $\bigotimes_{i=1}^{\ell} D_i$ and for every $n \in \omega$ there is $m \geq n$ such that $(D_1, \dots, D_\ell)$ is (n,m)-dense in $\bigotimes_{i=1}^{\ell} (T_i)_{t_i}$.*

This version of the Halpern-Läuchli Theorem follows easily from Theorem 1 in Halpern and Läuchli's original paper [10] and is essentially the version of the theorem that is quoted by Blass in [2]. See Section 3.1 in [20] for various other formulations of the theorem.

3. The main theorem

Theorem 3.1. *For every type τ of a finite induced subgraph of $G_{\max}$, every $G_{\max}$-tree T and every continuous coloring $c : \binom{G(T)}{\tau} \to 2$ there is a $G_{\max}$-subtree S of T such that c is constant on $\binom{G(S)}{\tau}$.*

We prove this theorem in a series of lemmas. First we fix a type τ of a finite induced subgraph of $G_{\max}$ and a continuous coloring $c : \binom{G_{\max}}{\tau} \to 2$.

We may assume that the type τ is skew. Otherwise, given a $G_{\max}$-tree T, we choose a skew $G_{\max}$-subtree S of T and then $\binom{G(S)}{\tau}$ is empty. In particular, c is constant on $\binom{G(S)}{\tau}$. This proves Theorem 3.1 in the case that τ is not skew.

Now fix a $G_{\max}$-tree T. If H is a finite induced subgraph of $G(T)$, let $\Delta(H)$ denote the maximal $\Delta(x,y)$ of two distinct vertices of H. For $n \in \omega$ let $H \restriction n = \{x \restriction n : x \text{ is a vertex of } H\}$

Lemma 3.2. *There is a $G_{\max}$-subtree S of T such that for the induced subgraphs H of $G(S)$ of type τ the color $c(H)$ depends only on $H \restriction (\Delta(H) + 1)$.*

Proof. First consider a single finite induced subgraph H of $G(T)$ of type τ. By our definition of $\Delta(H)$, the map $x \mapsto x \restriction (\Delta(H) + 1)$ is a bijection from the set $V(H)$ of vertices of H onto $H \restriction (\Delta(H)+1)$. Let $t_1,\dots,t_\ell$ denote the elements of $H \restriction (\Delta(H)+1)$. For all $\overline{x} = (x_1,\dots,x_\ell) \in [T_{t_1}] \times \cdots \times [T_{t_\ell}]$ the induced subgraph of $G(T)$ on the set $\{x_1,\dots,x_\ell\}$ is isomorphic to H. By the continuity of c, for all such $\overline{x}$ there are open neighborhoods $U_1^{\overline{x}} \ni x_1,\dots,U_\ell^{\overline{x}} \ni x_\ell$ such that for all $(y_1,\dots,y_\ell) \in U_1^{\overline{x}} \times \cdots \times U_\ell^{\overline{x}}$ for the induced subgraph H' of $G(T)$ on the vertices $y_1,\dots,y_\ell$ we have $c(H) = c(H')$.

We may assume that the $U_i^{\overline{x}}$ are basic open sets, i.e., sets of the form $[T_r]$ for some $r \in T$. Since the space $[T_{t_1}] \times \cdots \times [T_{t_\ell}]$ is compact, there is a finite set $F \subseteq [T_{t_1}] \times \cdots \times [T_{t_\ell}]$ such that

$$[T_{t_1}] \times \cdots \times [T_{t_\ell}] = \bigcup_{\overline{x} \in F} \prod_{i=1}^{\ell} U_i^{\overline{x}}.$$

Hence there is some $m \in \omega$, namely the maximal length of the r's with $[T_r] = U_i^{\overline{x}}$ for some $\overline{x} \in F$ and $i \in \{1,\dots,\ell\}$, such that for all induced subgraphs H' of $G(T)$ with $H' \restriction (\Delta(H)+1) = H \restriction (\Delta(H)+1)$ the color $c(H')$ only depends on $H' \restriction m$.

Since for each $m \in \omega$ there are only finitely many sets of the form $H \restriction m$ where H is a subgraph of $G(T)$, there is a function $f : \omega \to \omega$ such that for every finite induced subgraph H of $G(T)$ with $\Delta(H)+1 = n$, the color $c(H)$ only depends on $H \restriction f(n)$. Now let S be a $G_{\max}$-subtree of T such that whenever $s \in S$ is a splitting node of S of length n, then S has no splitting node t whose length is in the interval $(n, f(n)]$. Now for subgraphs H of $G(S)$ of type τ the color $c(H)$ only depends on $H \restriction (\Delta(H)+1)$. □

Lemma 3.3. *Assume that for all finite induced subgraphs H of $G(T)$ of type τ the color $c(H)$ only depends on $H \restriction (\Delta(H)+1)$. Then there is a $G_{\max}$-subtree S of T such that for the induced subgraphs H of $G(S)$ of type τ the color $c(H)$ only depends on $H \restriction \Delta(H)$.*

Proof. Let H be an induced subgraph of $G(T)$ of type τ. We call the unique node $t_0 \in \mathrm{Lev}_T(\Delta(H))$ that has at least two incomparable extensions in $H \restriction (\Delta(H)+1)$ the *highest splitting node* of H. Let L be the induced subgraph of $G_{t_0}^T$ whose vertices are the extensions of t_0 in $H \restriction (\Delta(H)+1)$.

Let $t_1,\dots,t_\ell$ be the elements of $H \restriction (\Delta(H)+1)$ that are not extensions of t_0. Let $\bar{t} = (t_1,\dots,t_\ell)$. The ℓ-tuple $\bar{t}$ determines a coloring $c_{t_0}^{\bar{t}}$ of the induced copies of L in $G_{t_0}^T$ by two colors:

Given an induced copy L' of L in $G_{t_0}^T$ let $s_1,\dots,s_k$ be the vertices of L'. Choose

$$(z_1,\dots,z_k,y_1,\dots,y_\ell) \in [T_{s_1}] \times \cdots \times [T_{s_k}] \times [T_{t_1}] \times \cdots \times [T_{t_\ell}].$$

Now $\{z_1,\dots,z_k,y_1,\dots,y_\ell\}$ is the set of vertices of a copy H' of H in $G(T)$ of type τ. Let $c_{t_0}^{\bar{t}}(L') = c(H')$. Since $c(H')$ depends only on $H' \restriction (\Delta(H)+1)$, $c(H')$ does not depend on the choices of the z_i and y_j.

We construct the required subtree S of T. We do that by choosing a fusion sequence $(T_k)_{k\in\omega}$ along with a strictly increasing sequence $(m_k)_{k\in\omega}$

of natural numbers witnessing that the T_k form a fusion sequence. First let $T_0 = T$ and $m_0 = 0$. Suppose T_k and m_k have been chosen already. Some book-keeping device tells us that for a certain node $t \in \mathrm{Lev}_{T_k}(m_k)$ and a certain finite ordered graph B there has to be an extension $t_0 \in T_{k+1}$ of length $< m_{k+1}$ such that B embeds into $G^{T_{k+1}}_{t_0}$.

We will choose T_{k+1} and m_{k+1} such that t is the only element of $\mathrm{Lev}_{T_k}(m_k)$ that has an extension of length $< m_{k+1}$ in T_{k+1} that is a splitting node. Also, t will only have a single extension of length $< m_{k+1}$ in T_{k+1} that is a splitting node. In particular, we know in advance the size of $\mathrm{Lev}_{T_{k+1}}(|t_0| + 1)$. This gives us a finite upper bound on the number of colorings of the form $c^{\bar{t}}_{t_0}$ of the induced copies of L in $G^{T_k}_{t_0}$. Let N denote this upper bound.

By the Nešetřil-Rödl theorem (Theorem 2.1 above) and the remark immediately following it, there is a finite ordered graph D such that for every collection of at most N colorings by two colors of the induced copies of L in D, there is an induced copy B' of B in D such that all induced copies of L in B' have the same colors with respect to all of the N colorings.

Since T_k is a $G_{\max}$-tree, there is an extension t_0 of t in T_k such that D embeds into $G^{T_k}_{t_0}$. Let $m_{k+1} = |t_0| + 1$. Choose a set $Z \subseteq \mathrm{Lev}_{T_k}(m_{k+1})$ such that each element of $\mathrm{Lev}_{T_k}(m_k)$ other than t has exactly one extension in Z and such that t has no extension in Z. Now for all $\bar{t} = (t_1, \dots, t_\ell) \in Z^\ell$ such that there is an induced subgraph H of $G(T_k)$ with

$$H \restriction |t_0| = \{t_0, t_1 \restriction |t_0|, \dots, t_\ell \restriction |t_0|\}$$

we consider the coloring $c^{\bar{t}}_{t_0}$. By the choice of D, $G^{T_k}_{t_0}$ contains an induced copy B' of B such that all the relevant colorings $c^{\bar{t}}_{t_0}$ are constant on the set of induced copies of L in B'.

Let Y be the set of vertices of B' and let $T_{k+1} = (T_k)_{Y \cup Z}$. This finishes the recursive construction of the trees T_k and of the natural numbers m_k. Let S be the fusion $\bigcap_{k \in \omega} T_k$ of the sequence $(T_k)_{k \in \omega}$. Then S is a $G_{\max}$-tree by our book-keeping.

Let H be an induced subgraph of $G(S)$ of type τ. Let t_0 be the highest splitting node of H. Choose $k \in \omega$ such that $m_k \leq |t_0| < m_{k+1}$. Let $t_1, \dots, t_\ell$ denote the elements of $H \restriction (|t_0| + 1)$ and let $\bar{t} = (t_1, \dots, t_\ell)$. Since S is skew by construction, the set $\{t_1, \dots, t_\ell\}$ is uniquely determined by $H \restriction |t_0|$. But since T_{k+1} was chosen so that all induced copies of L in $G^{T_{k+1}}_{t_0}$ have the same color with respect to $c^{\bar{t}}_{t_0}$, $c(H)$ actually does not depend on the copy of L inside $G^{T_{k+1}}_{t_0}$ that lives on the vertices $\mathrm{succ}_{T_{k+1}}(t_0)$. Any other copy of L in $G^{T_{k+1}}_{t_0}$ would yield the same color. Now $G^{T_{k+1}}_{t_0} = G^S_{t_0}$. It follows that $c(H)$ only depends on $H \restriction |t_0| = H \restriction \Delta(H)$. □

Remark: With Lemma 3.3 the proof of Theorem 3.1 can be completed for the generic case of graphs with no nontrivial modules: for such a graph H, $H \restriction \Delta(H)$ is a single point. For all nodes t for which there is some graph of type τ with $\{t\} = H \restriction \Delta(H)$ let the color of t be defined as the color of some such H. This definition does not depend on the choice of H, by Lemma 3.3, and assigns a color to all sufficiently high nodes t. Now either there is a node t such that all t' extending t are of the first color, or there is a dense set of nodes t' of the other color. Now it is straightforward to find a fusion sequence in which all splitting nodes are of the same color, thus proving the theorem in this case.

For the general case more work is needed, in which the Halpern-Läuchli Thoerem plays a crucial role. The general proof covers also the special case which was sketched in the remark above.

Definition 3.4. *For a finite induced subgraph H of $G(T)$ of type τ let $\Delta'(H)$ denote the minimal $n \in \omega$ such that $|H \restriction n| = |H \restriction \Delta(H)|$.*

Lemma 3.5. *Let T be a normal, skew $G_{\max}$-tree. Assume that for all finite induced subgraphs H of $G(T)$ of type τ the color $c(H)$ only depends on $H \restriction \Delta(H)$. Let H_0 be an induced subgraph of $G(T)$ of type τ. Let $n = \Delta'(H_0)$. Call an induced copy H of H_0 in $G(T)$ of type τ* compatible *with H_0 if $H \restriction n = H_0 \restriction n$. Then there is a $G_{\max}$-subtree S of T such that $\mathrm{Lev}_S(n) = \mathrm{Lev}_T(n)$ and c is constant on the set of subgraphs of $G(S)$ that are compatible with H_0.*

Proof. Let $t_1, \dots, t_\ell$ be an enumeration of $H_0 \restriction n$ without repetition such that the highest splitting node of $T(H_0)$ is an extension of t_1. Note that for every graph H of type τ that is compatible with H_0, the highest splitting node of H is an extension of t_1. We may assume that for every splitting node t of T_{t_1} there is a graph H of type τ, compatible with H_0, whose highest splitting node is t. This can be achieved by thinning out the tree T above level n in order to make sure that the graphs G_t are sufficiently large for all splitting nodes t of T that extend t_1.

We define an auxiliary coloring

$$\bar{c} : \bigotimes_{i=1}^{\ell} T_{t_i} \to 2$$

as follows:

Given $(s_1, \dots, s_\ell) \in \bigotimes_{i=1}^{\ell} T_{t_i}$, for every $i \in \{1, \dots, \ell\}$ let x_i be the minimal vertex of $G(T_{t_i})$ such that $s_i \subset x_i$. Let $m \geq n$ be minimal such that $x_1 \restriction m$ is a splitting node of T. Let $s' = x_1 \restriction m$ and choose a graph H that is compatible with H_0 such that s' is the highest splitting node of H. The graph H exists by our assumptions on T. We can choose H in such a way that $x_2, \dots, x_\ell$ are vertices of H. Now let $\bar{c}(s_1, \dots, s_\ell) = c(H)$.

By our assumptions on T, the color $c(H)$ only depends on $H \restriction m$. This means that $\bar{c}(s_1, \dots, s_\ell)$ depends on our choice of m and the sequences $x_1 \restriction m, x_2 \restriction m, \dots, x_\ell \restriction m$ and on the fact that $x_1 \restriction m$ is the highest splitting node of H, but it does not depend on the choice of H above the m-th level.

By the Halpern-Läuchli theorem, there are $r_1 \in T_{t_1}, \dots, r_\ell \in T_{t_\ell}$ and sets $D_1 \subseteq (T_{t_1})_{r_1}, \dots, D_\ell \subseteq (T_{t_\ell})_{r_\ell}$ such that $\bar{c}$ is constant on $\bigotimes_{i=1}^{\ell} D_i$ and for all $m \geq n$ there is $k \geq m$ such that $(D_1, \dots, D_\ell)$ is (m,k)-dense in $\bigotimes_{i=1}^{\ell}(T_{t_i})_{r_i}$. We may assume $t_i \subseteq r_i$ and hence $(T_{t_i})_{r_i} = T_{r_i}$ for all $i \in \{1, \dots, \ell\}$.

We now construct a fusion sequence $(T_j)_{j\in\omega}$ and a strictly increasing sequence $(m_j)_{j\in\omega}$ of natural numbers as follows:

Let

$$T_0 = \bigcup_{i=1}^{\ell} T_{r_i} \cup \{t \in T : t \text{ is not comparable with any } t_i,\ i \in \{1, \dots, \ell\}\}$$

and $m_0 = n$. Suppose we have chosen T_j and m_j for some $j \in \omega$. Choose $m > m_j$ such that all $t \in \mathrm{Lev}_{T_j}(m_j)$ have an extension s of length $< m$ that is a splitting node in T_j. Let $k \geq m$ be such that $(D_1, \dots, D_\ell)$ is (m,k)-dense in $\bigotimes_{i=1}^{\ell} T_{r_i}$.

Now choose a set $Z \subseteq \mathrm{Lev}_{T_j}(k)$ such that the following hold:

(1) For all $t \in \mathrm{Lev}_{T_j}(m)$, if t and t_1 are incomparable, then t has exactly one extension in Z.

(2) For all $t \in \mathrm{Lev}_{T_j}(m_j)$, if $t_1 \subseteq t$, then t has exactly one extension in Z.

(3) For each $i \in \{1, \dots, \ell\}$, $T_{t_i} \cap Z \subseteq D_i$.

Finally, we choose $m_{j+1} > k$ such that each element of $Z \cap (T_j)_{t_1}$ has an extension of length $< m_{j+1}$ that is a splitting node of T_j. We choose $Z' \subseteq \mathrm{Lev}_{T_j}(m_{j+1})$ such that the following hold:

(0′) Every $r \in Z'$ is an extension of some $t \in Z$.

(1′) Let $t \in Z \cap (T_j)_{t_1}$ and let s be the minimal splitting node of T_j that extends t. Then every immediate successor of s has exactly one extension in Z', namely the lexicographically minimal extension in $\mathrm{Lev}_{T_j}(m_{j+1})$.

(2′) For every $r \in Z' \cap (T_j)_{t_1}$ there are a splitting node s of T_j and $t \in Z$ such that $t \subseteq s \subseteq r$.

(3′) For all $t \in Z$ that are incomparable with t_1, $Z' \cap (T_j)_t$ consists of the lexicographically minimal extensions of elements of $Z \cap (T_j)_t$ in $\mathrm{Lev}_{(T_j)_t}(m_{j+1})$.

Now let

$$T_{j+1} = \{t \in T_j : t \text{ is comparable with an element of } Z'\}.$$

This finishes the construction of the sequences $(T_j)_{j\in\omega}$ and $(m_j)_{j\in\omega}$.

Now let $S = \bigcap_{j\in\omega} T_j$. The tree S is generated by the set $\bigcup_{j\in\omega} T_j \restriction m_j$. For every splitting node s of S we made sure that all the immediate successors of s in T are also in S. Hence S is a strong subtree of T. Since S is perfect, it follows from Lemma 2.3 that S is a $G_{\max}$-tree.

We now show that c is constant on the set of all induced subgraphs H of $G(S)$ that are compatible with H_0. Let H be an induced subgraph of $G(S)$ that is compatible with H_0. Let s be the highest splitting node of H and choose $j \in \omega$ such that $m_j < |s| < m_{j+1}$. Note that s is an extension of t_1. By conditions (2) and ($1'$), no restriction of s to some number in the interval $[m_j, |s|)$ is a splitting node of S.

In the construction of m_{j+1} and T_{j+1} we chose integers m and k such that $m_j < m \leq k < m_{j+1}$. By (2), $(T_{j+1})_{t_1}$ has no splitting node whose length is in the interval $[m_j, k)$ and thus $k \leq |s|$. By ($0'$) and ($3'$), for each vertex x of H with $t_1 \not\subseteq x$, $x \restriction |s|$ is the lexicographically minimal extension of $x \restriction k$ in T. But $\mathrm{Lev}_{T_{j+1}}(k) \cap \bigcup_{i=1}^{\ell} T_{t_i} \subseteq \bigcup_{i=1}^{\ell} D_i$. It follows that $c(H)$ is equal to the constant color that $\bar{c}$ assumes on the set $\bigotimes_{i=1}^{\ell} D_i$. □

Lemma 3.6. *Assume that for all finite induced subgraphs H of $G(T)$ of type τ the color $c(H)$ only depends on $H \restriction \Delta(H)$. Then there is a $G_{\max}$-subgraph S of T such that for all finite induced subgraphs H of type τ, $c(H)$ only depends on $H \restriction \Delta'(H)$.*

Proof. We may assume that T is normal and skew and that for every induced subgraph H of $G(T)$ of type τ and every splitting node $t \in T$, H embeds into G_t^T. Also, if for every subgraph H of $G(T)$ of type τ the tree of initial segments of vertices of H only has a single splitting node, Lemma 3.5 gives us a $G_{\max}$-subtree S of T such that c is constant on $G(S)$. Hence we can assume that the tree of initial segments of a graph H of type τ has at least two different splitting nodes.

Again we construct a fusion sequence $(T_k)_{k\in\omega}$ and a sequence $(m_k)_{k\in\omega}$ witnessing this. In our construction we make sure that for all $k \in \omega$, T_{k+1} has exactly one splitting node whose length is in the interval $[m_k, m_{k+1})$ and the length of this splitting node is exactly $m_{k+1} - 1$.

Let t be the minimal splitting node of T. Let $T_0 = T$ and $m_0 = |t| + 1$. Suppose T_k and m_k have been chosen. By Lemma 3.5, for every finite induced subgraph H_0 of $G(T)$ of type τ such that $\Delta'(H_0) = m_k$ there is a $G_{\max}$-subtree T_k' of T_k such all copies H of H_0 of type τ in $G(T_k)$ that are compatible with H_0 have the same color $c(H)$. Iterating this argument finitely many times, we find a G_{max}-subtree T_k'' of T_k such that for all induced subgraphs H of $G(T)$ of type τ with $\Delta'(H) = m_k$ the color $c(H)$ only depends on $H \restriction m_k$.

Now some book-keeping device tells us that a certain $t \in \mathrm{Lev}_{T_k}(m_k)$ should have an extension t_0 of length $< m_{k+1}$ such that a certain finite ordered graph F embeds into $G_{t_0}^{T_{k+1}}$. We choose an extension t_0 of t such that F embeds into $G_{t_0}^{T_k''}$ and let $m_{k+1} = |t_0| + 1$. Let $Z \subseteq \mathrm{Lev}_{T_k}(m_{k+1})$ be such that $\mathrm{succ}_{T_k}(t_0) \subseteq Z$ and each $s \in \mathrm{Lev}_{T_k}(m_k) \setminus \{t\}$ has exactly one extension in Z. Now let $T_{k+1} = (T_k)_Z$. This finishes the definition of the fusion sequence $(T_k)_{k\in\omega}$ and the sequence $(m_k)_{k\in\omega}$.

Finally let $S = \bigcap_{k\in\omega} T_k$. By our book-keeping, S is a $G_{\max}$-tree. Whenever H is an induced subgraph of $G(S)$ of type τ there is a unique $k \in \omega$ such that $\Delta'(H) = m_k$. Since S is a $G_{\max}$-subtree of T_{k+1}, by the choice of T_{k+1}, the color $c(H)$ only depends on $H \restriction \Delta'(H)$. □

Proof of Theorem 3.1. Let ℓ denote the number of vertices of graphs of type τ. We prove the theorem by induction on ℓ. If $\ell = 1$, then we just observe that every continuous coloring c of the subgraphs of $G(T)$ of type τ is constant on an open set $U \subseteq [T]$.

Now assume that $\ell > 1$ and for all types τ' of subgraphs of $G(T)$ with less than ℓ vertices the theorem holds. By Lemma 3.6 there is a $G_{\max}$-subtree T' of T such that on T' the color $c(H)$ of a graph of type τ only depends on $H \restriction \Delta'(H)$. Given such a graph H, let $t_1, \dots, t_k$ denote the distinct elements of $H \restriction \Delta'(H)$ and choose $(x_1, \dots, x_k) \in [T'_{t_1}] \times \cdots \times [T'_{t_k}]$. Now the type τ' of the induced subgraph of $G(T)$ on the vertices $x_1, \dots, x_k$ only depends on τ.

We define a coloring c' on the set of all subgraph H' of $G(T')$ of type τ'. Given such a subgraph, let H be a graph of type τ such that $H \restriction \Delta'(H) = H' \restriction (\Delta(H') + 1)$. Such a graph H exists since T' is a $G_{\max}$-tree. Now let $c'(H') = c(H)$. By our assumption on T', $c(H)$ only depends on $H \restriction \Delta'(H)$ and hence $c'(H')$ is independent of the choice of H. Clearly, graphs of type τ' have less than ℓ vertices and hence, by our inductive hypothesis, there is a $G_{\max}$-subtree S of T' such that c' is constant on subgraphs of S of type τ'. But now c is constant on subgraphs of S of type τ. This finishes the proof of the theorem. □

3.1. **The Baire measurable case.**

Definition 3.7. *Let τ be the type of a finite induced subgraph of $G_{\max}$ and let T be a $G_{\max}$-tree. A coloring $c : \binom{G(T)}{\tau} \to 2$ is* Baire measurable *if the sets $c^{-1}(0)$ and $c^{-1}(1)$ have the Baire property in the Polish space $\binom{G(T)}{\tau}$.*

Our main Theorem 3.1 can be extended to Baire measurable colorings using standard methods from descriptive set theory.

We need the following lemma.

Lemma 3.8. *Let τ be the type of a nonempty finite induced subgraph of $G_{\max}$ and let $c : \binom{G(T)}{\tau} \to 2$ be a Baire measurable measurable coloring. Then there is a $G_{\max}$-subtree S of T such that c is continuous on $\binom{G(S)}{\tau}$.*

Proof. We choose open sets $U, V \subseteq \binom{G(T)}{\tau}$ such that the symmetric differences $c^{-1}(0) \triangle U$ and $c^{-1}(1) \triangle V$ are meager. Let $(N_n)_{n\in\omega}$ be a sequence of closed nowhere dense subsets of $\binom{G(T)}{\tau}$ such that

$$(c^{-1}(0) \triangle U) \cup (c^{-1}(1) \triangle V) \subseteq \bigcup_{n\in\omega} N_n.$$

Our goal is to construct a $G_{\max}$-subtree S of T such that $\binom{G(S)}{\tau}$ is disjoint from $\bigcup_{n\in\omega} N_n$. In this case, the preimages of 0 and 1 of the restriction of c to the set $\binom{G(S)}{\tau}$ are the open subsets $U \cap \binom{G(S)}{\tau}$ and $V \cap \binom{G(S)}{\tau}$ of $\binom{G(S)}{\tau}$. It follows that c is continuous on $\binom{G(S)}{\tau}$.

It remains to find the $G_{\max}$-subtree S that avoids the set $\bigcup_{n\in\omega} N_n$. We construct a fusion sequence $(T_k)_{k\in\omega}$ of $G_{\max}$-subtrees of T and a strictly increasing sequence $(m_k)_{k\in\omega}$ of natural numbers and then put $S=\bigcap_{k\in\omega} T_k$.

Suppose T_k and m_k have already been chosen. We assume that for all $t\in \mathrm{Lev}_{T_k}(m_k)$ and all $s\in T$ with $t\subseteq s$ we have $s\in T_k$. Some book-keeping will tell us that we have to find a splitting node s above a certain $t\in \mathrm{Lev}_{T_k}(m_k)$ such that a certain finite ordered graph H embeds into $G_s^{T_{k+1}}$. Since T_k is a $G_{\max}$-tree, there is $m>m_k$ such that t has an extension $s\in T_k$ of length $<m$ such that H embeds into $G_s^{T_k}$.

Now suppose H is a subgraph of $G(T_k)$ of type τ such that $\Delta(H)<m$. Let ℓ be the number of vertices of H. The set $H\restriction m$ determines an open subset O of $\binom{G(T)}{\tau}$. The set $\bigcup_{n\leq k} N_n$ is closed and nowhere dense in $\binom{G(T)}{\tau}$. Hence there is a nonempty open subset of O that is disjoint from $\bigcup_{n\leq k} N_n$. It follows that the ℓ elements of $H\restriction m$ have extensions $s_1,\dots,s_\ell\in T_k$ such that the open subset of $\binom{G(T)}{\tau}$ determined by $s_1,\dots,s_\ell$ is disjoint from $\bigcup_{n\leq k} N_n$. We may assume that $s_1,\dots,s_\ell$ are all of the same length $m'>m$.

Let $Z\subseteq \mathrm{Lev}_{T_k}(m')$ be a set that contains exactly one extension of every element of $\mathrm{Lev}_{T_k}(m)$ and in particular the elements $s_1,\dots,s_\ell$. Now consider the $G_{\max}$-tree T' consisting of all elements of T_k that are comparable to one of the elements of Z. Whenever H' is a subgraph of $G(T')$ of type τ with $H'\restriction m=H\restriction m$, then $H'\restriction m'=\{s_1,\dots,s_\ell\}$. In particular, H' is not an element of $\bigcup_{n\leq k} N_n$.

We can iterate this argument and obtain $m_{k+1}>m'$ and a set $X\subseteq \mathrm{Lev}_{T_k}(m_{k+1})$ with the following property: If H' is a subgraph of $G(T_k)$ with $\Delta(H')<m$ such that $H'\restriction m_{k+1}\subseteq X$, then H' is not an element of $\bigcup_{n\leq k} N_n$.

Let

$$T_{k+1}=\{t\in T_k:\exists s\in X(s\subseteq t\vee t\subseteq s)\}.$$

Now for every subgraph H' of $G(T_{k+1})$ of type τ with $\Delta(H')<m$ we have $H'\notin\bigcup_{n\leq k} N_n$. This finishes the recursive definition of the sequences $(T_k)_{k\in\omega}$ und $(m_k)_{k\in\omega}$.

Finally let $S=\bigcap_{k\in\omega} T_k$. We use the book-keeping in the construction of the T_k to make sure that S is a $G_{\max}$-tree. Let $n\in\omega$ and suppose H is a subgraph of $G(S)$ of type τ. Then there is $k\in\omega$ such that $\Delta(H)<m_k$. We can choose $k\geq n$. Note that $\mathrm{Lev}_S(m_k)=\mathrm{Lev}_{T_k}(m_k)$. By the choice of T_{k+1} and since $S\subseteq T_{k+1}$, $H\notin\bigcup_{i\leq k} N_i$. In particular, $H\notin N_n$. This shows that $\binom{G(S)}{\tau}$ is disjoint from $\bigcup_{n\leq\omega} N_n$. It follows that c is continuous on the set $\binom{G(S)}{\tau}$. □

The generalization of Theorem 3.1 to Baire measurable colorings now follows easily from Lemma 3.8.

Theorem 3.9. *For every type τ of a finite induced subgraph of $G_{\max}$, every $G_{\max}$-tree T and every Baire measurable coloring $c : \binom{G(T)}{\tau} \to 2$ there is a $G_{\max}$-subtree S of T such that c is constant on $\binom{G(S)}{\tau}$.*

Proof. By Lemma 3.8, there is a $G_{\max}$-subtree T' of T such that c is continuous on $\binom{G(T')}{\tau}$. Now by Theorem 3.1 there is a $G_{\max}$-subtree S of T' such that c is constant on $\binom{G(S)}{\tau}$. □

Let A be a finite ordered graph. Let $k(A)$ denote the number of different (skew) types of A.

Theorem 3.10. *For every finite graph A and a universal inverse limit graph G, for every finite Baire measurable partition of $\binom{G}{A}$, there is a closed ordered copy G' of G in G such that the type of each $B \in \binom{G}{A}$ determines its cell in the partition.*

This theorem follows by iterating Theorem 3.9.

By Corollary 2.6, the probability that a finite ordered graph A can have only one skew type in any universal limit graph G tends to 1 with the size of A, thus we get:

Theorem 3.11. *With high probability, a finite ordered graph A satisfies for every universal inverse limit graph G,*

$$G \to_{Baire} (G)^A.$$

3.2. **Concluding remarks.** Sauer [18] obtained the Ramsey theorem for partitioning subgraphs of Rado's homogeneous and universal countable graph. It is interesting to note that both Sauer's theorems and Blass' theorem can be derived from Milliken's theorem on strong subtrees (see [20]). Theorem 3.1 above, though, does not follow readily from Milliken's theorem, as the $T_{\max}$ subtrees that are gotten in various stages, most importantly when applying the Nešetril-Rödl theorem, fail to be strong subtrees.

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E-mail address: `frick.stefanie@googlemail.com`

Department of Mathematics, University of Hamburg, Bundesstr. 55, 20249 Hamburg, Germany

E-mail address: `stefan.geschke@uni-hamburg.de`

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva, 84105 Israel

E-mail address: `kojman@math.bgu.ac.il`