A DUAL OPEN COLORING AXIOM

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Abstract. We discuss a dual of the Open Coloring Axiom ($\mathrm{OCA}_{[\mathrm{ARS}]}$) introduced by Abraham, Rubin, and Shelah [2] and show that it follows from a statement about continuous colorings on Polish spaces that is known to be consistent. We mention some consequences of the new axiom and show that $\mathrm{OCA}_{[\mathrm{ARS}]}$ implies that all cardinal invariants in Cichoń's diagram are at least $\aleph_2$.

1. Introduction

There are two versions of the Open Coloring Axiom, the one introduced by Abraham, Rubin, and Shelah [2] and the one introduced by Todorcevic [13]. We deal with the first axiom, which we denote by $\mathrm{OCA}_{[\mathrm{ARS}]}$, following the notation in [11].

For every topological space X, the set $[X]^2$ of two-element subsets of X carries a natural topology, the one inherited from X^2. The basic open sets of $[X]^2$ are the sets of the form $\{\{x, y\} \in [X]^2 : x \in U, y \in V\}$, where U and V are disjoint open sets in X.

Let $U_1, \dots, U_n$ be open sets with $[X]^2 = U_1 \cup \dots \cup U_n$. We refer to $C = (U_1, \dots, U_n)$ as a finite *open pair cover* on X. A set $H \subseteq X$ is *C-homogeneous* (or just *homogeneous* if C is clear from the context) if for some $i \in \{1, \dots, n\}$, $[H]^2 \subseteq U_i$.

$\mathrm{OCA}_{[\mathrm{ARS}]}$ is the statement "for every separable metric space X of size $\aleph_1$ and every finite open pair cover C on X, X is covered by countably many C-homogeneous sets".

If we consider, for a given finite open pair cover C on an uncountable Polish space X, the σ-ideal $\mathcal{I}_C$ generated by the C-homogeneous subsets of X, the axiom $\mathrm{OCA}_{[\mathrm{ARS}]}$ easily implies $\mathrm{non}(\mathcal{I}_C) > \aleph_1$. Here $\mathrm{non}(\mathcal{I}_C)$ is the uniformity of the ideal $\mathcal{I}_C$, i.e., the least size of a subset of X not in $\mathcal{I}_C$. Dual to $\mathrm{non}(\mathcal{I}_C)$ is covering number $\mathrm{cov}(\mathcal{I}_C)$, the least size of a subset $\mathcal{F}$ of $\mathcal{I}_C$ such that $X = \bigcup \mathcal{F}$.

Dualizing $\mathrm{OCA}_{[\mathrm{ARS}]}$ we therefore obtain the statement "for every Polish space X and every finite open pair cover C on X, $\mathrm{cov}(\mathcal{I}_C) < 2^{\aleph_0}$". We will refer to this statement as the *dual open coloring axiom*.

A special case of finite open pair covers are the so called *continuous pair colorings* studied in [8] and [6]. A continuous pair coloring on a Polish space X is simply a continuous map $c : [X]^2 \to n$ or equivalently, letting $U_i = c^{-1}(i)$ for $i \in n$, a finite open pair-cover $(U_0, \dots, U_{n-1})$ on X with the U_i pairwise disjoint.

One of the drawbacks of the concept of continuous pair colorings is that connected Polish spaces admit only constant continuous pair colorings. This is the

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main reason why we are interested in the more general concept of finite open pair covers.

In [8] it was shown that the dual open covering axiom is consistent when restricted to continuous pair colorings. In the present paper we give a proof of this fact that gives a bit more information than the original consistency proof. Moreover, we show that the general dual open coloring axiom follows from its restriction to continuous pair colorings.

2. Covering X^{n+1} by n-ary functions

One reason to study finite open pair covers is their connection to coverings of powers of a space X by graphs of functions.

Definition 2.1. Let X be a set and $n \in \omega$. A point $(x_0, \dots, x_n) \in X^{n+1}$ is *covered* by a function $f : X^n \to X$ if there is a permutation σ of $n+1$ such that $f(x_{\sigma(0)}, \dots, x_{\sigma(n-1)}) = x_{\sigma(n)}$.

Let $A \subseteq X^{n+1}$. A family $\mathcal{F}$ of functions from X^n to X *covers* A if every point in A is covered by a function in $\mathcal{F}$.

By a theorem of Kuratowski [10], for every infinite cardinal κ, exactly κ n-ary functions are needed to cover all of $(\kappa^{+n})^{n+1}$. Here κ^{+n} denotes the n-th cardinal successor of κ. (Kuratowski originally formulated his theorem in a slightly different way. The formulation used here and its proof can be found in [1].)

Example 2.2. Let X be a Polish space and $n \geq 1$. Consider the following open pair cover on X^{n+1}. For every $i < n+1$ let

$$U_i = \{\{(x_0, \dots, x_n), (y_0, \dots, y_n)\} \in [X^{n+1}]^2 : x_i \neq y_i\}.$$

Let $C = (U_0, \dots, U_n)$. Clearly, a set $H \subseteq X^{n+1}$ is C-homogeneous iff there is a function $f : X^n \to X$ and a permutation σ of $n+1$ such that $\sigma(n) = i$ and

$$H \subseteq \{(x_0, \dots, x_n) : x_{\sigma(n)} = f(x_{\sigma(0)}, \dots, x_{\sigma(n-1)})\}.$$

If $|X| = 2^{\aleph_0} \leq \kappa^{n+1}$ for some infinite cardinal κ, then, by Kuratowski's theorem,

$$\mathfrak{hm}(C) \leq \kappa.$$

Example 2.2 shows that while maximal homogeneous sets exist for all finite open pair covers by Zorn's lemma, these maximal homogeneous sets do not have to be nice in the sense of being Borel, analytic etc. On the other hand, maximal homogeneous sets for continuous pair colorings are always closed since in that case closures of homogeneous sets are again homogeneous by continuity.

Definition 2.3. Let X be a metric space and let d denote the metric on X. For $c \in \mathbb{R}$ we say that a function $f : X \to X$ is *Lipschitz of class* $< c$ if for all $x_0, x_1 \in X$ with $x_0 \neq x_1$,

$$\left| \frac{d(f(x_0), f(x_1))}{d(x_0, x_1)} \right| < c.$$

We say that f is *Lipschitz of class* $\leq c$ if for all $x_0, x_1 \in X$ with $x_0 \neq x_1$,

$$\left| \frac{d(f(x_0), f(x_1))}{d(x_0, x_1)} \right| \leq c.$$

Example 2.4. Let $\varepsilon > 0$. We consider an open-pair cover on $\mathbb{R}^2$. Let

$$U_0 = \left\{ \{(x_0, x_1), (y_0, y_1)\} \in [\mathbb{R}^2]^2 : x_0 \neq y_0 \wedge \left| \frac{x_1 - y_1}{x_0 - y_0} \right| < 1 \right\}$$

and

$$U_1 = \left\{ \{(x_0, x_1), (y_0, y_1)\} \in [\mathbb{R}^2]^2 : x_1 \neq y_1 \wedge \left| \frac{x_0 - y_0}{x_1 - y_1} \right| < 1 + \varepsilon \right\}.$$

Put $C = (U_0, U_1)$. Then $H \subseteq \mathbb{R}^2$ is (U_0, U_1)-homogeneous if either there is a function $f : \mathbb{R} \to \mathbb{R}$ that is Lipschitz of class < 1 such that $H \subseteq \{(x, y) : f(x) = y\}$ or there is a function $f : \mathbb{R} \to \mathbb{R}$ that is Lipschitz of class $< 1 + \varepsilon$ such that $H \subseteq \{(x, y) : x = f(y)\}$.

In particular, $\mathbb{R}^2$ can be covered by $\mathfrak{hm}(C)$ functions that are Lipschitz of class $< 1 + \varepsilon$.

3. Continuous pair colorings

First let us note that, as long as we are interested in uncountable homogeneity numbers of continuous pair colorings, we may restrict our attention to colorings that only use two colors. This is because every continuous pair coloring on a Polish space that uses n colors can be decomposed into colorings that only use two colors.

Namely, replace a coloring $c = [X]^2 \to n$ on a Polish space by $h \circ c$ where $h : n \to 2$ is onto. Then consider the colorings $c \restriction [H]^2$ for every closed $(h \circ c)$-homogeneous set H. The coloring $c \restriction [H]^2$ is again a continuous pair coloring on a Polish space but uses less than n colors. Iterating this we obtain continuous colorings that only use two colors.

This argument does not work for finite open pair covers because homogeneous sets for a finite open pair cover on a Polish spaces are not necessarily included in a homogeneous set that carries a Polish space topology (see the remark after Example 2.2).

We mention two important examples of continuous pair colorings.

Definition 3.1. For $\{x, y\} \in [\omega^\omega]^2$ let

$$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$$

and let

$$c_{\text{parity}}(x, y) = \Delta(x, y) \mod 2.$$

Let $c_{\min} = c_{\text{parity}} \restriction [2^\omega]^2$. As it turns out, $\mathfrak{hm}(c_{\text{parity}}) = \mathfrak{hm}(c_{\min})$ [6, Lemma 2.10]. We define $\mathfrak{hm} = \mathfrak{hm}(c_{\min})$.

It was shown in [8] that $\mathfrak{hm}$ is minimal among the uncountable homogeneity numbers of continuous pair colorings on Polish spaces.

Let us mention a connection between c_{parity}-homogeneous sets and certain functions from ω^ω to itself. This connection was observed in [8].

Remark 3.2. Let d be the metric on ω^ω defined by

$$d(x, y) = \begin{cases} 2^{-\Delta(x,y)}, \text{ if } x \neq y \\ 0, \text{otherwise.} \end{cases}$$

For $x, y \in \omega^\omega$ let $x \otimes y = (x(0), y(0), x(1), y(1), \dots)$. The mapping $\otimes$ is a homeomorphism between $(\omega^\omega)^2$ and ω^ω.

If $H \subseteq \omega^\omega$ is c_{parity}-homogeneous of color 0, then for every $x \in \omega^\omega$ there is at most one $y \in \omega^\omega$ with $x \otimes y \in H$. If H is maximal homogeneous, then there is some y with $x \otimes y \in H$. Thus, a maximal c_{parity}-homogeneous set H of color 0 gives rise to a function $f_H : \omega^\omega \to \omega^\omega$ with $H = \{x \otimes f(x) : x \in \omega^\omega\}$.

Similarly, every maximal c_{parity}-homogeneous set H of color 1 gives rise to a function $f_H : \omega^\omega \to \omega^\omega$ with $H = \{f(x) \otimes x : x \in \omega^\omega\}$. A straight forward calculation shows that if H is of color 0, then f_H is Lipschitz of class ≤ 1 and if H is of color 1, then f_H is Lipschitz of class $\leq 1/2$.

In particular, $(\omega^\omega)^2$ can be covered by $\mathfrak{hm}$ Lipschitz functions of class ≤ 1. Actually, the number of Lipschitz functions of arbitrary class needed to cover $(\omega^\omega)^2$ is exactly $\mathfrak{hm}$ (see [6] for a proof of the latter statement).

From this remark it follows that by Kuratowski's theorem, $\mathfrak{hm}^+ \geq 2^{\aleph_0}$ [8]. On the other hand, Example 2.2 shows that uncountable homogeneity numbers of finite open pair covers on Polish spaces can be more than just one cardinal away from $2^{\aleph_0}$. Moreover, if the size of the continuum is κ^+, then there is a finite open pair cover on $\mathbb{R}^2$ whose homogeneity number is exactly κ.

This is not true for continuous pair colorings on Polish spaces. For instance, after adding, for some infinite cardinal κ, κ^+ Sacks reals to a model of CH using a countable support product, we obtain a model of $\mathfrak{hm} = 2^{\aleph_0} = \kappa^+$ [8].

Also, from Remark 3.2 it follows that there is a family of size $\mathfrak{hm}$ of continuous functions from ω^ω to ω^ω that covers $(\omega^\omega)^2$. Since the size of such a family is at least $\mathfrak{d}$ [6], $\mathfrak{d} \leq \mathfrak{hm}$.

Using this inequality, in [6] it was shown that there is a continuous pair coloring $c_{\max}$ on 2^ω such that for every continuous pair coloring c on a Polish space X we have $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\max})$. We improve this result and show

Theorem 3.3. *Let C be a finite open pair cover on a Polish space X. Then $\mathfrak{hm}(C) \leq \mathfrak{hm}(c_{\max})$.*

This theorem is perhaps a bit surprising since the natural lower bound for uncountable homogeneity numbers of continuous pair colorings, namely $\mathfrak{hm}$, is not (at least not provably in ZFC) a lower bound of the uncountable homogeneity numbers of finite open pair covers on Polish spaces.

The only property of $c_{\max}$ that will be used in the proof of Theorem 3.3 is that its homogeneity number is maximal among the homogeneity numbers of continuous pair colorings on 2^ω.

The proof of Theorem 3.3 uses a series of lemmas. The first lemma is essentially Exercise 13.5 in [9].

Lemma 3.4. *Let X a Polish space. Then there is a Polish space topology on X which refines the original topology and is zero-dimensional.*

Lemma 3.5. *Every Polish space X can be covered by $\leq \mathfrak{d}$ sets that are either singletons or copies of 2^ω.*

Proof. Let τ be the original topology on X. Let τ' be a zero-dimensional Polish topology on X that refines τ. Such a topology exists by Lemma 3.4. Let Y denote the space X with the topology τ'.

Being a Polish space, Y is a continuous image of ω^ω. Since ω^ω can be covered by $\mathfrak{d}$ compact sets, there is a family $\mathcal{K}$ of compact subsets of Y such that $|\mathcal{K}| \leq \mathfrak{d}$ and $\bigcup \mathcal{K} = Y$.

Using Cantor-Bendixson analysis, every $K \in \mathcal{K}$ decomposes into at most countably many points and a (possibly empty) compact set without isolated points. Since Y is zero-dimensional, every non-empty compact subset of Y without isolated points is homeomorphic to 2^ω. Let $K \subseteq Y$ be a copy of 2^ω.

The topology τ' refines the topology τ. But since K is compact with respect to τ', the two topologies coincide on K. It follows that K is homeomorphic to 2^ω as a subspace of X. Thus, the family $\mathcal{K}$ gives rise to a family $\mathcal{K}'$ of size $\leq \mathfrak{d}$ such that $\bigcup \mathcal{K}' = X$ and $\mathcal{K}'$ consists of singletons and copies of 2^ω. $\square$

Lemma 3.6. *Let $C = (U_0, \dots, U_{n-1})$ be an open pair cover on 2^ω. Then there is a continuous coloring $c : [2^\omega]^2 \to n$ such that for all $i < n$, $c^{-1}(i) \subseteq U_i$. In particular, $\mathfrak{hm}(C) \leq \mathfrak{hm}(c)$.*

Proof. Let $\{s, t\} \in [2^\omega]^2$. Let $m \in \omega$ be minimal such that for some $i < n$ the following holds: for all $x, y \in 2^\omega$ with $s \restriction m \subseteq x$ and $t \restriction m \subseteq y$, $\{x, y\} \in U_i$. Let $c(s, t) = i$ where $i < n$ is minimal with the property that for all $x, y \in 2^\omega$ with $s \restriction m \subseteq x$ and $t \restriction m \subseteq y$, $\{x, y\} \in U_i$.

This defines the continuous coloring $c : [2^\omega]^2 \to n$. It is easily checked that it has the desired properties. $\square$

Proof of Theorem 3.3. Let X and C be as in the statement of the theorem. Let $\mathcal{K}$ be a family of size $\leq \mathfrak{d}$ consisting of singletons and copies of 2^ω such that $\bigcup \mathcal{K} = X$.

The singletons in $\mathcal{K}$ are trivially C-homogeneous. Every other element of $\mathcal{K}$ can be covered by $\leq \mathfrak{hm}(c_{\max})$ C-homogeneous sets by Lemma 3.6. Since $\mathfrak{d} \leq \mathfrak{hm}(c_{\max})$, it follows that X can be covered by $\leq \mathfrak{hm}(c_{\max})$ C-homogeneous sets. $\square$

4. Homogeneity numbers are big

We show that $\mathfrak{hm}$ is at least $\mathrm{cof}(\mathcal{N})$, the cofinality of the ideal of measure zero subsets of the real line. The cardinal $\mathrm{cof}(\mathcal{N})$ the biggest cardinal that appears in the Cichoń diagram. Recall the combinatorial characterization of $\mathrm{cof}(\mathcal{N})$ (see [3] or [4, Theorem 2.3.9]):

A function $S : \omega \to [\omega]^{<\omega}$ is a *slalom* (or, more precisely, a 2^n-*slalom*) if for all $n \in \omega$, $|S(n)| \leq 2^n$. A real $r \in \omega^\omega$ *goes through* a slalom S if for all but finitely many $n \in \omega$, $r(n) \in S(n)$. Now $\mathrm{cof}(\mathcal{N})$ is the least size of a family $\mathcal{F}$ of slaloms such that every real $r \in \omega^\omega$ goes through an element of $\mathcal{F}$.

Theorem 4.1. $\mathfrak{hm} \geq \mathrm{cof}(\mathcal{N})$

Proof. We may assume that $\mathfrak{hm} < 2^{\aleph_0}$. By Remark 3.2, there is a family $\mathcal{F}$ of size $\mathfrak{hm}$ of functions from ω^ω to ω^ω such that for every pair $(x, y) \in (\omega^\omega)^2$ there is a function $f \in \mathcal{F}$ such that f is Lipschitz of class ≤ 1 and $f(x) = y$ or f is Lipschitz of class $\leq 1/2$ and $f(y) = x$.

Let χ be a sufficiently large cardinal and fix Skolem functions for the structure $(\mathcal{H}_\chi, \in)$. For $M \subseteq \mathcal{H}_\chi$ and $x \in \mathcal{H}_\chi$ let $M[x]$ denote the closure of $M \cup \{x\}$ under the Skolem functions for $\mathcal{H}_\chi$.

Let M be an elementary submodel of $\mathcal{H}_\chi$ of size $\mathfrak{hm}$ such that $\mathcal{F} \subseteq M$. Let $x \in \omega^\omega$ be arbitrary. We show that x goes through a slalom that belongs to M.

Since $|M[x]|=|M|= \mathfrak{hm} < 2^{\aleph_0}$, there is a real $y \in 2^\omega \setminus M[x]$. Since $\mathcal{F} \subseteq M$, there is a function $f \in M$ such that f is Lipschitz of class ≤ 1 and $f(x) = y$ or f is Lipschitz of class $\leq 1/2$ and $f(y) = x$. But if $f(x) = y$ for any function $f \in M$, then $y \in M[x]$, contradicting the choice of y.

It follows that there is a Lipschitz function $f \in M$ of class $\leq 1/2$ such that $f(y) = x$. For a function $g : \omega^\omega \to \omega^\omega$ being Lipschitz of class $\leq 1/2$ means that for every n, the first $n+1$ coordinates of $g(z)$ only depend on the first n coordinates of z. It follows that for every $n \in \omega$ the set $S(n) = \{f(z)(n) : z \in 2^\omega\}$ is of size at most 2^n

Since $x = f(y) \in f[2^\omega]$, x goes through the slalom S. Since S can be defined using parameters in M, namely f, we have $S \in M$. $\square$

Corollary 4.2. *Assume the dual open coloring axiom. Then*

1. *All cardinal invariants mentioned in Cichoń's diagram are $< 2^{\aleph_0}$.*
2. *For every $\varepsilon > 0$, $\mathbb{R}^2$ can be covered by $< 2^{\aleph_0}$ functions from $\mathbb{R}$ to $\mathbb{R}$ that are Lipschitz of class $< 1 + \varepsilon$.*
3. *Every closed set $S \subseteq \mathbb{R}^2$ either has a (nonempty) perfect 3-clique or it can be covered by $< 2^{\aleph_0}$ of its convex subsets. Here a subset $C \subseteq S$ is a* 3-clique[1] *of S if for any three distinct points in C the triangle spanned by the three points is not a subset of S.*

[1]The name "clique" was chosen since a 3-clique is a clique (in the graph theoretic sense) in the hypergraph $(S, \{A \in [S]^3 : A \text{ is defected in } S\})$. Here $A \subseteq S$ is *defected* in S if the convex hull of A is not included in S.

Proof. (1) follows directly from Theorem 4.1. (2) is Example 2.4. For (3) we have to refer to [8], where is was proved that every closed set S in the real plane that does not have a perfect 3-clique can be covered by $\mathfrak{hm}(c)$ convex subsets for some continuous pair coloring c on ω^ω. $\square$

5. $\mathrm{OCA}_{[\mathrm{ARS}]}$ and cardinal invariants

We show that $\mathrm{OCA}_{[\mathrm{ARS}]}$ implies that all cardinal invariants in Cichoń's diagram are big. The argument for this is a dualization of the argument used in the proof of Theorem 4.1.

The *additivity* $\mathrm{add}(\mathcal{N})$ of the ideal of measure zero subsets of the real line is the least size of a family of measure zero sets whose union is not in the ideal. This cardinal invariant is the smallest in Cichoń's diagram. The combinatorial characterization of $\mathrm{add}(\mathcal{N})$ dual to the one of $\mathrm{cof}(\mathcal{N})$ mentioned before Theorem 4.1 is as follows:

$\mathrm{add}(\mathcal{N})$ is the least size of a subset A of ω^ω, such that there is no countable family of slaloms such that every element of A goes through at least one of these slaloms.

Theorem 5.1. *Assume* $\mathrm{OCA}_{[\mathrm{ARS}]}$. *Then* $\mathrm{add}(\mathcal{N}) \geq \aleph_2$ *and hence all cardinal invariants in Cichoń's diagram are at least* $\aleph_2$.

Proof. Let $A \subseteq \omega^\omega$ be of size $\aleph_1$. We show that there are countably many slaloms such that every real in A goes through one of these slaloms.

By enlarging A if necessary, we may assume that $|A \cap 2^\omega| = \aleph_1$. By $\mathrm{OCA}_{[\mathrm{ARS}]}$, A is covered by countably many c_{parity}-homogeneous sets. The translation between c_{parity}-homogeneous sets and Lipschitz functions (Remark 3.2) shows that there is a countable family $\mathcal{F}$ such that for all $x, y \in A$ there is a function $f \in \mathcal{F}$ such that f is Lipschitz of class ≤ 1 and $f(x) = y$ or f is Lipschitz of class $\leq 1/2$ and $f(y) = x$. Let χ be a sufficiently large cardinal. As in the proof of Theorem 4.1, we fix Skolem functions for the structure $(\mathcal{H}_\chi, \in)$.

Let M be a countable elementary submodel of $(\mathcal{H}_\chi, \in)$ containing $\mathcal{F}$. We claim that every real in A goes through a slalom from M.

Let $x \in A$ be arbitrary. Choose $y \in (A \cap 2^\omega) \setminus M[x]$. This is possible since $M[x]$ is countable and $|A \cap 2^\omega| = \aleph_1$. Clearly, no function in M maps x to y. Since $\mathcal{F} \subseteq M$, there is a function $f \in M$ such that f is Lipschitz of class $\leq 1/2$ and $f(y) = x$. In particular, $x \in f[2^\omega] \in M$. Exactly as in the proof of Theorem 4.1 it follows that x goes through a slalom from M. $\square$

6. A consistency result

In [8] it was shown that in the iterated Sacks model, for every continuous coloring c on a Polish space X, $\mathfrak{hm}(c) \leq \aleph_1$. (Recall that $2^{\aleph_0} = \aleph_2$ in the Sacks model.) By Theorem 3.3, this implies that the Sacks model is a model of the dual open coloring axiom.

We will construct another model of set theory in which the continuum is $\aleph_2$ but $\mathfrak{hm}(c_{\max}) = \aleph_1$. The reason for constructing this model is that until now, no reasonable cardinal characteristic of the continuum was known that can be strictly bigger than $\mathfrak{hm}(c_{\max})$ (except for $2^{\aleph_0}$, of course).

Definition 6.1. For $n \in \omega$ a set $X \subseteq \omega^\omega$ is *n-ary* if there is no $F \in [X]^{n+1}$ such that Δ is constant on $[F]^2$. In other words, no node in

$$T(X) = \{s \in \omega^{<\omega} : \exists x \in X (s \subseteq x)\}$$

has more than n immediate successors. Note that the closure of an n-ary set is n-ary.

Let $\mathfrak{l}_{n,\omega}$ be the least size of a family of n-ary sets that covers ω^ω. For every $m > n$ let $\mathfrak{l}_{n,m}$ be the least size of a family of n-ary sets that covers m^ω. We will refer to the numbers $\mathfrak{l}_{n,\omega}$ and $\mathfrak{l}_{n,m}$ as *localization numbers*.

Obviously, for all $n, m \in \omega$ with $n < m$, $\mathfrak{l}_{n,m} \leq \mathfrak{l}_{n,\omega}$ and $\mathfrak{d} \leq \mathfrak{l}_{n,\omega}$. Apart from the trivial monotonicity properties of the localization numbers we have $\mathfrak{l}_{n+1,n+2} \leq \mathfrak{l}_{n,n+1}$ [12]. By induction on m this implies $\mathfrak{l}_{n,m} = \mathfrak{l}_{n,n+1}$ for all $n, m \in \omega$ with $n < m$.

In [7] it was shown that for all $n \in \omega$ we have $\mathfrak{l}_{n,n+1} \leq \mathfrak{hm}$. However, we will show that $\mathfrak{hm}(c_{\max}) < \mathfrak{l}_{n,\omega}$ for all $n \in \omega$ is consistent. In particular, we will get a model where $\mathfrak{l}_{n,n+1}, \mathfrak{d} < \mathfrak{l}_{n,\omega}$ for every $n \in \omega$.

We will use a countable support iteration of length ω_2 over a model of CH of a forcing notion that we call the *Miller lite forcing*.

6.1. **The finite version of Miller forcing.**

Definition 6.2. *Miller lite forcing* ($\mathbb{ML}$) consists of subtrees T of $\omega^{<\omega}$ such that for every $n \in \omega$ and every node $s \in T$ there is $t \in T$ such that $s \subseteq t$ and t has at least n immediate successors in T. The elements of $\mathbb{ML}$ are ordered by set-theoretic inclusion.

It should be clear that the finitely splitting trees, i.e., trees in which every node only has finitely many successors, are dense in $\mathbb{ML}$.

If G is an $\mathbb{ML}$-generic filter over some model of set theory, then the trees in G have a unique common branch, the *generic real* added by $\mathbb{ML}$. The filter G can be recovered from the generic real.

The main technical device used for analyzing the Miller lite forcing is *fusion*. For $p \in \mathbb{ML}$ and $t \in p$ let

$$p_t = \{s \in p : s \subseteq t \vee s \supseteq t\}$$

and let $\mathrm{succ}_p(t)$ denote the set of immediate successors of t in p. For $n \in \omega$ and $p \in \mathbb{ML}$ let

$$p^n = \{t \in p : t \in \mathrm{succ}_p(s) \text{ for some } s \in p \text{ that is minimal with } |\mathrm{succ}_p(s)| > n\}.$$

For $p, q \in \mathbb{ML}$ let $p \leq_n q$ if $p \leq q$ and $p^n = q^n$. A sequence $(p_n)_{n\in\omega}$ in $\mathbb{ML}$ is a *fusion sequence* if for all $n \in \omega$, $p_{n+1} \leq_n p_n$. The *fusion* of the the sequence $(p_n)_{n\in\omega}$ is the condition $p = \bigcap_{n\in\omega} p_n \in \mathbb{ML}$. For every $n \in \omega$ we have $p \leq_n p_n$.

Lemma 6.3. *$\mathbb{ML}$ satisfies Axiom A as defined in* [5].

Proof. Axiom A is witnessed by the sequence $(\leq_n)_{n\in\omega}$ defined above. □

We say that $p \in \mathbb{ML}$ is *normal* if for all $s \in p$ with $|\,\mathrm{succ}_p(s)\,| > 1$ and all $t \in p$ such that t is a minimal proper extension of s with $|\mathrm{succ}_p(t)| > 1$ we have $|\mathrm{succ}_p(t)| = |\mathrm{succ}_p(s)| + 1$.

For every $n \in \omega$ let $I_n = \prod_{k=0}^{n}(k+1)$ and let $I_\omega = \prod_{k\in\omega}(k+1)$. For notational convenience we put $I_{-1} = \{\emptyset\}$. For sequences σ and τ we use $\sigma^\frown\tau$ to denote the concatenation of the two sequences. For $i \in \omega$ we denote by $\sigma^\frown i$ the concatenation of σ and the sequence of length 1 with value i.

If $p \in \mathbb{ML}$ is normal, then for every $n \in \omega$ there is a natural bijection

$$I_n \to p^n; \sigma \mapsto t_\sigma,$$

namely the one that preserves the lexicographic order. The map

$$\bigcup_{n\in\omega} I_n \to p; \sigma \mapsto t_\sigma$$

preserves $\subseteq$ and induces a homeomorphism $h : I_\omega \to [p]$, which preserves the lexicographic order. Here

$$[p] = \{x \in \omega^\omega : \forall n \in \omega(x \restriction n \in p)\}$$

is the set of all branches of p. For every $\sigma \in I_n$ let $p * \sigma = p_{t_\sigma}$.

Note that the normal conditions are dense in $\mathbb{ML}$. We will therefore tacitly assume that all conditions under consideration are normal. However, there is one point where one has to be careful. Given a condition $p \in \mathbb{ML}$ we will often construct a condition $q \leq_n p$ in the following way:

For each $\sigma \in I_n$ we choose a condition $q_\sigma \leq p * \sigma$. It is easily checked that $q = \bigcup_{\sigma \in I_n} q_\sigma$ is again a condition in $\mathbb{ML}$ and that $q \leq_n p$. Moreover, for every $\sigma \in I_n$ we have $q_\sigma = q * \sigma$.

Even if p and all q_σ are normal, q is usually not normal. But there is a normal condition $q' \leq_n q$. There is a canonical way of constructing such a condition q'. Passing from q to q' is *normalization of q below n*. Normalization below n will always be carried out without mentioning in the situation just described.

$\mathbb{ML}$ has the following Ramsey theoretic property:

Lemma 6.4. *Let $p \in \mathbb{ML}$ and let $c : [[p]]^2 \to 2$ be a continuous pair coloring. Then there is a condition $q \leq p$ such that $[q]$ is c-homogeneous.*

Proof. We start by constructing a condition $r \leq p$ such that for any two distinct branches $x, y \in [r]$ the color $c(x, y)$ is already determined by $x \restriction \Delta(x, y) + 1$ and $y \restriction \Delta(x, y) + 1$. We say that c is an *almost node coloring* on $[r]$.

The condition r will be the fusion of a fusion sequence $(r_n)_{n\in\omega}$. Let $r_0 = p$.

Suppose we have constructed r_n. Let $\sigma \in I_{n-1}$ and suppose that i and j are distinct elements of $n+1$. By thinning out the parts $r_n * (\sigma^\frown i)$ and $r_n * (\sigma^\frown j)$ of r_n we obtain a condition $r_{n,1} \leq_n r_n$ such that $c(x, y)$ is the same for all $x \in [r_{n,1} * (\sigma^\frown i)]$ and all $y \in [r_{n,1} * (\sigma^\frown j)]$. This is possible by the continuity of c.

Iterating this argument we can construct a condition $r_{n+1} \leq_n r_n$ such that for all $\sigma \in I_{n-1}$ and all distinct $i, j \in n + 1$ the coloring c is constant on

$$\{\{x, y\} : x \in [r_{n+1} * (\sigma^\frown i)] \wedge y \in [r_{n+1} * (\sigma^\frown j)]\}.$$

This finishes the inductive construction of the r_n.

Put $r = \bigcap_{n\in\omega} r_n$. Now c is an almost node coloring on $[r]$.

A node $s \in r$ is a *splitting node* of r if it has more than one immediate successor. For every splitting node $s \in r$ we define a coloring $c_s : [\mathrm{succ}_r(s)]^2 \to 2$ by letting $c_s(t_0, t_1) = c(x, y)$ where $x, y \in [r]$ are chosen such that $t_0 \subseteq x$ and $t_1 \subseteq y$. Note that this definition is independent of the particular choice of x and y since c is an almost node coloring on $[r]$.

We now decrease the condition r further in order to get a condition r' such that for every splitting node $s \in r'$ the set $\mathrm{succ}_{r'}(s)$ is c_s-homogeneous.

The condition r' will be the fusion of a fusion sequence $(r'_n)_{n\in\omega}$. Let $r'_0 = r$. Suppose r'_n has been defined. We construct $r'_{n+1} \leq_n r'_n$ as follows.

By the finite Ramsey theorem, there is $m \in \omega$ such that every pair coloring (with two colors) on a set of size m has a homogeneous set of size n. For every $\sigma \in I_n$ we choose a splitting node s_σ of $r'_n * \sigma$ with $|\mathrm{succ}_{r'_n}(s_\sigma)| \geq m$. By the choice of m, there is a c_{s_σ}-homogeneous set $H_\sigma \subseteq \mathrm{succ}_{r'_n}(s_\sigma)$ of size n. Let

$$r'_{n+1} = \{t \in r'_n : \exists \sigma \in I_n \exists s \in H_\sigma(s \subseteq t \vee t \subseteq s)\}.$$

Clearly, $r'_{n+1} \leq_n r'_n$.

Let $r' = \bigcap_{n\in\omega} r'_n$. For every splitting node $s \in r'$ the set of immediate successors is c_s-homogeneous, say of color i_s.

Our next goal is to find a condition $q \leq r'$ such that for every splitting node $s \in q$ we get the same color i_s. If r' has a cofinal set of splitting nodes s with $i_s = 0$, then we can use a fusion argument as above in order to find a condition $q \leq r'$ such that all splitting nodes s of q have $i_s = 0$.

If r' does not have a cofinal set of splitting nodes s with $i_s = 0$, then there is $t \in r'$ such that no splitting node s of r'_t has $i_s = 0$. In this case $q = r'_t$ is a condition with $i_s = 1$ for all splitting nodes s of q.

In either case $[q]$ is c-homogeneous. □

Lemma 6.4 can be used to show that forcing with $\mathbb{ML}$ does not add new reals that avoid all ground model homogeneous sets.

Lemma 6.5. *Let $c : [\omega^\omega]^2 \to 2$ be a continuous coloring in the ground model M. If G is $\mathbb{ML}$-generic over M, then in $M[G]$, ω^ω is covered by the c-homogeneous sets coded in M.*

The proof of this lemma needs some preparation.

Let c be a continuous pair coloring on ω^ω as in Lemma 6.5. Then c induces a mapping

$$\overline{c} : (\omega^{<\omega})^2 \to \{0, 1, \text{undefined}\}$$

as follows:

For $s, t \in \omega^{<\omega}$ let $\overline{c}(s, t) = i \in 2$ if for all $x, y \in \omega^\omega$ with $s \subseteq x$ and $t \subseteq y$ we have $x \neq y$ and $c(x, y) = i$. Otherwise let $\overline{c}(s, t) = \text{undefined}$. Note that $\overline{c}(s, t) \in 2$ implies that s and t are incomparable.

For a forcing notion $\mathbb{P}$, a name $\dot{x}$ for a new element of ω^ω, and a condition $p \in \mathbb{P}$ let

$$T_p(\dot{x}) = \{s \in \omega^{<\omega} : \exists q \leq p (q \Vdash s \subseteq \dot{x})\}$$

be the *tree of p-possibilities for $\dot{x}$*. Let $\dot{x}[p]$ denote the longest initial segment of $\dot{x}$ that is decided by p, i.e., the stem of the tree $T_p(\dot{x})$. Note that, since $\dot{x}$ is a name for a new real, $T_p(\dot{x})$ is a perfect tree and $\dot{x}[p]$ is finite. Cleary, p forces $\dot{x}$ to be a branch of $T_p(\dot{x})$.

Now let $\dot{x}$ be an $\mathbb{ML}$-name for a new element of ω^ω and let $c : [\omega^\omega]^2 \to 2$ be continuous. We say that a condition $p \in \mathbb{ML}$ is *accurate* (with respect to $\dot{x}$) if for all $n \in \omega$ and all $\sigma, \tau \in I_n$ with $\sigma \neq \tau$ we have $\overline{c}(\dot{x}[p * \sigma], \dot{x}[p * \tau]) \in 2$.

Proof of Lemma 6.5. Let c be as in the formulation of the lemma. Let $\dot{x}$ be an $\mathbb{ML}$-name for a new element of ω^ω. We have to show that the set of conditions in $\mathbb{ML}$ that force $\dot{x}$ to be an element of some c-homogeneous ground model set is dense.

Let $p \in \mathbb{ML}$. It is sufficient to find a condition $r \leq p$ such that the set $[T_q(\dot{x})]$ is c-homogeneous. We show slightly more.

Claim 6.6. There is an accurate condition $r \leq p$ such that $[T_r(\dot{x})]$ is c-homogeneous.

In order to prove the claim, it is actually sufficient to show that there is any accurate condition $q \leq p$.

For suppose that $q \leq p$ is accurate. Then $\overline{c}$ gives rise to a mapping

$$\overline{d} : q^2 \to \{0, 1, \text{undefined}\}$$

by letting $\overline{d}(s, t) = \overline{c}(\dot{x}[q_s], \dot{x}[q_t])$ for all $s, t \in q$. Since q is accurate, $\overline{d}$ comes from a continuous pair coloring $d : [[q]]^2 \to 2$.

Now by Lemma 6.4, there is a condition $r \leq q$ such that $[r]$ is d-homogeneous. Clearly, r is accurate and the d-homogeneity of $[r]$ implies that $[T_r(\dot{x})]$ is c-homogeneous.

It remains to find an accurate condition $q \leq p$. First of all we observe that for all $q_0, q_1 \leq p$ there are $q_0' \leq q_0$ and $q_1' \leq q_1$ such that $\overline{c}(\dot{x}[q_0'], \dot{x}[q_1']) \in 2$. This can be seen as follows:

Choose decreasing sequences $(q_{j,n})_{n\in\omega}$, $j \in 2$, in $\mathbb{ML}$ such that $\dot{x}[q_{0,0}]$ and $\dot{x}[q_{1,0}]$ are incomparable and for all $j \in 2$ and all $n \in \omega$, $q_{j,n} \leq q_j$ and $q_{j,n}$ decides $\dot{x} \restriction n$. This is possible since $\dot{x}$ is a name for a new real. For $j \in 2$ let $x_j = \bigcup_{n\in\omega} \dot{x}[q_{j,n}]$. By the continuity of c, there is $n \in \omega$ such that $c(x_0, x_1)$ depends only on $x_0 \restriction n$ and $x_1 \restriction n$. In particular, $\overline{c}(\dot{x}[q_{0,n}], \dot{x}[q_{1,n}]) = c(x_0, x_1) \in 2$. Now $q_0' = q_{0,n}$ and $q_1' = q_{1,n}$ have the desired properties.

The accurate condition $q \leq p$ will be the fusion of a fusion sequence $(q_n)_{n\in\omega}$. Let $q_0 = p$. Suppose q_n has been defined. Let $\sigma \in I_{n-1}$ and let i, j be distinct elements of $n + 1$. By what we have said before, there are conditions $q_{n,i} \leq q_n * (\sigma^\frown i)$ and $q_{n,j} \leq q_n * (\sigma^\frown j)$ such that $\overline{c}(\dot{x}[q_{n,i}], \dot{x}[q_{n,j}]) \in 2$. It follows that we can thin out the parts $q_n * (\sigma^\frown i)$ and $q_n * (\sigma^\frown j)$ of q_n in order to obtain a condition $q_n' \leq_n q_n$ such that $\overline{c}(\dot{x}[q_n' * (\sigma^\frown i)], \dot{x}[q_n' * (\sigma^\frown j)]) \in 2$.

Iterating this argument we arrive at a condition $q_{n+1} \leq_n q_n$ such that for all $\sigma \in I_{n-1}$ and distinct $i, j \in n + 1$ we have $\overline{c}(\dot{x}[q_{n+1} * (\sigma^\frown i)], \dot{x}[q_{n+1} * (\sigma^\frown j)]) \in 2$. This finishes the inductive construction of the fusion sequence $(q_n)_{n\in\omega}$.

It is easily checked that $q = \bigcap_{n\in\omega} q_n$ is indeed accurate. □

6.2. **Iterating $\mathbb{ML}$.** For every ordinal α let $\mathbb{ML}_\alpha$ denote the countable support iteration of $\mathbb{ML}$ of length α. Since $\mathbb{ML}$ satisfies Axiom A, $\mathbb{ML}_\alpha$ does not collapse $\aleph_1$. We will use the analogue of Lemma 6.5 for $\mathbb{ML}_\alpha$.

Lemma 6.7. *Let $c : [\omega^\omega]^2 \to 2$ be a continuous coloring in the ground model M. Let α be an ordinal. Then for every $\mathbb{ML}_\alpha$-generic filter G over M, $(\omega^\omega)^{M[G]}$ is covered by c-homogeneous sets coded in the ground model.*

Before we give the proof of Lemma 6.7, let us derive from it

Corollary 6.8. *Forcing with $\mathbb{ML}_{\omega_2}$ over a model of CH yields a model of the dual open coloring axiom.*

Proof. Let G be an $\mathbb{ML}_{\omega_2}$-generic filter over the ground model M. Assume that M is a model of CH. In $M[G]$ let C be a finite open pair cover on a Polish space X. By Theorem 3.3, $\mathfrak{hm}(C) \leq \mathfrak{hm}(c_{\max})$. By Lemma 6.7, in $M[G]$, 2^ω is covered by the $c_{\max}$-homogeneous sets coded in the ground model. Since M satisfies CH, there are only $\aleph_1$ Borel sets in the ground model. It follows that $\mathfrak{hm}(C) \leq \mathfrak{hm}(c_{\max}) = \aleph_1$ in $M[G]$. By the usual arguments, $M[G] \models 2^{\aleph_0} = \aleph_2$. □

In [8] Lemma 6.7 is proved for the countable support iteration of Sacks forcing instead of $\mathbb{ML}$. In the case of $\mathbb{ML}$ one has to deal with finitely splitting trees that split more and more as we go down the tree as opposed to binary trees in the case of Sacks forcing. This issue has been addressed in the proof of the consistency of $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$ presented in [6].

We put together the techniques used in [6] and in [8] in order to prove Lemma 6.7. We first have to extend out notion of fusion to countable support iterations of $\mathbb{ML}$.

Let α be an ordinal. For $F \in [\alpha]^{<\aleph_0}$, $\eta : F \to \omega$, and $p, q \in \mathbb{ML}_\alpha$ let $q \leq_{F,\eta} p$ if $q \leq p$ and for all $\beta \in F$, $q \restriction \beta \Vdash q(\beta) \leq_{\eta(\beta)} p(\beta)$. Roughly speaking, $q \leq_{F,\eta} p$ means that on each coordinate from F, q is $\leq_n$-below p where n is given by η.

A sequence $(p_n)_{n\in\omega}$ of conditions in $\mathbb{ML}_\alpha$ is a *fusion sequence* if there is an increasing sequence $(F_n)_{n\in\omega}$ of finite subsets of α and a sequence $(\eta_n)_{n\in\omega}$ such that for all $n \in \omega$, $\eta_n : F_n \to \omega$, $p_{n+1} \leq_{F_n,\eta_n} p_n$, for all $\gamma \in F_n$ we have $\eta_n(\gamma) \leq \eta_{n+1}(\gamma)$, and for all $\gamma \in \mathrm{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$.

This notion is precisely what is needed in countable support iterations to get suitable fusions. It essentially means that once we have touched (i.e., decreased) a coordinate of p_0, we have to build a fusion sequence in that coordinate.

If $(p_n)_{n\in\omega}$ is a fusion sequence in $\mathbb{ML}_\alpha$, its fusion p_ω is defined inductively. Let $F_\omega = \bigcup F_n$.

Suppose $p_\omega(\gamma)$ has been defined for all $\gamma < \beta$ for some $\beta < \alpha$. If $\beta \notin F_\omega$, let $p_\omega(\beta)$ be a name for $1_{\mathbb{ML}}$. If $\beta \in F_\omega$, then $p_\omega \restriction \beta$ forces $(p_n(\beta))_{n\in\omega}$ to be a fusion sequence in $\mathbb{ML}$. Let $p_\omega(\beta)$ be a name for the fusion of the $p_n(\beta)$'s.

Now fix a continuous coloring $c : [\omega^\omega]^2 \to 2$. If G is $\mathbb{ML}_\alpha$-generic over the ground model M and $x \in (\omega^\omega)^{M[G]}$, then there is some ordinal $\beta \le \alpha$ and an $\mathbb{ML}_\beta$-name $\dot{x}$ for an element of ω^ω not added before stage β of the iteration such that $\dot{x}_G = x$. In this last equation $\dot{x}$ is considered as an $\mathbb{ML}_\alpha$-name in the natural way. This shows that in order to prove Lemma 6.7 it suffices to show the following:

Lemma 6.9. *Let α be an ordinal and suppose that $\dot{x}$ is an $\mathbb{ML}_\alpha$-name for an element of ω^ω that is not added in an initial stage of the iteration. Then for every condition $p \in \mathbb{ML}_\alpha$ there is a condition $q \le p$ such that $[T_q(\dot{x})]$ is c-homogeneous.*

The way to build a condition q for which $[T_q(\dot{x})]$ is c-homogeneous is the following: q will be the fusion of a fusion sequence $(p_n)_{n\in\omega}$ with witness $(F_n, \eta_n)_{n\in\omega}$. For each n, (p_n, F_n, η_n) will determine a finite initial segment T_n of $T_q(\dot{x})$. We have to make sure that $T_q(\dot{x})$ is the union of the T_n and that the T_n are good enough to guarantee the c-homogeneity of $[T_q(\dot{x})]$. The latter will be ensured by the (F_n, η_n)-faithfulness of each p_n defined below.

Definition 6.10. Let $i \in 2$ be a fixed color. For F and η as before, a condition $q \in \mathbb{ML}_\alpha$ is (F, η)-*faithful* if for all $\sigma, \tau \in \prod_{\gamma\in F} I_{\eta(\gamma)}$ with $\sigma \ne \tau$, $\bar{c}(x[q * \sigma], x[q * \tau]) = i$.

The color i that appears in Definition 6.10 will be chosen so that it is possible to construct $q \le p$ such that $[T_q(\dot{x})]$ is homogeneous of color i.

There are two cases. If α is a successor ordinal, i.e., $\alpha = \beta + 1$ for some ordinal β, then by Claim 6.6, we may assume that $p \restriction \beta$ forces that $[T_{p(\beta)}(\dot{x})]$ is c-homogenous and that $p(\beta)$ is accurate. Moreover, we may assume that $p \restriction \beta$ decides the color of $[T_{p(\alpha)}(\dot{x})]$ to be $i \in 2$. This is how we choose i if α is a successor ordinal.

If α is a limit ordinal, then we can find the color i using the following Lemma. The Lemma was proved in [8, Lemma 30] for countable support iterations of Sacks forcing, but is was pointed out that the same proof goes through for other forcing iterations as well.

Lemma 6.11. *For $i \in 2$ let*

$$E_i = \{p \in \mathbb{ML}_\alpha : \forall \beta < \alpha \forall q \le p \exists q' \le q \exists q_0, q_1 \in \mathbb{ML}_{\beta,\alpha}$$
$$(q' \restriction \beta \Vdash q_0, q_1 \le q' \restriction [\beta, \alpha) \land \bar{c}(x[q_0], x[q_1]) = i)\}$$

where $\mathbb{ML}_{\beta,\alpha}$ denotes the natural $\mathbb{ML}_\beta$-name for the rest of the iteration up to α. Then E_0 and E_1 are open and $E_0 \cup E_1$ is dense in $\mathbb{ML}_\alpha$.

Using this lemma, we may assume that p already is an element of some E_i. That is how we choose i if α is a limit ordinal.

We now state and prove the two Lemmas that we use in the inductive contruction of the fusion sequence $(p_n)_{n\in\omega}$.

Lemma 6.12. *Let α be a limit ordinal and let $\dot{x}$ be a $\mathbb{ML}_\alpha$-name for an element of ω^ω which is not added by an initial stage of the iteration. Let F, η, and i be as in Definition 6.10 and suppose that $q \in \mathbb{ML}_\alpha$ is (F, η)-faithful.*

a) Let $\beta \in \alpha \setminus F$ and let $F' = F \cup \{\beta\}$ and $\eta' = \eta \cup \{(\beta, 0)\}$. Then q is (F', η')-faithful.

b) Suppose $q \in E_i$. Let $\beta \in F$ and let $\eta' = \big(\eta \restriction (F \setminus \{\beta\})\big) \cup \{(\beta, \eta(\beta)+1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

Proof. a) follows immediately from the definitions.

For b) let $\delta = \max(F) + 1$ and $n = \eta(\beta) + 1$.

Claim 6.13. There is a condition $q' \leq_{F,\eta} q$ such that for each $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ there are sequences $q_{\sigma,0}, \dots, q_{\sigma,n}$ of names for conditions such that for all $k \leq n$,

$$q' * \sigma \restriction \delta \Vdash q_{\sigma,k} \leq q \restriction [\delta, \alpha),$$

$q' * \sigma \restriction \delta$ decides $\dot{x}[q_{\sigma,k}]$, and for all $l \leq n$ with $k \neq l$,

$$q' * \sigma \restriction \delta \Vdash \overline{c}(x[q_{\sigma,k}], x[q_{\sigma,l}]) = i.$$

For the proof of the claim, let $\{\sigma_1, \dots, \sigma_m\}$ be an enumeration of $\prod_{\gamma \in F} I_{\eta(\gamma)}$. We build a $\leq_{F,\eta}$-decreasing sequence $(q_j)_{j \leq m}$ such that $q_0 = q$ and $q' = q_m$ works for the claim. As we construct q_j, we find suitable $q_{\sigma_j,k}$ for all $k < n$.

Let $j \in \{1, \dots, m\}$ and assume that q_{j-1} has already been constructed. Since $q \in E_i$ and E_i is open, there are $q_j^* \leq q_{j-1} * \sigma_j$ and sequences $q_{\sigma_j,0}$ and $q^*_{\sigma_j,1}$ of names of conditions such that

$$q_j^* \restriction \delta \Vdash q_{\sigma_j,0}, q^*_{\sigma_j,1} \leq q \restriction [\delta, \alpha) \wedge \overline{c}(\dot{x}[q_{\sigma_j,0}], \dot{x}[q^*_{\sigma_j,1}]) = i.$$

Iterating this process by splitting $q^*_{\sigma_j,1}$ into $q_{\sigma_j,1}$ and $q^*_{\sigma_j,2}$ and so on and decreasing q_j^*, we finally obtain $q_j^* \leq q_{j-1} * \sigma_j$ and $q_{\sigma_j,k}$, $k \leq n$, such that for all $k \leq n$.

$$q_j^* \restriction \delta \Vdash q_{\sigma_j,k} \leq q \restriction [\delta, \alpha)$$

and for all $l \leq n$ with $l \neq k$,

$$q_j^* \restriction \delta \Vdash \overline{c}(\dot{x}[q_{\sigma_j,k}], \dot{x}[q_{\sigma_j,l}]) = i.$$

We may assume that $q_j^* \restriction \delta$ decides $\dot{x}[q_{\sigma_j,k}]$ for all $k \leq n$. Let $q_j \leq_{F,\eta} q_{j-1}$ be such that $q_j * \sigma_j \restriction \delta = q_j^* \restriction \delta$ and $q_j \restriction [\delta, \alpha) = q \restriction [\delta, \alpha)$. This finishes the construction, and it is easy to check that it works.

Continuing the proof of lemma 6.12, let $q_{\sigma,k}$ and q' be as in the claim. For $\rho \in I_{\eta(\beta)}$ let $r^{\rho^\frown 0}, \dots, r^{\rho^\frown n}$ be sequences of names for conditions such that for all $k \leq n$ and all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ with $\sigma(\beta) = \rho$,

$$q' * \sigma \restriction \delta \Vdash r^{\rho^\frown k} = q_{\sigma,k}.$$

Let r be a sequence of names for conditions such that $r \restriction \delta = q' \restriction \delta$ and for all $\sigma \in \prod_{\gamma \in F} I_{\eta'(\gamma)}$,

$$q' * \sigma \restriction \delta \Vdash r \restriction [\delta, \alpha) = r^{\sigma(\beta)}.$$

With this choice of r we have $r \leq_{F,\eta} q$. It follows from the construction that r is (F, η')-faithful. □

A similar lemma is true if the new real is added in a successor step.

Lemma 6.14. *Let α be a successor ordinal, say $\alpha = \delta + 1$ and let $\dot{x}$ be a $\mathbb{ML}_\alpha$-name for an element of ω^ω which is not added by an initial stage of the iteration. Let F, η, and i be as in Definition 6.10 and suppose that $q \in \mathbb{ML}_\alpha$ is (F, η)-faithful.*

a) Let $\beta \in \alpha \setminus F$ and let $F' = F \cup \{\beta\}$ and $\eta' = \eta \cup \{(\beta, 0)\}$. Then q is (F', η')-faithful.

b) Suppose

$$q \restriction \delta \Vdash \text{“}[T_{q(\delta)}] \textit{ is c-homogeneous of color } i \textit{ and } q(\delta) \textit{ is accurate”}.$$

Let $\beta \in F$ and let $\eta' = \eta \restriction F \setminus \{\beta\} \cup \{(\beta, \eta(\beta)+1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

Proof. As in Lemma 6.12, a) follows directly from the definitions.

For the proof of b) let $n = \eta(\beta) + 1$. We have to consider two cases. First suppose $\beta = \delta$. In this case let $r \leq_{F,\eta} q$ be such that $r \restriction \delta \Vdash r(\delta) = q(\delta)$ and for all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ and all $k \leq n$, $r * \sigma \restriction \delta$ decides $\dot{x}[r(\delta) * (\sigma(\delta)^\frown k)]$.

Note that r is indeed (F, η')-faithful since we assumed $q \restriction \delta$ to force that $[T_{q(\delta)}]$ is c-homogeneous of color i and that $q(\delta)$ is accurate.

If $\beta \neq \delta$, the argument will be similar to the one used for Lemma 6.12. For all $k \leq n$ and all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ let $q_{\sigma,k}$ be a name for a condition such that

$$q * \sigma \restriction \delta \Vdash q_{\sigma,k} \leq q(\delta) * \sigma(\delta)$$

and for all $l \leq n$ with $l \neq k$

$$q * \sigma \restriction \delta \Vdash \overline{c}(\dot{x}[q_{\sigma,k}(\delta)], \dot{x}[q_{\sigma,l}(\delta)]) = i.$$

Now fix $q' \leq_{F,\eta} q$ such that for all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ and all $k \leq n$, $q' * \sigma \restriction \delta$ decides $\dot{x}[q_{\sigma,k}]$. Note that for all $k, l \leq n$ with $k \neq l$ we have that

$$q' * \sigma \restriction \delta \Vdash \overline{c}(\dot{x}[q_{\sigma,k}], \dot{x}[q_{\sigma,l}]) = i.$$

Choose r such that $r \restriction \delta = q' \restriction \delta$ and for all $\sigma \in \prod_{\gamma \in F} I_{\eta'(\gamma)}$

$$r * \sigma \restriction \delta \Vdash r(\delta) * \sigma(\delta) = q_{\sigma,k}$$

where $k = \sigma(\beta)(n-1)$ (i.e., k is the last digit of $\sigma(\beta)$).

It follows from the definition of r that $r \leq_{F,\eta} q$. It is easily checked that r is (F, η')-faithful. $\square$

Using the last two lemmas, we can prove Lemma 6.9, which finishes the proof of Lemma 6.7.

Proof of Lemma 6.9. Since $\dot{x}$ is a name for a real not added in an initial stage of the iteration $\mathbb{ML}_\alpha$, $\mathrm{cf}(\alpha) \leq \aleph_0$. Let $p \in \mathbb{ML}_\alpha$. If α is a limit ordinal, using Lemma 6.11, we can decrease p such that for some $i \in 2$, $p \in E_i$.

If α is a successor ordinal, say $\alpha = \delta + 1$, we can use Claim 6.6 to decrease p such that for some $i \in 2$

$$p \restriction \delta \Vdash \text{“}[T_{p(\delta)}] \text{ is } c\text{-homogeneous of color } i \text{ and } p(\delta) \text{ is accurate”}.$$

By induction, we define a sequence $(p_n, F_n, \eta_n)_{n \in \omega}$ such that

(1) for all $n \in \omega$, $p_n \in \mathbb{ML}_\alpha$, $p_n \leq p$, $F_n \in [\alpha]^{<\aleph_0}$, $\eta_n : F_n \to \omega$, and p_n is (F_n, η_n)-faithful,
(2) for all $n \in \omega$, $F_n \subseteq F_{n+1}$, $p_{n+1} \leq_{F_n,\eta_n} p_n$, and for all $\gamma \in F_n$ we have $\eta_n(\gamma) \leq \eta_{n+1}(\gamma)$, and
(3) for all $n \in \omega$ and all $\gamma \in \mathrm{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$.

This construction can be done using parts a) and b) of Lemma 6.12 and Lemma 6.14 respectively, depending on whether α is a limit ordinal or not, to extend F_n or to make η_n bigger, together with some bookkeeping to ensure (3). Now $(p_n)_{n \in \omega}$ is a fusion sequence. Let q be the fusion of this sequence. For each $n \in \omega$ let T_n be the tree generated by $\{\dot{x}[p_n * \sigma] : \sigma \in \prod_{\gamma \in F_n} I_{\eta(\gamma)}\}$. It is easily seen that $T_q(\dot{x}) = \bigcup_{n \in \omega} T_n$.

It now follows from the faithfulness of the p_n that $[T_q(\dot{x})]$ is c-homogeneous of color i. $\square$

On the other hand, $\mathbb{ML}$ adds a generic real that for all $n \in \omega$ avoids every n-ary set in the ground model.

Lemma 6.15. *Let G be $\mathbb{ML}$-generic over the ground model M and let $n \in \omega$. Then in $M[G]$, there is an element of ω^ω that is not covered by an n-ary set coded in the ground model.*

Proof. Let $\dot{x}$ be a name for the generic real added by $\mathbb{ML}$. We show that for every n-ary set $X \subseteq \omega^\omega$, $\dot{x}$ is forced to be an element of $\omega^\omega \setminus X$.

Let $p \in \mathbb{ML}$. Consider the tree $p \cap T(X)$. Since every element of $p \cap T(X)$ has at most n immediate successors, there is $t \in p$ such that $t \notin p \cap T(X)$. Now $p_t \Vdash \dot{x} \notin X$. □

Corollary 6.16. *Forcing with $\mathbb{ML}_{\omega_2}$ over a model of CH yields a model where for every $n \in \omega$ we have $\mathfrak{l}_{n,\omega} = 2^{\aleph_0} = \aleph_2$.*

Combining Corollary 6.8 and Corollay 6.16, we obtain

Corollary 6.17. *The dual open coloring axiom is consistent with $\mathfrak{l}_{n,\omega} = 2^{\aleph_0}$.*

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