2-DIMENSIONAL CONVEXITY NUMBERS AND P_4-FREE GRAPHS

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Abstract. For $S \subseteq \mathbb{R}^n$ a set $C \subseteq S$ is an *m-clique* if the convex hull of no m-element subset of C is contained in S. We show that there is essentially just one way to construct a closed set $S \subseteq \mathbb{R}^2$ without an uncountable 3-clique that is not the union of countably many convex sets. In particular, all such sets have the same convexity number; that is, they require the same number of convex subsets to cover them. The main result follows from an analysis of the convex structure of closed sets in $\mathbb{R}^2$ without uncountable 3-cliques in terms of clopen, P_4-free graphs on Polish spaces.

1. Introduction

A coarse measure of the nonconvexity of a set $S \subseteq \mathbb{R}^n$ is its *convexity number* $\gamma(S)$, the least size of a family $\mathcal{F}$ of convex sets with $\bigcup \mathcal{F} = S$. S is *countably convex* if its convexity number is countable. Otherwise S is *uncountably convex.* In this paper we continue the study of the convexity numbers of closed, uncountably convex subsets of $\mathbb{R}^2$ that was started in [7] and continued in [6].

Caratheodory's theorem states that the convex hull of a set $T \subseteq \mathbb{R}^n$ is the union of the convex hulls of the $(n+1)$-element subsets of T. A subset T of a set $S \subseteq \mathbb{R}^n$ is *defected* (in S) if the convex hull of T is not a subset of S. By Caratheodory's theorem, $T \subseteq S$ is defected iff some $(n+1)$-element subset of T is. It follows that the convex structure of a set $S \subseteq \mathbb{R}^n$ can be analyzed by looking at the *defectedness hypergraph*

$$(S, \{T \subseteq S : |T| = n+1 \text{ and } T \text{ is defected}\})$$

of S. The convexity number of S is exactly the chromatic number of the defectedness hypergraph of S.

In [6] it was shown that a closed uncountably convex subset S of $\mathbb{R}^2$ either has a (nonempty) perfect subset C such that the convex hull of any three distinct points of C is not contained in S, i.e., a perfect *3-clique*, or the convexity number of S is equal to the *homogeneity number* of a continuous coloring of the two-element subsets of a Polish space with two colors. Here the homogeneity number $\mathfrak{hm}(c)$ of a coloring $c : [X]^2 \to 2$ is the least size of a family of c-homogeneous subsets of X that covers X. The homogeneity number of a coloring $c : [X]^2 \to 2$ is the same as the *cochromatic number* of the graph $(X, c^{-1}(1))$.

In other words, the problem of studying the 3-uniform, open defectedness hypergraph of S and its chromatic number can be reduced to studying certain clopen graphs on Polish spaces and their cochromatic numbers. Here an n-uniform hypergraph on a Polish space X is *open* (*closed*, *clopen*) if its n-ary edge relation is open

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(closed, clopen) as a subset of

$$\{(x_1,\dots,x_n)\in X^n : x_1,\dots,x_n \text{ are pairwise distinct}\}.$$

Continuous colorings of the two-element subsets of Polish spaces have been investigated in [5]. It turned out that there are continuous colorings

$$c_{\min}, c_{\max} : [2^\omega]^2 \to 2$$

such that $\mathfrak{hm}(c_{\min})$ is minimal among all the uncountable homogeneity numbers of continuous colorings on Polish spaces, while $\mathfrak{hm}(c_{\max})$ is maximal. We write $\mathfrak{hm}$ for $\mathfrak{hm}(c_{\min})$. Consistently, $\mathfrak{hm} < \mathfrak{hm}(c_{\max})$. Moreover, somewhat surprizingly, $\mathfrak{hm}^+ \geq 2^{\aleph_0}$. This last inequality was proved already in [6] and shows that there are at most two different uncountable homogeneity numbers of continuous colorings on Polish spaces, namely $2^{\aleph_0}$ and its predecessor, provided the predecessor exists at all. It follows that there are at most two different uncountable convexity numbers of closed subsets of $\mathbb{R}^2$.

We say that a closed set $S \subseteq \mathbb{R}^2$ *realizes* the homogeneity number $\mathfrak{hm}(c)$ of a continuous coloring $c : [X]^2 \to 2$ on a Polish space X if in every forcing extension of the set-theoretic universe we have $\gamma(S) = \mathfrak{hm}(c)$. Note that a closed set $S \subseteq \mathbb{R}^2$ that has a perfect 3-clique satisfies $\gamma(S) = 2^{\aleph_0}$ in every forcing extension. An example of Kubiś in [6] shows that $\mathfrak{hm} = \mathfrak{hm}(c_{\min})$ is realized by a closed subset of $\mathbb{R}^2$. It was an open question whether any homogeneity number other than $\mathfrak{hm}$ can be realized by a closed subset of $\mathbb{R}^2$.

We show that this is not the case. The only homogeneity number that can be realized as the convexity number of a closed subset of $\mathbb{R}^2$ is $\mathfrak{hm}(c_{\min})$. Our proof of this fact shows that Kubiś's example of an uncountably convex closed subset of $\mathbb{R}^2$ without a perfect 3-clique is in some sense the only possible example. The key point of our proof is the fact that the graphs arising in the context of two-dimensional convexity as sketched above do not have induced subgraphs that are paths with four vertices, i.e., these graphs are *P_4-free*.

Using basic properties of finite P_4-free graphs and the technology developed in [5], we show that every clopen P_4-free graph on a Polish space can be decomposed into a small number of induced subgraphs that embed into $(2^\omega, c^{-1}_{\min}(1))$. This implies that these graphs have cochromatic numbers $\leq \mathfrak{hm}$.

2. The decomposition theorem for closed planar sets

In this section, let S denote an uncountably convex, closed subset of $\mathbb{R}^2$ without a perfect 3-clique. We review the results on the structure of S from [6].

$C \subseteq S$ is a *semi-clique* if for all $x \in C$ and all open neighborhoods U of x the set $S \cap U$ is defected in S. Let $B(S)$ denote the *convexity radical* of S, i.e., the set of $x \in S$ such that for every open neighborhood U of x, $S \cap U$ is uncountably convex. Clearly, $B(S)$ is a closed semi-clique of S.

Two subsets of $\mathbb{R}^2$ are *affinely isomorphic* if there is an *affine isomorphism*, i.e., a composition of a linear automorphism of $\mathbb{R}^2$ and a translation, that maps one set to the other.

Lemma 2.1. [6, Lemma 17] *Let B be a G_δ semi-clique in S. Then there is an open set $U \subseteq \mathbb{R}^2$ such that $B \cap U$ is nonempty and affinely isomorphic to the graph of a Lipschitz function.*

Lemma 2.2. [6, Lemma 18] *Let $B \subseteq B(S)$ be a G_δ set without isolated points which is affinely isomorphic to the graph of a Lipschitz function f. Then f is differentiable on a dense subset of its domain.*

Now, for some nonempty set $A \subseteq \mathbb{R}$ let $f : A \to \mathbb{R}$ be a Lipschitz function such that f is a G_δ semi-clique in S that has no isolated points. In this case A is a G_δ subset of $\mathbb{R}$ without isolated points. Since the topology of a G_δ subset of $\mathbb{R}$ is compatible with a separable complete metric, for every open set U that intersects A, $A \cap U$ is uncountable.

Let D denote the set of differentiability points of f, which is, by the previous lemma, a dense subset of A. For $x, y \in D$ with $x < y$ we say that $\{x, y\}$ is in *configuration* $\sim$ if either $f'(x), f'(y) < \frac{f(x)-f(y)}{x-y}$ or $f'(x), f'(y) > \frac{f(x)-f(y)}{x-y}$.

Lemma 2.3. [6, Lemma 20] *There is a nonempty open set $U \subseteq A$ that contains no pairs in configuration $\sim$.*

We say that f does not contain pairs in configuration $\sim$ if A does not contain pairs in configuration $\sim$. Assume that f does not contain pairs in configuration $\sim$. Let J be the set of $x \in A$ such that for some $\varepsilon > 0$ one of the intervals $(x, x+\varepsilon)$ and $(x-\varepsilon, x)$ is disjoint from A. J is a countable set. As pointed out before, every open set that intersects A has an uncountable intersection with A. This together with the countability of J implies that for any two distinct points of $A \setminus J$ there is a third point between the two. In particular, $A \setminus J$ has no isolated points. Since A is G_δ and J is countable, $A \setminus J$ is G_δ. Hence from now on we may assume that A is dense as a linear order, i.e., any two distinct points of A have a third point from A in between.

We say that $\{x, y\} \in [A]^2$ with $x < y$ is in *configuration* $\sqcap$ (respectively $\sqcup$) if for all $z \in (x, y)$, $(z, f(z))$ is either on or strictly above (below) the line segment joining $(x, f(x))$ and $(y, f(y))$.

Lemma 2.4. [6, Lemma 21] *Suppose f contains no pair in configuration $\sim$. Then the following hold:*

1. *Every pair in A is either in configuration $\sqcap$ or in configuration $\sqcup$.*
2. *If $x, y \in D$ and $x < y$, then $\{x, y\}$ is in configuration $\sqcup$ iff*

$$f'(x) \leq \frac{f(y) - f(x)}{y - x} \leq f'(y).$$

3. *The coloring $c : [A]^2 \to \{\sqcap, \sqcup\}$ that assigns to each pair its configuration is continuous.*

Note that (1) in particular says that the two configurations $\sqcap$ and $\sqcup$ are exclusive. This comes from the fact that S has no uncountable 3-clique and is explained in [6] before Lemma 16.

We call $\{x, y, z\} \in [A]^2$ *decided* iff it is homogeneous with respect to the coloring c defined in (3) of Lemma 2.4 and undecided otherwise.

Lemma 2.5. [6, Lemma 22] *Suppose that f has no pair in configuration $\sim$. Then the following hold:*

1. *Each nonempty open subset of A has pairs in configuration $\sqcap$ and pairs in configuration $\sqcup$.*
2. *For every undecided set $\{x, y, z\} \in [A]^3$, the set*

$$\{(x, f(x)), (y, f(y)), (z, f(z))\}$$

is defected in S.

3. *There is a nonempty open set $U \subseteq \mathbb{R}$ such that whenever an unordered triple $\{x, y, z\} \in A \cap U$ is decided, then*

$$\{(x, f(x)), (y, f(y)), (z, f(z))\}$$

undefected in S.

Now unfix the variables A and f. We slightly diverge from the definitions in [6]. We call a set $K \subseteq S$ *special* if K is a G_δ semi-clique and there is an affine isomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that

(1) $h[K]$ is the graph of a Lipschitz function $f : A \to \mathbb{R}$,
(2) $A \subseteq R$ is a G_δ set without isolated points and dense as a linear order,
(3) A contains no pairs in configuration $\sim$,
(4) A set $\{x, y, z\} \in [A]^3$ is undecided iff the set

$$\{(x, f(x)), (y, f(y)), (z, f(z))\}$$

is defected in S.

Our definition of special sets is more restrictive than the one used in [6]. However, the special sets constructed in the proof of the *Decomposition Theorem for* $\mathbb{R}^2$ [6, Theorem 15] are actually special in our narrower sense.

Theorem 2.6 (Decomposition Theorem for $\mathbb{R}^2$)**.** *Let S be a closed uncountably convex subset of $\mathbb{R}^2$ without a perfect 3-clique. Then there are a countably convex set A and a countable union B of special sets such that $S = A \cup B$.*

Now let S be as in Theorem 2.6, let $K \subseteq S$ be special, and let f and A witness this as in the definition of a special set. Then A is G_δ and hence a Polish space. Let

$$c_K : [A]^2 \to \{\sqcap, \sqcup\}$$

be the coloring that assigns to each pair its (unique) configuration. By Lemma 2.4 (3), c_K is continuous.

By Caratheodory's theorem, a set $T \subseteq S$ is defected iff T has a defected subset of size at most 3. It follows that the least size of a family of convex subsets of S that covers K is precisely $\mathfrak{hm}(c_K)$. This in particular shows that even though f, A, and c_K depend on the choice of the affine isomorphism h, $\mathfrak{hm}(c_K)$ does not.

Theorem 2.6 shows that $\gamma(S)$ is the supremum of a countable set of homogeneity numbers of continuous colorings on Polish spaces. However, as mentioned in the introduction, there are at most two possible homogeneity numbers of continuous colorings on Polish spaces, namely the continuum $2^{\aleph_0}$ and its predecessor, provided that exists. Since $\gamma(S)$ is uncountable, $\gamma(S)$ is actually the maximum of a set of at most two homogeneity numbers. Hence $\gamma(S)$ is of the form $\mathfrak{hm}(c_K)$ for some special set K.

3. Restrictions on convexity colorings

We show that there are some severe restrictions on which continuous colorings can be isomorphic to colorings of the form c_K, where K is a special subset of S and S is a closed, uncountably convex subset of $\mathbb{R}^2$ without a perfect 3-clique.

The crucial observations are the following transitivity properties of colorings by configuration.

Lemma 3.1. *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be any function with the property that every unordered pair $\{a, b\} \in [A]^2$ is either in configuration $\sqcap$ or in configuration $\sqcup$. Let $c : [A]^2 \to \{\sqcap, \sqcup\}$ be the coloring by configuration.*

a) Let $x_1, x_2, x_3 \in A$ be such that $x_1 < x_2 < x_3$. If $c(x_1, x_2) = c(x_2, x_3) = \sqcap$, then $c(x_1, x_3) = \sqcap$. If $c(x_1, x_2) = c(x_2, x_3) = \sqcup$, then $c(x_1, x_3) = \sqcup$.

b) Let $x_1, x_2, x_3, x_4 \in A$ be such that $x_1 < x_2 < x_3 < x_4$. If $c(x_1, x_3) = c(x_2, x_4) = \sqcap$, then $c(x_1, x_4) = \sqcap$. If $c(x_1, x_3) = c(x_2, x_4) = \sqcup$, then $c(x_1, x_4) = \sqcup$.

Proof. a) Suppose $c(x_1, x_2) = c(x_2, x_3) = \sqcap$. Then $(x_2, f(x_2))$ is not below the line segment from $(x_1, f(x_1))$ to $(x_3, f(x_3))$. For no $y \in (x_1, x_2)$, $(y, f(y))$ is below the line segment from $(x_1, f(x_1))$ to $(x_2, f(x_2))$. For no $y \in (x_2, x_3)$, $(y, f(y))$

is below the line segment from $(x_2, f(x_2))$ to $(x_3, f(x_3))$. It follows that for no $y \in (x_1, x_3)$, $(y, f(y))$ is below the line segment from $(x_1, f(x_1))$ to $(x_3, f(x_3))$. Hence $c(x_1, x_3) = \sqcap$. The symmetric argument shows

$$c(x_1, x_2) = c(x_2, x_3) = \sqcup \quad \Rightarrow \quad c(x_1, x_3) = \sqcup.$$

b) Suppose $c(x_1, x_3) = c(x_2, x_4) = \sqcap$. Then

$$\frac{f(x_1) - f(x_3)}{x_1 - x_3} > \frac{f(x_2) - f(x_3)}{x_2 - x_3}$$

and

$$\frac{f(x_2) - f(x_3)}{x_2 - x_3} > \frac{f(x_2) - f(x_4)}{x_2 - x_4}.$$

It follows that

$$\frac{f(x_2) - f(x_4)}{x_2 - x_4} < \frac{f(x_1) - f(x_3)}{x_1 - x_3}.$$

On the other hand, if $c(x_1, x_4) = \sqcup$, then

$$\frac{f(x_1) - f(x_3)}{x_1 - x_3} < \frac{f(x_1) - f(x_4)}{x_1 - x_4} < \frac{f(x_2) - f(x_4)}{x_2 - x_4}.$$

A contradiction. Hence $c(x_1, x_4) = \sqcap$. □

Given a function $f : A \to \mathbb{R}$ for some set $A \subseteq \mathbb{R}$ such that every unordered pair $\{a, b\} \in [A]^2$ is either in cofiguration $\sqcap$ or in configuration $\sqcup$, let G_f denote the graph with the set A of vertices and $\{a, b\} \in [A]^2$ an edge iff $\{a, b\}$ is in configuration $\sqcap$.

Recall that P_4 is the path with four vertices (and length three). A graph is *P_4-free* if it has no induced subgraph isomorphic to P_4, i.e., if it has no *induced 4-path*. A coloring c of the two-element subsets of a set X with two colors is *P_4-free* if for one color i the graph $(X, c^{-1}(i))$ is P_4-free. Since P_4 is isomorphic to its complement, it does not matter which color i we actually use in this definition.

Theorem 3.2. *Let f be a function from a set $A \subseteq \mathbb{R}$ to $\mathbb{R}$. If every $\{a, b\} \in [A]^2$ is either in configuration $\sqcap$ or in configuration $\sqcup$, then the graph G_f and hence the coloring by configuration on A are P_4-free.*

For the proof of this theorem, we will use the following lemma several times.

Lemma 3.3. *Let $a, b \in A$ be such that $a < b$ and such that in G_f there is a path from a to b that does not use vertices below a or above b. Then the vertices a and b are connected by an edge in G_f.*

Proof. For every edge e of G_f let $\ell(e) \in A$ be the left endpoint of e and $r(e) \in A$ the right endpoint with respect to the order on A.

Claim 3.4. Let $a, b \in A$ be such that $a < b$ and there is a path in G_f from a to b that has no vertices outside $[a, b]$. Then there is a finite sequence $(e_1, \dots, e_k)$ of edges of G_f such that $\ell(e_1) = a$, $r(e_k) = b$, and either $k = 1$ or for each $i \in \{1, \dots, k-1\}$ we have

$$a \leq \ell(e_i) \leq \ell(e_{i+1}) \leq r(e_i) \leq r(e_{i+1}) \leq b.$$

We prove the claim simultaneously for all a and b by induction on the length of a path from a to b that does not leave the interval $[a, b]$.

If there is a path of length 1 from a to b, then a and b are connected by an edge and the claim holds with $k = 1$ and $e_1 = \{a, b\}$. Now suppose for some $n \geq 1$ we have shown the claim for all a and b that are connected by a path of length $\leq n$ that stays inside $[a, b]$. Let $a, b \in A$ be such that $a < b$ and there is a path $a, x_1, \dots, x_n, b$ of length $n + 1$ such that $x_1, \dots, x_n \in [a, b]$.

We may assume that a and b are not among the $x_1, \dots, x_n$ since otherwise there would be a shorter path from a to b that stays inside $[a, b]$ and to which the inductive hypothesis applies. Let $j \in \{1, \dots, n\}$ be such that $x_j = \max\{x_1, \dots, x_n\}$. Now $x_j \in (a, b)$ and $a, x_1, \dots, x_j$ is a path of length $\leq n$ from a to x_j that does not use the vertex b and therefore, by the maximality of x_j, does not use any vertex outside the interval $[a, x_j]$. Hence, by the inductive hypothesis, there is a sequence $(e_1, \dots, e_k)$ of edges of G_f such that $\ell(e_1) = a$, $r(e_k) = x_j$, and either $k = 1$ or for each $i \in \{1, \dots, k-1\}$ we have

$$a \leq \ell(e_i) \leq \ell(e_{i+1}) \leq r(e_i) \leq r(e_{i+1}) \leq x_j.$$

Let $e_{k+1} = \{x_n, b\}$. By the maximality of x_j, $x_n \leq x_j$. Hence $(e_1, \dots, e_{k+1})$ is a sequence of edges that shows that the claim holds for a and b. This finishes the inductive proof of the claim for all a and b.

Given a and b as in the claim, let k be minimal with the property that there is a sequence $(e_1, \dots, e_k)$ as in the conclusion of the claim. If $k = 1$, a and b are connected by an edge and we are done. If $k > 1$, then by a) and b) of Lemma 3.1, $\{\ell(e_1), r(e_2)\}$ is an edge in G_f. Replacing e_1 and e_2 in $(e_1, \dots, e_k)$ by the single edge $\{\ell(e_1), r(e_2)\}$ we see that k was not minimal, a contradiction. □

Proof of Theorem 3.2. Observe that due to the symmetry between $\sqcap$ and $\sqcup$ Lemma 3.3 also holds for non-edges of G_f: if there is a path from a to b in the complement of G_f that uses only vertices in $[a, b]$, then $\{a, b\}$ is not an edge of G_f.

Now suppose that $v_1, \dots, v_4 \in A$ are the vertices of an induced 4-path of G_f. We may assume that $v_1 < \dots < v_4$. By Lemma 3.3, $\{v_1, v_4\}$ is an edge in G_f. We distinguish two cases:

Case 1. v_1 is an endpoint of the 4-path.

In this case there is a path from v_2 to v_4 that uses only vertices inside $[v_2, v_4]$. By Lemma 3.3, $\{v_2, v_4\}$ is an edge of G_f. It follows that v_3 is the other endpoint of the 4-path and that $\{v_2, v_3\}$ is an edge of G_f. In this situation, $\{v_1, v_3\}$ and $\{v_3, v_4\}$ are non-edges of G_f. Hence by Lemma 3.3 for non-edges, $\{v_1, v_4\}$ is a non-edge, a contradiction.

Case 2. v_2 is an endpoint of the 4-path and neither v_1 nor v_4 are endpoints.

Obviously, in this case v_3 is the other endpoint of the 4-path. If $\{v_2, v_4\}$ is an edge of G_f, then also $\{v_1, v_3\}$ is an edge. It follows that $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_3, v_4\}$ are non-edges. By Lemma 3.3 for non-edges, $\{v_1, v_4\}$ is a non-edge, a contradiction. Hence $\{v_2, v_4\}$ is not an edge of the 4-path. But then $\{v_1, v_2\}$ has to be an edge. It follows that also $\{v_3, v_4\}$ is an edge. This implies that $\{v_1, v_3\}$, $\{v_2, v_3\}$ and $\{v_2, v_4\}$ are non-edges. Again by the Lemma 3.3 for non-edges, $\{v_1, v_4\}$ is a non-edge, also a contradiction.

There is only one case that has not been discussed yet, namely when v_3 and v_4 are endpoints of the path. But this is symmetric to a situation discussed in Case 1. It follows that $v_1, \dots, v_4$ is not an induced 4-path. □

Recall that a finite graph is *perfect* if for every induced subgraph the chromatic number is equal to the maximal size of a complete subgraph. We call an infinite graph perfect if every finite induced subgraph is perfect. We call a coloring c of the two-element subsets of a set X with two colors *perfect* if for one color i the graph $(X, c^{-1}(i))$ is perfect. Since the complement of a perfect graph is also perfect by Lovász' Perfect Graph Theorem [8], in this definition the actual choice of the color i does not matter.

The Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas [3] easily implies that every P_4-free graph is perfect. However, there is a much more elementary proof of the perfectness of P_4-free graphs by Seinsche [9]. Now Lemma 3.2 immediately gives:

Corollary 3.5. *If $f : A \to \mathbb{R}$ is a function as in Lemma 3.2, then the graph G_f and hence the coloring by configuration on $[A]^2$ are perfect.*

Now fix a closed, uncountably convex set $S \subseteq \mathbb{R}^2$ that does not have a perfect clique. Let $K \subseteq S$ be special and let $f : A \to \mathbb{R}$ be the function witnessing this. Let $c_K : [A]^2 \to \{\sqcap, \sqcup\}$ be the corresponding continuous coloring. From Lemma 2.4 a) and Lemma 3.2 we obtain:

Corollary 3.6. *The graph G_K and the coloring c_K are P_4-free and hence perfect.*

4. Continuous P_4-free colorings

In the previous section we showed that the continuous colorings that occur in the context of closed uncountably convex planar sets without perfect 3-cliques are P_4-free. We use the methods developed in [5] in order to analyze P_4-free continuous colorings. Let us start with the definition of $c_{\min}$.

Definition 4.1. For $\{x, y\} \in [\omega^{\leq\omega}]^2$ with $x \not\subseteq y$ and $y \not\subseteq x$ let

$$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}.$$

Let $c_{\text{parity}}(x, y) = \Delta(x, y) \mod 2$, $c_{\min} = c_{\text{parity}} \restriction [2^\omega]^2$, and $G_{\min} = (2^\omega, c_{\min}^{-1}(1))$.

In [9], Seinsche characterized finite P_4-free graphs as graphs having no induced subgraphs G of size at least 2 with both G and the complement $\overline{G}$ connected[1]. This implies that the class of finite P_4-free graphs coincides with the smallest class of graphs that contains the one-vertex graph K_1 and is closed under disjoint unions and complements. The graphs in this latter class are called *cographs*[2] Every cograph is represented by a *cotree* that records how the graph has been constructed from K_1 using complementation and disjoint union [2].

Alon [1] observed that finite induced subgraphs of $G_{\min}$ are perfect. Using Seinsche's characterization of P_4-free graphs, we can actually show the following:

Lemma 4.2. *A finite graph G is P_4-free iff it is isomorphic to an induced subgraph of $G_{\min}$.*

Proof. Given an infinite graph G, let age(G) denote the *age* of G, i.e., the class of all finite graphs isomorphic to an induced subgraph of G.

$G_{\min}$ is isomorphic to an induced subgraph of its complement $\overline{G}_{\min}$ and vice versa. Also, $G_{\min}$ contains two disjoint copies of itself that are not connected by an edge. This implies that age$(G_{\min})$ is closed under complements and disjoint unions. Hence every cograph is in age$(G_{\min})$.

For the other direction of the equivalence, let $F \subseteq 2^\omega$ be finite. Let

$$m = \min\{\Delta(x, y) : x, y \in F \wedge x \neq y\}.$$

For $i \in 2$ let $F_i = \{x \in F : x(m) = i\}$. If m is even, then no edge of $G_{\min}$ runs between F_0 and F_1. If m is odd, then no vertex in F_0 is connected to any vertex in F_1 by an edge in the complement of $G_{\min}$.

It follows that $G_{\min}$ has no finite induced subgraph G of size at least 2 such that both G and $\overline{G}$ are connected. Hence G is a P_4-free. □

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[1]We thank the anonymous referee for pointing out that Seinsche's characterization fails in the infinite. Consider the graph on the natural numbers where each even number is connected to all smaller numbers and no odd number is connected to any smaller number. Both this graph and its complement are connected, yet the graph is P_4-free.

[2]We would like to thank François Dorais for pointing out the connection between P_4-free graphs and cographs and for referring to Seinsche's article.

Before we can say more about P_4-free continuous colorings, we need to quote a number of results related to continuous colorings.

The cardinal invariant $\mathfrak{d}$ is the least cardinality of a family of compact sets that covers ω^ω. The following lemma is folklore.

Lemma 4.3 (Lemma 3.5 in [4])**.** *Every Polish space X can be covered by a family of size at most $\mathfrak{d}$ of sets that are either singletons or copies of 2^ω.*

Definition 4.4. If $c : [X]^2 \to 2$ and $d : [Y]^2 \to 2$ are continuous colorings on Polish spaces X and Y, we write $c \le d$ if there is a topological embedding $e : X \to Y$ such that for all $\{x_1, x_2\} \in [X]^2$, $c(x_1, x_2) = d(e(x_1), e(x_2))$.

Clearly, $c \le d$ implies $\mathfrak{hm}(c) \le \mathfrak{hm}(d)$. We collect the fundamental results on continuous colorings and homogeneity numbers.

Lemma 4.5. *a) $\mathfrak{hm}^+ \ge 2^{\aleph_0}$ (Lemma 8 in [6])*

b) $\mathfrak{d} \le \mathfrak{hm}$ (Section 3 of [5])

c) If $c : [X]^2 \to 2$ is a continuous coloring on a Polish space X and $\mathfrak{hm}(c) > \aleph_0$, then $c_{\min} \le c$. In particular, $\mathfrak{hm} \le \mathfrak{hm}(c)$ (Theorem 10 in [6]).

d) $c_{\text{parity}} \le c_{\min}$. In particular, $\mathfrak{hm} = \mathfrak{hm}(c_{\text{parity}})$ (Lemma 2.10 in [5]).

Definition 4.6. Let $X \subseteq \omega^\omega$ be a closed set. A continuous coloring $c : [X]^2 \to 2$ is an *almost node-coloring* if for all $\{x, y\} \in [X]^2$ and all $a, b \in X$ with $\Delta(x, a), \Delta(y, b) > \Delta(x, y)$ we have $c(a, b) = c(x, y)$.

Lemma 4.7 (Lemma 2.17 in [5])**.** *Let $c : [2^\omega]^2 \to 2$ be a continuous coloring. Then there is a topological embedding $e : 2^\omega \to \omega^\omega$ such that for every c_{parity}-homogeneous set $H \subseteq e[2^\omega]$, the coloring c^e which is induced on H by c via e is an almost node-coloring.*

Corollary 4.8. *For every continuous coloring $c : [X]^2 \to 2$ on a Polish space X, X is the union of $\mathfrak{hm}$-many sets $Y \subseteq X$ such that $c \restriction [Y]^2$ is isomorphic to an almost node-coloring on a compact subset of ω^ω. In particular, if $\mathfrak{hm}(c)$ is uncountable, then $\mathfrak{hm}(c) = \mathfrak{hm}(d)$ for some almost node-coloring d on a compact subset of ω^ω such that $d \le c$.*

Proof. This follows from Lemma 4.3, Lemma 4.5, and Lemma 4.7. □

Corollary 4.9. *Let $c : [X]^2 \to 2$ be a continuous coloring on a Polish space X with $\mathfrak{hm}(c) > \aleph_0$. If c is P_4-free, then there is a P_4-free almost node-coloring $d : [Y]^2 \to 2$ on a compact subset Y of ω^ω such that $\mathfrak{hm}(c) \le \mathfrak{hm}(d)$.*

Proof. If c is P_4-free, then so are the colorings $c \restriction [Y]^2$, $Y \subseteq X$. Now the Corollary follows from Corollary 4.8. □

Lemma 4.10. *Let $d : [X]^2 \to 2$ be a P_4-free almost node-coloring on a compact set $X \subseteq \omega^\omega$. Then $d \le c_{\min}$.*

Proof. Let $T = T(X) = \{s \in \omega^{<\omega} : \exists x \in X (s \subseteq x)\}$. For each $s \in T$ let $\text{succ}_T(s)$ denote the set of immediate successors of s in the tree T. Since X is compact, $\text{succ}_T(s)$ is finite for every $s \in T$. For $x, y \in X$ with $x \neq y$ let $x \wedge y = x \restriction \Delta(x, y)$.

Since d is an almost node coloring, for each $s \in T$ there is a coloring

$$d_s : [\text{succ}_T(s)]^2 \to 2$$

such that for all $x, y \in X$ with $s = x \wedge y$,

$$d_s(x \restriction \Delta(x, y) + 1, y \restriction \Delta(x, y) + 1) = d(x, y).$$

By recursion on the length of $s \in T$, we define a map $f : T \to 2^\omega$ with the following properties:

(1) For all $s, t \in T$, if $s \subsetneq t$, then $f(s) \subsetneq f(t)$
(2) For all $s, t \in T$, if s and t are incomparable with respect to $\subseteq$, then so are $f(s)$ and $f(t)$.
(3) For all $s \in T$ and all $t_0, t_1 \in \mathrm{succ}_T(s)$ with $t_0 \neq t_1$,

$$d_s(t_0, t_1) = \Delta(f(t_0), f(t_1)) \mod 2.$$

We start the definition of f by letting $f(\emptyset) = \emptyset$. Now suppose that $f(s)$ has been defined for some $s \in T$. Since d is P_4-free, d_s is P_4-free. By Lemma 4.2, $d_s \leq c_{\min}$. Let $U \subseteq 2^\omega$ be the open set of all binary sequences that start with $f(s)$. Since $c_{\min} \leq c_{\min} \restriction [U]^2$, we have $d_s \leq c_{\min} \restriction [U]^2$. Fix an embedding $e_s : \mathrm{succ}_T(s) \to U$ witnessing this. Choose $m \in \omega$ such that the finite sequences $e(t) \restriction m$, $t \in \mathrm{succ}_T(s)$, are pairwise distinct. For each $t \in \mathrm{succ}_T(s)$ let $f(t) = e(t) \restriction m$. This finishes the recursive definition of f.

Now $e : X \to 2^\omega$ defined by $e(x) = \bigcup_{n\in\omega} f(x \restriction n)$ witnesses $d \leq c_{\min}$. □

Corollary 4.11. *Let $c : [X]^2 \to 2$ be a P_4-free continuous coloring on a Polish space X. Then $\mathfrak{hm}(c) \leq \mathfrak{hm}$.*

Proof. By Corollary 4.9, there is a P_4-free almost node coloring d on a compact subset of ω^ω such that $\mathfrak{hm}(c) \leq \mathfrak{hm}(d)$. By Lemma 4.10, $d \leq c_{\min}$ and hence $\mathfrak{hm}(d) \leq \mathfrak{hm}$. □

Corollary 4.12. *Let $S \subseteq \mathbb{R}^2$ be a closed, uncountably convex set without a perfect 3-clique. Then $\gamma(S) = \mathfrak{hm}$.*

Proof. By Theorem 2.6 there are a countably convex set A and a sequence $(K_n)_{n\in\omega}$ of special sets such that $S = A \cup \bigcup_{n\in\omega} K_n$. For each $n \in \omega$ let c_n be the continuous coloring associated with the special set K_n. By Corollary 3.6, each c_n is P_4-free. By Corollary 4.11, $\mathfrak{hm}(c_n) \leq \mathfrak{hm}$ for all $n \in \omega$. It follows that $\gamma(S) \leq \mathfrak{hm}$.

Since S is uncountably convex, at least one of the colorings c_n has an uncountable homogeneity number. Since $\mathfrak{hm}$ is the minimal uncountable homogeneity number of all continuous colorings on a Polish space, we have $\mathfrak{hm} \leq \gamma(S)$. □

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