CONVEX OPEN SUBSETS OF $\mathbb{R}^n$ ARE HOMEOMORPHIC TO n-DIMENSIONAL OPEN BALLS

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It is wellknown that convex open subsets of $\mathbb{R}^n$ are homeomorphic to n-dimensional open balls, but a full proof of this fact seems to be difficult to find in the literature.

Theorem 1. *Let $n \in \mathbb{N}$ and let $U \subseteq \mathbb{R}^{n+1}$ be nonempty, open, and convex. Then U is homeomorphic to the open unit ball D^{n+1} in $\mathbb{R}^{n+1}$.*

Proof. Translating U if necessary, we may assume $0 \in U$. Let $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. We define $d : S^n \to [0,\infty) \cup \{\infty\}$ as follows:

For each $v \in S^n$ let R_v denote the ray $\{rv : r \in [0,\infty)\}$. Let $d(v) = \infty$ if $R_v \subseteq U$. If $R_v \not\subseteq U$ let $d(v) = \inf\{r \in [0,\infty) : rv \notin U\}$.

The set $[0,\infty) \cup \{\infty\}$ with its usual topology is homeomorphic to the interval $[0,\pi/2]$, witnessed by the map $f : [0,\infty) \cup \{\infty\} \to [0,\pi/2]$ that maps $r \in [0,\infty)$ to $\arctan(r)$ and ∞ to $\pi/2$. Let $v \in S^n$. We show that d is continuous in v with respect to the usual topology on $[0,\infty) \cup \{\infty\}$.

First case: $d(v) = \infty$. In this case $Rv \subseteq U$. Since U is open, there is a nonempty open ball B around the origin with $B \subseteq U$. Since U is convex, the convex hull C of $B \cup R_v$ is contained in U. For every $N > 0$ the set of elements of C that have distance at least N from the origin is open. It follows that the set of directions $w \in S^n$ such that the $R_w \cap C$ has elements of distance at least N from the origin is open. It follows that d is continuous in v.

Second case: $d(v) < \infty$, i.e., $R_v \not\subseteq U$. Let $\varepsilon > 0$. Choose $x \in R_v \cap U$ such that $d(v) - |x| < \varepsilon/2$. Since U is open, for some $\delta > 0$, the δ-ball $B_\delta(x)$ around x is contained in U. The set of directions $w \in S^n$ such

Date: July 4, 2012.

that $R_w \cap B_\delta(x) \neq \emptyset$ is open in S^n and contains v. For all w with $R_w \cap B_\delta(x) \neq \emptyset$ we have $d(w) \geq d(v) - \varepsilon$.

Now let $x \in R_v$ be such that $|x| > d(v)$.

Claim 2. The point x is not contained in the closure of U.

Assume x is in the closure of U. Let $y \in R_v$ be such that $|y| = d(v)$. We have $|y| < |x|$. By the definition of $d(v)$ there are points $z \in R_v$ arbitrarily close to y such that $z \notin U$. It follows that y is on the boundary of U and that x is not contained in the open ball B around the origin contained in U that we have chosen above.

Choosing $\varepsilon > 0$ sufficiently small, we have the following: For all $z \in B_\varepsilon(x)$, y is in the convex hull C_z of $B \cup \{z\}$. Since x is in the closure of U, there is $z \in U$ such that $|x - z| < \varepsilon$. Since $z \in B_\varepsilon(x)$, $y \in C_z$. By the convexity of U, $C_z \subseteq U$. But $C_z \setminus \{z\}$ is an open neighborhood of y that is contained in U. This shows that y is not on the boundary of U, a contradiction. This finishes the proof of the claim.

Since x is not in the closure of U, there is an open ball around x that is disjoint from U. It follows that there an open set neighborhood of v in S^n such that for all directions w in that neighborhood $d(w) \leq |x|$. This shows that d is continuous in v.

We are now ready to define a homeomorphism $h : U \to \mathbb{D}^{n+1}$. Let $h(0) = 0$. Recall the definition of $f : [0, \infty) \cup \{\infty\} \to [0, \pi/2]$. For $x \in U \setminus \{0\}$ let $v = x/|x|$ and put

$$h(x) = \frac{f(|x|)}{f(d(v))} \cdot v.$$

Since U is an open neighborhood of 0, $d(v)$ and hence $f(d(v))$ is bounded away from 0. Moreover, on $U \setminus \{0\}$, v depends continuously on x. Since f and d are continuous, h is continuous on $U \setminus \{0\}$. For $x \to 0$ we have $f(|x|) \to 0$. It follows that $h(x) \to 0$ as $x \to 0$. Hence h is continuous in 0 as well. Clearly, h is 1-1. By the definition of d, h is onto. It is easily checked that h is actually a homeomorphism. $\square$

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