CONTINUOUS RAMSEY THEORY ON POLISH SPACES AND COVERING THE PLANE BY FUNCTIONS

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ABSTRACT. We investigate the Ramsey theory of continuous graph-structures on complete, separable metric spaces and apply the results to the problem of covering a plane by functions.

Let the *homogeneity number* $\mathfrak{hm}(c)$ of a pair-coloring $c : [X]^2 \to 2$ be the number of c-homogeneous subsets of X needed to cover X. We isolate two continuous pair-colorings on the Cantor space 2^ω, $c_{\min}$ and $c_{\max}$, which satisfy $\mathfrak{hm}(c_{\min}) \le \mathfrak{hm}(c_{\max})$ and prove:

Theorem. (1) *For every Polish space X and every continuous pair-coloring $c : [X]^2 \to 2$ with $\mathfrak{hm}(c) > \aleph_0$,*

$$\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min}) \quad \text{or} \quad \mathfrak{hm}(c) = \mathfrak{hm}(c_{\max}).$$

(2) *There is a model of set theory in which $\mathfrak{hm}(c_{\min}) = \aleph_1$ and $\mathfrak{hm}(c_{\max}) = \aleph_2$.*

The consistency of $\mathfrak{hm}(c_{\min}) = 2^{\aleph_0}$ and of $\mathfrak{hm}(c_{\max}) < 2^{\aleph_0}$ follows from [20].

We prove that $\mathfrak{hm}(c_{\min})$ is equal to the covering number of $(2^\omega)^2$ by graphs of Lipschitz functions and their reflections on the diagonal. An iteration of an optimal forcing notion associated to $c_{\min}$ gives:

Theorem. *There is a model of set theory in which*
(1) $\mathbb{R}^2$ *is coverable by $\aleph_1$ graphs and reflections of graphs of* continuous *real functions;*
(2) $\mathbb{R}^2$ *is not coverable by $\aleph_1$ graphs and reflections of graphs of* Lipschitz *real functions.*

Diagram 1 in the introduction summarizes the ZFC results in Part I of the paper. The independence results in Part II show that any two rows in Diagram 1 can be separated if one excludes $\mathrm{Cov}(\mathcal{L}ip(\mathbb{R}))$ from row (3).

1. INTRODUCTION

In this paper we study Ramsey properties of continuous graph structures on Polish spaces. A continuous graph-structure on a Polish space X is a pair-coloring $c : [X]^2 \to \{0, 1\}$ which is continuous with respect to the natural topology on unordered pairs from X. The Ramsey invariant which we use to classify such colorings is the *homogeneity number*: the least number of homogeneous subsets (of both colors) required to cover X.

Let us explain briefly how set-theoretic study of continuous pair-colorings leads to the consideration of homogeneity numbers. It is not hard to check that for every continuous pair-coloring on an uncountable Polish space there is a homogeneous subset which is perfect, and hence of size continuum, and that the *chromatic number*

of the coloring — that is, the number of homogeneous subsets of color 0 required to cover the space — is either countable or continuum[1].

This means that, from a set-theoretic point of view, these standard invariants are degenerate. The homogeneity number of continuous pair-colorings is a natural generalization of the chromatic number which is *not* set-theoretically degenerate. The classification of homogeneity numbers on Polish spaces leads both to an interesting theory in ZFC and to interesting meta-mathematics.

A second reason for the consideration of homogeneity numbers comes from convexity theory. For certain closed subsets of $\mathbb{R}^2$ the number of convex subsets required to cover them is equal to the homogeneity number $\mathfrak{hm}(c)$ of some continuous pair-coloring c on the Baire space [20]. Thus, homogeneity numbers measure the complexity of convex decompositions of certain closed planar sets.

Homogeneity numbers of continuous pair-colorings relate to at least two more subjects. The crucial inequality (Theorem 3.9 below), which reduces general continuous pair-colorings to compact ones, involves the notion of *covering a plane by functions*. About half of the paper is devoted to that subject. The connection between continuous pair-colorings and covering a plane by functions works in both ways: after establishing the classifications of homogeneity numbers we have at hand an *optimal forcing* for proving the consistency of "more Lipschitz functions are required to cover $(2^\omega)^2$ than continuous ones". This consistency result implies the consistency of "more Lipschitz functions are required to cover $\mathbb{R}^2$ than continuous ones".

Finally, it is perhaps a bit surprising that the classification of continuous homogeneity numbers is related to *perfect graphs* (see the recent survey [11] and [12]). The forcing construction that separates $\mathfrak{hm}(c_{\min})$ from $\mathfrak{hm}(c_{\max})$ (Section 4 below) makes a crucial use of a Ramsey connection between perfect graphs and random graphs [4].

Within set theory, continuous pair-colorings are naturally related to the broader class of *open colorings*, that has been a focus of interest for set theorists for three decades now, and motivated several important developments in the technique of forcing [7, 8, 2]. *Open coloring axioms*, which are statements in the Ramsey theory of open colorings, are among the more frequently used set-theoretic axioms in the theory of the continuum (see [33, 32, 17, 27] and the references therein).

1.1. **The results.** Two simple pair-colorings $c_{\min}$ and $c_{\max}$ are defined on the Cantor space, and are shown to satisfy for every Polish space X and every continuous $c : [X]^2 \to 2$ with uncountable $\mathfrak{hm}(c)$:

$$\mathfrak{hm}(c_{\min}) \leq \mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\max}) \tag{1}$$

To state the remaining results concisely, we briefly introduce some notation. A function $f : X \to X$ *covers* a point $(x, y) \in X^2$ if $f(x) = y$ or $f(y) = x$. For a metric space (X, dist) let $\text{Cov}(\mathcal{L}ip(X))$ denote the number of Lipschitz functions from X to X required to cover X^2 and $\text{Cov}(\mathcal{C}ont(X))$ denote the analogous numbers for continuous functions. The *Baire space* ω^ω and the *Cantor space* 2^ω are considered with the standard metric $\text{dist}(x, y) = \frac{1}{2^{\Delta(x,y)}}$, where $\Delta(x, y) = \min\{n : x(n) \neq y(n)\}$ for $x \neq y$.

The remaining ZFC equalities and inequalities are summarized in Diagram 1.

Homogeneity numbers are on the right column and covering-by-functions cardinals are on the middle column. We draw attention to the fact that the rows (2)–(6) have to share at most two consecutive cardinals since $\text{Cov}(\mathcal{C}ont(2^\omega))$ cannot be

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[1]This holds for every open coloring on an analytic set [18].

$$(6) \qquad \left(\mathrm{Cov}(\mathcal{C}ont(2^\omega)) \right)^+$$

$$(5) \qquad 2^{\aleph_0}$$

$$(4) \qquad \mathfrak{hm}(c_{\max})$$

$$(3) \quad \mathrm{Cov}(\mathcal{L}ip(\mathbb{R})) \geq \mathrm{Cov}(\mathcal{L}ip(\omega^\omega)) = \mathrm{Cov}(\mathcal{L}ip(2^\omega)) \quad = \quad \mathfrak{hm}(c_{\min})$$

$$(2) \quad \mathrm{Cov}(\mathcal{C}ont(\mathbb{R})) = \mathrm{Cov}(\mathcal{C}ont(\omega^\omega)) = \mathrm{Cov}(\mathcal{C}ont(2^\omega))$$
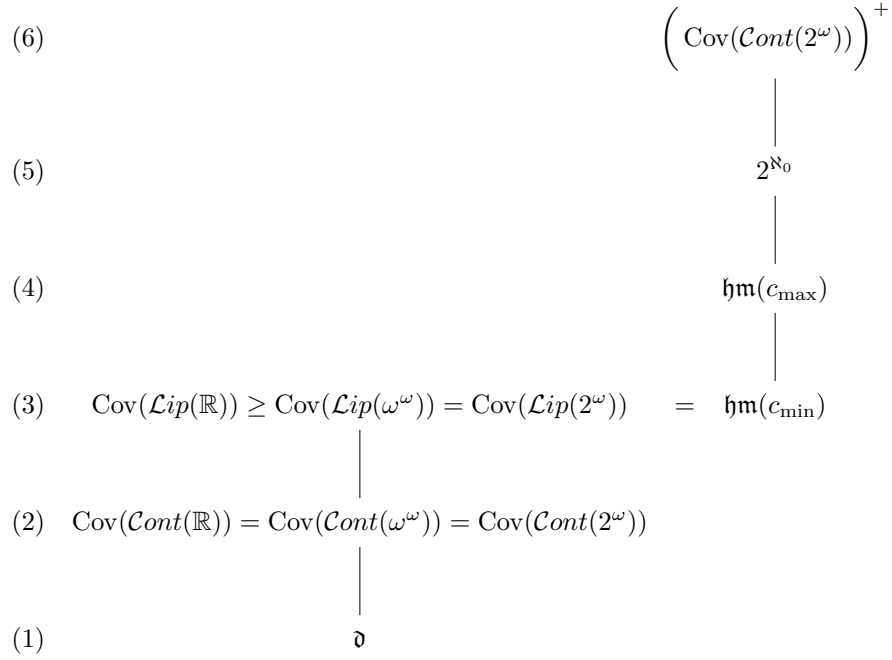
$$(1) \qquad \mathfrak{d}$$

Diagram 1

more than one cardinal below $2^{\aleph_0}$; thus, four different models of set theory are required to separate them from each other.

The independence results in Part II of the paper show that after excluding $\mathrm{Cov}(\mathcal{L}ip(\mathbb{R}))$ from row (3) for each of the rows (1)–(5) it is consistent that the value at the row is $\aleph_1$ and at all rows above the value is $\aleph_2$. The forcing for separating (2) from (3) is a new example of an *optimal forcing* in the sense of Zapletal [34] for increasing a cardinal invariant while leaving small everything that can be left small.

The inequality $\mathrm{Cov}(\mathcal{L}ip(2^\omega)) \leq \mathfrak{hm}(c_{\min})$ and the consistency of $\mathfrak{hm}(c) < 2^{\aleph_0}$ for every Polish space X and continuous $c : [X]^2 \to 2$ were proved in [20].

The last inequality cannot hold for all *open* colorings. In [2] an example of an open pair-coloring on the square of any uncountable Polish space X is given such that X^2 cannot be covered by fewer than $2^{\aleph_0}$ homogeneous sets. Let us present a slightly simplified version of this coloring.

An unordered pair $\{(x_0, y_0), (x_1, y_1)\}$ of elements of X^2 is of color 0 if it is a 1-1-function and of color 1 otherwise. The set of pairs of color 0 is open. If $H \subseteq X^2$ is homogeneous of color 1, then it is either (a part of) a row or (a part of) a column in the square. The homogeneous sets of color 0 are graphs of (partial) injective functions. It is easily checked that X^2 cannot be covered by less than $2^{\aleph_0}$ homogeneous sets.

1.1.1. *Structure of the paper.* The paper is divided to two parts. Absolute ZFC results are in Part I and independence results are in Part II. Notation, preliminaries and background material are included at the beginning of each section. The first part employs elementary techniques and does not require any specialized knowledge.

Although we are supposed to assume that every reader will read the whole paper, we suspect that those who will read the second part are knowledgeable in forcing notation. For those readers who read the first part and decide that they have to learn forcing so that they can read the second part, we recommend the standard [26, 6] as sources for notation and introduction to forcing.

We tried to keep notation as standard as possible.

PART I: RESULTS IN ZFC

2. THE STRUCTURE OF CONTINUOUS PAIR-COLORINGS ON POLISH SPACES

2.1. Basic definitions and preliminary facts.

2.1.1. *Colorings, chromatic numbers, and homogeneity numbers.* The symbol $[A]^2$ denotes the set of all two-element subsets of a set A. Ramsey's theorem states that if A is infinite, then for every function $c : [A]^2 \to 2 := \{0,1\}$ there is an infinite set $B \subseteq A$ so that c is constant on $[B]^2$. A function $c : [A]^2 \to 2$ is called a *pair-coloring*, and a set $B \subseteq A$ for which $c \restriction [B]^2$ is constant is called *c-homogeneous* or *c-monochromatic*. In the future we write just $c \restriction B$ instead of $c \restriction [B]^2$. A set H is *c-homogeneous of color i* for $i \in 2$, if the constant color on H is i.

A pair coloring c on A can be thought of as (the characteristic function of) the edge relation of a graph $G = (A, c)$. In this setting Ramsey's theorem states that every infinite graph contains either an infinite *clique* — a subgraph in which any pair of vertices forms an edge — or an infinite *independent set* — a subset in which no two vertices form an edge.

Recall that the *chromatic number* of a graph is the least number of independent sets required to cover the set of vertices.

Definition 2.1. *For a coloring $c : [A]^2 \to 2$ the* homogeneity number of c, *denoted by $\mathfrak{hm}(c)$, is the minimal number of c-homogeneous subsets required to cover A.*

The difference between chromatic and homogeneity numbers is that in the definition of the latter covering is by homogeneous sets of both colors.

2.1.2. *Continuous colorings on Polish spaces.* Let X be a topological space and let $X^2 := X \times X$ with the product topology. We identify $[X]^2$ with the quotient space $(X^2 \setminus \{(x,x) : x \in X\})/\sim$, where $(x,y) \sim (w,z)$ iff $(x,y) = (w,z)$ or $(x,y) = (z,w)$.

Theorem 2.2 (Eilenberg 1941). *If X is a connected topological space with more than one point, then $[X]^2$ is connected.*

Proof. Eilenberg's Theorem 1 in [14] states that if X is connected and contains more than one point, then either $X^2 \setminus \{(x,x) : x \in x\}$ is connected or else X can be linearly ordered by a relation $<$ so that the topology of X refines the order topology defined by $<$ and so that the connected components of $X^2 \setminus \{(x,x) : x \in X\}$ are $\{(x,y) : x < y\}$ and $\{(x,y) : y < x\}$. In the former case $[X]^2$ is connected as a continuous image of a connected space and in the latter case $[X]^2$ is connected because it is homeomorphic to the connected component $\{(x,y) : x < y\}$. □

We thank Apollo Hogan for pointing out Theorem 2.2 and referring us to Eilenberg's paper.

A coloring $c : [X]^2 \to 2$ is continuous if the preimages of 0 and of 1 are open. Equivalently, c is continuous if for all $\{x,y\} \in [X]^2$, there are disjoint open neighborhoods U and V of x and y, respectively, such that c is constant on $U \times V$. Here we identify c with the corresponding symmetric function from $X^2 \setminus \{(x,x) : x \in X\}$ to 2. By Theorem 2.2, there are no non-constant continuous colorings of $[X]^2$ if X is connected.

A topological space X is *Polish* if it is homeomorphic to a separable and complete metric space. Every Polish space is a disjoint union of a countable open *scattered* subset with a *perfect* subset (where either of the two components may be empty). Since every nonempty perfect subset of a Polish space has the cardinality of the continuum, every uncountable Polish space is equinumerous with the continuum.

Definition 2.3. (1) *A pair-coloring c on X is* reduced *if c is continuous and no nonempty open subset of X is c-homogeneous.*

(2) *A coloring $c : [X]^2 \to 2$ is* trivial *if $\mathfrak{hm}(c) \leq \aleph_0$.*

Fact 2.4. *If X is a Polish space and $c : [X]^2 \to 2$, then $X = X_0 \cup X_1$ such that X_0 is open, X_1 is perfect, $X_0 \cap X_1 = \emptyset$, $c \restriction X_0$ is trivial and $c \restriction X_1$ is reduced.*

Proof. Let X_0 be the union of all open sets $U \subseteq X$ for which $c \restriction U$ is trivial. X_0 is open and since X has a countable basis, c is trivial on X_0. Let $X_1 = X \setminus X_0$. $\square$

Fact 2.5. *A continuous pair-coloring on a Polish space X satisfies $\mathfrak{hm}(c) > \aleph_0$ if and only if there exists a nonempty perfect $Y \subseteq X$ so that $\mathfrak{hm}(c) = \mathfrak{hm}(c \restriction Y)$ and $c \restriction Y$ is reduced.*

Proof. Suppose $\mathfrak{hm}(c) > \aleph_0$ and write $X = X_0 \cup X_1$ as stated in the previous Fact. So $c \restriction X_1$ is reduced. Since $\mathfrak{hm}(c \restriction X_0) \leq \aleph_0$ it follows that $\mathfrak{hm}(c) = \mathfrak{hm}(c \restriction X_1)$ and clearly $X_1 \neq \emptyset$. On the other hand, suppose $Y \subseteq X$ is perfect and nonempty, that $\mathfrak{hm}(c) = \mathfrak{hm}(c \restriction Y)$ and $c \restriction Y$ is reduced. Continuity of c gives that the closure of every c-homogeneous set is again c-homogeneous; so if $Y \subseteq X$ is perfect and $c \restriction Y$ is reduced, every c-homogeneous subset of Y is nowhere dense and by the Baire theorem $\mathfrak{hm}(c) > \aleph_0$. $\square$

2.1.3. *Notation.* Let ω^ω denote the set of all (infinite) sequences of natural numbers. Let $\omega^{<\omega}$ denote the set of all finite sequences of natural numbers and let $\omega^{\leq\omega} = \omega^{<\omega} \cup \omega^\omega$. Similarly, $2^\omega, 2^{<\omega}, 2^{\leq\omega}$ are the analogous sets for sequences over $\{0, 1\}$.

Definition 2.6. *For $x, y \in \omega^{\leq\omega}$ let $\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$ if there is some $n \in \omega$ such that $x(n) \neq y(n)$. Otherwise $\Delta(x, y)$ is undefined.*
If $\Delta(x, y)$ is defined for $x, y \in \omega^{\leq\omega}$, put

$$\mathrm{dist}(x, y) := \frac{1}{2^{\Delta(x,y)}}$$

If $\Delta(x, y)$ is not defined, put $\mathrm{dist}(x, y) := 0$.

The function dist satisfies the triangle inequality. In fact, it satisfies a stronger inequality: $\mathrm{dist}(x, z) \leq \max\{\mathrm{dist}(x, y), \mathrm{dist}(y, z)\}$ for all x, y, z. (This makes dist an *ultra-metric*.)

The following Polish spaces play an important role in this section: the *Cantor space* $(2^\omega, \mathrm{dist})$ and the *Baire space* $(\omega^\omega, \mathrm{dist})$. These spaces are indeed complete, separable metric spaces. The Cantor space is homeomorphic to the usual *Cantor set* and the Baire space is homeomorphic the space of *irrational numbers*.

2.1.4. *The minimal coloring $c_{\min}$.*

Definition 2.7. *If X and Y are topological spaces and c and d are continuous pair-colorings on X and Y, respectively, then we write $c \leq d$ if there is a topological embedding $e : X \to Y$, such that for all $\{x_0, x_1\} \in [X]^2$, $c(x_0, x_1) = d(e(x_0), e(x_1))$.*

Clearly, if $c \leq d$ via an embedding $e : X \to Y$, then $e^{-1}[A]$ is c-homogeneous for every d-homogeneous $A \subseteq Y$. Hence, $c \leq d$ implies that $\mathfrak{hm}(c) \leq \mathfrak{hm}(d)$.

We introduce next a pair coloring $c_{\min}$ on the Cantor space which satisfies $c_{\min} \leq c$ for all reduced c.

Definition 2.8.   (1) *Let $\mathrm{parity}(x, y)$ denote the parity of $\Delta(x, y)$ for $x, y \in \omega^{\leq\omega}$ such that $\Delta(x, y)$ is defined.*
(2) *Let $c_{\mathrm{parity}} := \mathrm{parity} \restriction \omega^\omega$.*
(3) *Let $c_{\min} := \mathrm{parity} \restriction 2^\omega$.*

Clearly, c_{parity} is a reduced pair-coloring on ω^ω and $c_{\min}$ is a reduced pair-coloring on 2^ω.

If $H \subseteq 2^\omega$ is $c_{\min}$-homogeneous of color 0, then all splittings in $T(H)$, the tree of all finite initial segments of members of H, occur on even levels. If T is a subtree

of $\omega^{<\omega}$, we identify every infinite branch of T with its union, a point in ω^ω. A set $H \subseteq 2^\omega$ is, then, maximal $c_{\min}$-homogeneous of color 0 is if and only if H is the set of all infinite branches of a tree T in which $t \in T$ has two immediate successors if $|t|$ is even and one immediate successor if $|t|$ is odd. Similarly, H is maximal $c_{\min}$-homogeneous of color 1 if and only if it is the set of all infinite branches of a tree T such that $t \in T$ has two immediate successors in T if $|t|$ is odd and one immediate successor in T if $|t|$ is even.

Lemma 2.9. *For every reduced pair-coloring c on a Polish space we have:*

$$c_{\min} \le c.$$

Consequently, $\mathfrak{hm}(c_{\min}) \le \mathfrak{hm}(c)$ for every reduced c.

Proof. Suppose $c : [X]^2 \to 2$ is reduced and X is Polish. Since no nonempty open set is c-homogeneous in X, X has no isolated points.

By induction on n choose, for every $t \in 2^n$, an open set $U_t \ne \emptyset$ of diameter $< 1/n$ such that

– $t \subseteq s \Rightarrow \mathrm{cl}(U_s) \subseteq U_t$,
– $\Delta(t_1, t_2)$ defined implies that $\mathrm{cl}(U_{t_1}) \cap \mathrm{cl}(U_{t_2}) = \emptyset$ and
  for all $x_1 \in \mathrm{cl}(U_{t_1})$, $x_2 \in \mathrm{cl}(U_{t_2})$: $c(x_1, x_2) \equiv \Delta(t_1, t_2) \mod 2$.

At the induction step, for a given $t \in 2^n$ find $x_1, x_2 \in U_t$ which satisfy $c(x_1, x_2) \equiv n \mod 2$ (possible since U_t is not c-homogeneous) and inflate x_1, x_2 to a sufficiently small open balls $U_{t^\frown 0}, U_{t^\frown 1}$.

The map e mapping each $x \in 2^\omega$ to the unique element of $\bigcap_n U_{x \restriction n}$ is an embedding of 2^ω into X which preserves $c_{\min}$. □

In [20] $\mathfrak{hm}(c_{\min})$ was denoted simply by $\mathfrak{hm}$. We will also sometimes write $\mathfrak{hm}$ for $\mathfrak{hm}(c_{\min})$.

Before we proceed, let us remark that c_{parity} is not more complicated than $c_{\min}$:

Lemma 2.10. $c_{\mathrm{parity}} \le c_{\min}$

Proof. We have to define an embedding $e : \omega^\omega \to 2^\omega$ witnessing $c_{\mathrm{parity}} \le c_{\min}$.

For $x \in \omega^\omega$, let $e(x)$ be the concatenation of the sequences b_n, $n \in \omega$, which are defined as follows.

If n is even, then let b_n be the sequence of length $2 \cdot x(n) + 2$ which starts with $2 \cdot x(n)$ zeros and then ends with two ones. If n is odd, let b_n be the sequence of length $2 \cdot x(n) + 2$ starting with $2 \cdot x(n) + 1$ zeros and ending with a single one.

It is clear that e is continuous and it is easy to check that e is an embedding witnessing $c_{\mathrm{parity}} \le c_{\min}$. □

2.2. **Classification of homogeneity numbers.** We begin now the classification of homogeneity numbers of continuous pair colorings on Polish spaces. The purpose of this Section is to find two continuous pair-colorings on the Cantor spaces whose homogeneity numbers capture all possible homogeneity numbers of continuous pair colorings. The following sequence of reductions will be performed: First, it will be shown that any homgneity number occurs on some compact space; then, that every homogeneity number occurs on the Cantor space; and finally that every number occurs as a homogeneity number of a particularly simple coloring on the Cantor space — an *almost node coloring*. At the end of the section, we shall be able to isolate a pair-coloring $c_{\max}$ with a maximal homogeneity number in the class of continuous pair-colorings on Polish spaces and show that every homgneity number of an arbitrary reduced coloring on a Polish space is equal to $\mathfrak{hm}(c_{\min})$ or is equal to $\mathfrak{hm}(c_{\max})$.

2.2.1. *Reduction to compact spaces.* The following two fundamental inequalities hold for $c_{\min}$:

$$(\mathfrak{hm}(c_{\min}))^+ \geq 2^{\aleph_0} \tag{2}$$

$$\mathfrak{hm}(c_{\min}) \geq \mathfrak{d} \tag{3}$$

The first inequality was proved in [20] and the second one which, really, is the starting point of the present paper, will be proved in Section 3. Although these inequalities are central for this Section, their proofs belong to the setting of covering a square by functions.

From the first inequality it follows that there is room for at most one more homogeneity number above $\mathfrak{hm}(c_{\min})$ — since either $\mathfrak{hm}(c_{\min})$ or its immediate successor cardinal is the continuum. In [20] it was proved consistent that for all reduced pair-colorings c,

$$\mathfrak{hm}(c) = \aleph_1 < 2^{\aleph_0} = \aleph_2. \tag{4}$$

The second inequality relates $\mathfrak{hm}(c_{\min})$ to the *domination number* $\mathfrak{d}$. This number is the least number of functions from ω to ω needed to eventually dominate every such function. It is not difficult to see that $\mathfrak{d}$ is the least size of a family of compact sets that covers ω^ω. It is well-known that every Polish space is a continuous image of ω^ω. Therefore every Polish space can be covered by $\mathfrak{d}$ compact sets.

Lemma 2.11. *For every Polish space X and a continuous pair-coloring $c : [X]^2 \to 2$ with uncountable $\mathfrak{hm}(c)$ there is a compact subspace $Y \subseteq X$ so that $\mathfrak{hm}(c) = \mathfrak{hm}(c \restriction Y)$.*

Proof. Suppose, without loss of generality, that c is reduced on X. Cover X by compact subspaces Y_α, $\alpha \leq \mathfrak{d}$, and denote $c_\alpha := c \restriction Y_\alpha$. For each $\alpha < \mathfrak{d}$ fix a collection $\mathcal{U}_\alpha$ of c_α-homogeneous subsets of Y_α which covers Y_α and such that $|\mathcal{U}_\alpha| = \mathfrak{hm}(c_\alpha)$. Thus $\mathcal{U} = \bigcup_{\alpha < \mathfrak{d}} \mathcal{U}_\alpha$ is a collection of c-homogeneous sets which covers X, so $\mathfrak{hm}(c) \leq |\mathcal{U}|$.

In the case that for all $\alpha < \mathfrak{d}$ it holds that $\mathfrak{hm}(c_\alpha) \leq \mathfrak{hm}(c_{\min})$ we have that $\mathfrak{hm}(c) \leq |\mathcal{U}| \leq \mathfrak{d} \cdot \mathfrak{hm}(c_{\min})$, so by (3), $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\min})$. Since c is reduced, $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min})$ and $Y \subseteq X$ can be chosen as a copy of the Cantor space by Lemma 2.9

In the remaining case $\mathfrak{hm}(c) > \mathfrak{hm}(c_{\min})$, therefore there necessarily exists $\alpha < \mathfrak{d}$ for which $\mathfrak{hm}(c_\alpha) > \mathfrak{hm}(c_{\min})$, and consequently, by (2), $\mathfrak{hm}(c_\alpha) = \mathfrak{hm}(c)$.   □

2.2.2. *Reduction to colorings on 2^ω.* For a compact space X let $\mathrm{Comp}(X)$ be the set of connected components of X. For $x \in X$ let $\mathrm{comp}(x, X)$ denote the component of x in X and $\mathrm{comp}(x) = \mathrm{comp}(x, X)$ when X is clear from the context. $\mathrm{Comp}(X)$ becomes a compact space when equipped with the quotient topology. If X is a compact metric space, then so is $\mathrm{Comp}(X)$. Recall that compact metric spaces are Polish.

The components of $\mathrm{Comp}(X)$ are singletons. Since $\mathrm{Comp}(X)$ is compact, it is zero-dimensional. (See [15] for this. We assume compact spaces to be Hausdorff.)

Lemma 2.12. *Let X be compact and $c : [X]^2 \to 2$ continuous. Define a coloring $\bar{c} : [\mathrm{Comp}(X)]^2 \to 2$ by*

$$\bar{c}\big(\mathrm{comp}(x), \mathrm{comp}(y)\big) = c(x, y)$$

for all $x, y \in X$ with $\mathrm{comp}(x) \neq \mathrm{comp}(y)$. Then $\bar{c}$ is a well-defined continuous pair-coloring on $\mathrm{Comp}(X)$.

Proof. Suppose $x_0, x_1, y_0, y_1 \in X$ are such that $x_1 \in \mathrm{comp}(x_0)$, $y_1 \in \mathrm{comp}(y_0)$, and x_0 and y_0 are in different components. Then $c(x_0, y_0) = c(x_1, y_0)$ since x_0 and x_1 are in the same component of $X \setminus \{y_0\}$ and $c(\cdot, y_0) : X \setminus \{y_0\} \to 2$ is continuous. By the same argument, $c(x_1, y_0) = c(x_1, y_1)$. Thus $c(x_0, y_0) = c(x_1, y_1)$, showing that $\bar{c}$ is well-defined.

For every $x \in \mathrm{comp}(x_0), y \in \mathrm{comp}(y_0)$ fix, by continuity of c, disjoint open $U_{x,y} \ni x$, $V_{x,y} \ni y$ so that c is constant on $U_{x,y} \times V_{x,y}$. Now $\{U_{x,y} \times V_{x,y} : x \in \mathrm{comp}(x_0), y \in \mathrm{comp}(y_0)\}$ is an open cover of $\mathrm{comp}(x_0) \times \mathrm{comp}(y_0)$. Since the latter is compact, there is a finite subcover $\{U_{x_i, y_i} \times V_{x_i, y_i} : i < n\}$ of this cover, which can be shrunk so that $\bigcup_{i<n} U_{x_i, y_i} \cap \bigcup_{i<n} V_{x_i, y_i} = \emptyset$. Thus we found two disjoint open neighborhoods of $\mathrm{comp}(x)$, $\mathrm{comp}(y)$ respectively so that c is constant on their product. This proves the continuity of $\bar{c}$. $\qquad\square$

Lemma 2.13. *Let X be a compact metric space and suppose $c : [X]^2 \to 2$ is continuous. Then there exists a continuous $\bar{c} : [2^\omega]^2 \to 2$ such that $\mathfrak{hm}(c) \leq \mathfrak{hm}(\bar{c})$.*

Proof. Let $Y := \mathrm{Comp}(X)$ and let $f : X \to Y$ be the mapping that maps every $x \in X$ to $\mathrm{comp}(x, X)$. Let $\bar{c}$ be as in Lemma 2.12. Observe that Y is of countable weight.

Assume that Y is uncountable. Cantor-Bendixson analysis of Y gives us a decomposition of Y into countably many points and a perfect set. Since for every isolated point $y \in Y$ the set $f^{-1}(y)$ is c-homogeneous in X by Theorem 2.2, we may replace Y by a perfect subset of Y at the cost of removing countably many c-homogeneous subset of X.

Y is now zero-dimensional, compact, without isolated points and of countable weight. Therefore Y is the Cantor space.

Claim 2.14. $\mathfrak{hm}(c) \leq \mathfrak{hm}(\bar{c})$

By the continuity of $\bar{c}$, every maximal $\bar{c}$-homogeneous set in Y is closed. Now using Cantor-Bendixson analysis again, every uncountable maximal $\bar{c}$-homogeneous set can be decomposed into countably many singletons and a perfect set.

The preimages under f of singletons are c-homogeneous by Theorem 2.2. Also,

Claim 2.15. *For any perfect $\bar{c}$-homogeneous set $H \subseteq Y$, $f^{-1}[H]$ is c-homogeneous.*

Proof. For the claim let $H \subseteq Y$ be perfect and $\bar{c}$-homogeneous of color $i \in 2$. If $x, y \in f^{-1}[H]$ are in different components of X, then clearly $c(x, y) = i$. Now let z be one of the components of X. Assume $|z| > 1$. By Theorem 2.2, c is constant on z. Let $j \in 2$ be the constant value of c on z. We have to show $i = j$.

Let $(z_n)_{n \in \omega}$ be a sequence in $H \setminus \{z\}$ that converges to z. Pick $(x_n)_{n \in \omega}$ in X such that for all $n \in \omega$, $f(x_n) = z_n$. By compactness, $(x_n)_{n \in \omega}$ has a convergent subsequence. We may assume that $(x_n)_{n \in \omega}$ itself converges.

Let x be the limit of $(x_n)_{n \in \omega}$. Clearly, $x \in z$. Let $y \in z$ be different from x. Then $c(x, y) = j$. By continuity, $c(x, y) = \lim_{n \to \infty} c(x_n, y) = i$. Thus $i = j$, which finishes the proof of the claim. $\qquad\square$

Thus, the preimage under f of every $\bar{c}$-homogeneous subset of Y is a countable union of c-homogeneous subsets of X. This establishes $\mathfrak{hm}(c) \leq \mathfrak{hm}(\bar{c})$ and proves Lemma 2.13. $\qquad\square$

2.2.3. *Reduction to simple colorings on 2^ω.* We are now fishing in a much smaller tank: we can consider only colorings on the Cantor space. The next reduction will show that we can consider only "coarse" pair-colorings on the Cantor space.

Notation 2.16. *For a tree T and $t \in T$ let $\mathrm{succ}_T(t)$ be the set of immediate successors of t in T. Recall that if A is a subset of ω^ω, then $T(A)$ denotes the set*

of finite initial segments of the element of A, a subtree of $\omega^{<\omega}$. If T is a subtree of $\omega^{<\omega}$, then $[T]$ denotes the set of all elements of ω^ω which have all their finite initial segments in T. $[T]$ is a closed subset of ω^ω. In this way closed subsets of ω^ω correspond to subtrees of $\omega^{<\omega}$ without finite maximal branches.

A natural way to construct continuous pair-colorings on a subset A of ω^ω is the following: To each $t \in T(A)$ assign a coloring $c_t : [\mathrm{succ}_{T(A)}(t)]^2 \to 2$. Now for all $\{x, y\} \in [A]^2$ let t be the longest common initial segment of x and y and put $c(x, y) := c_t(x \restriction n + 1, y \restriction n + 1)$ where $n = \mathrm{dom}(t)$. Clearly, c is continuous. We call a coloring which is defined in this way an *almost node-coloring*.

A *node-coloring* on A is obtained by assigning a color to every node $t \in T(A)$ and then defining the color of $\{x, y\} \in [A]^2$ to be the color of the longest common initial segment of x and y. Equivalently, a node-coloring is an almost node-coloring in which $c_t : [\mathrm{succ}_{T(A)}(t)]^2 \to 2$ is constant for all $t \in T$.

Both $c_{\min}$ and c_{parity} are node-colorings.

Not every continuous pair-coloring on ω^ω is an almost node-coloring. However, the following holds:

Lemma 2.17. *Let $c : [2^\omega]^2 \to 2$ be continuous. Then there is a topological embedding $e : 2^\omega \to \omega^\omega$ such that for every c_{parity}-homogeneous set $H \subseteq e[2^\omega]$, the coloring $c^e \restriction H$ which is induced on H by c via e is an almost node-coloring.*

Proof. Let $n \in \omega$ and let $s, t \in 2^{n+1}$ be such that $\Delta(s, t) = n$. Let O_s and O_t denote the basic open subsets of 2^ω determined by s and t, respectively.

Since $O_s \times O_t$ is compact and c is continuous, there is $m > n$ such that for all $(x, y) \in O_s \times O_t$, $c(x, y)$ only depends on $x \restriction m$ and $y \restriction m$.

It follows that there is a function $f : \omega \to \omega$ such that for all $\{x, y\} \in [2^\omega]^2$, $c(x, y)$ only depends on $x \restriction f(\Delta(x, y))$ and $y \restriction f(\Delta(x, y))$. We can choose f strictly increasing and such that $f(0) \geq 1$. For $n \in \omega$ let $g(n) := f^n(0)$.

Identifying $2^{<\omega}$ and ω, we define the required embedding $e : 2^\omega \to \omega^\omega$ by letting $e(x) := (x \restriction g(0), x \restriction g(1), \dots)$. Let $E := e[2^\omega]$. c induces a continuous pair-coloring c^e on E via e. By the choice of f, for $\{u, v\} \in [E]^2$, $c^e(u, v)$ only depends on $u \restriction (\Delta(u, v) + 2)$ and $v \restriction (\Delta(u, v) + 2)$. This is because if $n = \Delta(u, v)$ and $x, y \in 2^\omega$ are such that $e(x) = u$ and $e(y) = v$, then $\Delta(x, y) < g(n)$ and thus $c(x, y)$ only depends on $x \restriction f(\Delta(x, y))$ and $y \restriction f(\Delta(x, y))$. But since f is strictly increasing, $f(\Delta(x, y)) < f(g(n)) = g(n + 1)$.

Now let H be a c_{parity}-homogeneous subset of E. The c_{parity}-homogeneity of H implies that for all $\{u, v\} \in [H]^2$, the restrictions of u and v to $\Delta(u, v) + 1$ uniquely determine the restrictions to $\Delta(u, v) + 2$. Therefore, for all $\{u, v\} \in [H]^2$, $c^e(u, v)$ only depends on $u \restriction (\Delta(u, v) + 1)$ and $v \restriction (\Delta(u, v) + 1)$.

It follows that $c^e \restriction H$ is an almost node-coloring. $\square$

Corollary 2.18. *For every continuous pair-coloring $c : [2^\omega]^2 \to 2$, there is an almost node-coloring d on some compact subset of ω^ω such that $\mathfrak{hm}(c) \leq \mathfrak{hm}(d)$.*

Proof. By the previous Lemma, 2^ω can be presented as a union of $\leq \mathfrak{hm}(c_{\min})$ sets on each of which c is reducible to an almost node-coloring. The rest of the proof is as in the proof of Lemma 2.11. $\square$

2.2.4. *The coloring $c_{\max}$.* We shall now define a maximal almost node-coloring.

Recall that the *random* graph on ω is, up to isomorphism, the only homogeneous and universal graph in the class of all graphs on ω. (See [16] for some information on the random graph.) Universality means: every graph (ω, E) is embeddable as an induced subgraph into the random graph (in particular, every finite graph is embeddable as an induced subgraph into a finite initial segment of the random graph).

Definition 2.19. *Let $\chi_{\mathrm{random}} : [\omega]^2 \to 2$ be the (characteristic function of the) edge relation of the random graph. For $s, t \in \omega^{\leq\omega}$ write $\mathrm{random}(s,t) = i$ iff $n := \Delta(s,t)$ exists and $i = \chi_{\mathrm{random}}(s(n+1), t(n+1))$. Let $c_{\mathrm{random}} : [\omega^\omega]^2 \to 2$ be defined by $c_{\mathrm{random}}(x,y) := \mathrm{random}(x,y)$. Finally, let*

$$c_{\mathrm{max}} := c_{\mathrm{random}} \upharpoonright \prod_{n \in \omega} (n+1) \tag{5}$$

Clearly, c_{random} and c_{max} are almost node-colorings. Since $\prod_{n \in \omega}(n+1)$ is homeomorphic to 2^ω, we regard c_{max} as a coloring on 2^ω.

It is interesting to point out:

Fact 2.20. *Whenever c is an almost node-coloring on a compact subspace of ω^ω, then: $c_{\mathrm{random}} \not\leq c$.*

Proof. Let $(x_n)_{n \in \omega}$ be an infinite path in c_{random}, i.e.,

$$\forall n < m : \qquad c_{\mathrm{random}}(x_n, x_m) = 1 \iff m = n + 1.$$

Since every countable graph embeds into $(\omega^\omega, c_{\mathrm{random}})$, such a sequence can be easily found.

On the other hand, if $Y \subseteq \omega^\omega$ is compact and $c : [Y]^2 \to 2$ is an almost node-coloring, there is no infinite path in (Y, c). Suppose to the contrary that $(y_n)_{n < \omega}$ is a path in (Y, c). Observe that $\Delta(y_{n+1}, y_{n+2}) > \Delta(y_n, y_{n+1})$ implies that $c(y_n, y_{n+2}) = 1$; and that $\Delta(y_{n+1}, y_{n+2}) < \Delta(y_n, y_{n+1})$ implies $c(y_{n+1}, y_{n+2}) = 0$. Thus, $\Delta(y_n, y_{n+1})$ is constant for all n — contrary to the compactness of Y.

The fact now follows. $\square$

Lemma 2.21. *a) If c is an almost node-coloring on a subset of ω^ω, then $c \leq c_{\mathrm{random}}$ via a level preserving embedding (isometry) of ω^ω into ω^ω.*

b) If c is an almost node-coloring on a compact subset of ω^ω, then $c \leq c_{\mathrm{max}}$.

Proof. Let us prove b) first. Suppose c is an almost node-coloring on a compact subset A of ω^ω. Then $T(A)$ is a finitely branching subtree of $\omega^{<\omega}$. For each $t \in T(A)$ fix a coloring $c_t : [\mathrm{succ}_{T(A)}(t)]^2 \to 2$ such that the c_t witnesses the fact that c is an almost node-coloring. For $s, t \in T$ let $\bar{c}(s,t) := c(x,y)$ if s and t are incomparable and $x, y \in [T]$ are such that $s \subseteq x$ and $t \subseteq y$. If s and t are comparable, then $\bar{c}(s,t)$ is undefined.

Let $T_k = \{t \in T(A) : |t| = k\}$. We construct a monotone (i.e., $\subseteq$-preserving) map $e : \bigcup_{k \in \omega} T_k \to T(\prod_{n \in \omega}(n+1))$ which induces the required embedding of A into $\prod_{n \in \omega}(n+1)$.

Argue by induction on k. Suppose that $e(s) \in \prod_{n \leq n(k)}(n+1)$ is defined for all $s \in T_k$, and for all $s, t \in T_k$ we already have $\mathrm{random}(e(s), e(t)) = \bar{c}(s,t)$.
Find $n(k+1) > n(k)$ such that for all $s \in T_k$ there is $t \in \prod_{n < n(k+1)}(n+1)$ with $e(s) \subseteq t$ and $c_s \leq \mathrm{random} \upharpoonright \mathrm{succ}_{T(\prod_{n \in \omega}(n+1))}(t)$. Now it is obvious how to define e on T_{k+1} with images in $\prod_{n \leq n(k+1)}(n+1)$.

a) is proved similarly, using the fact that every countable graph occurs as an induced subgraph of $(\mathrm{succ}_{\omega^{<\omega}}(s), \mathrm{random})$ for every $s \in \omega^{<\omega}$. $\square$

Corollary 2.22. *For every Polish X and every continuous $c : [X]^2 \to 2$:*

$$\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\mathrm{max}}).$$

Proof. Let c be an arbitrary reduced continuous pair-coloring on a Polish X. By Lemma 2.11 there exists a compact $Y \subseteq X$ so that $\mathfrak{hm}(c) = \mathfrak{hm}(c \upharpoonright Y)$. By Lemma 2.13 there is a coloring $\bar{c}$ on 2^ω so that $\mathfrak{hm}(c) \leq \mathfrak{hm}(\bar{c})$ and by Corollary 2.18 there is an almost node-coloring d on 2^ω so that $\mathfrak{hm}(\bar{c}) \leq \mathfrak{hm}(d)$. Finally, $d \leq c_{\mathrm{max}}$ by Lemma 2.21 above. $\square$

Finally,

Theorem 2.23. *For every reduced continuous pair-coloring c:*

$$\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min}) \ or \ \mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$$

Proof. By now we have that $\mathfrak{hm}(c_{\min}) \leq \mathfrak{hm}(c) \leq \mathfrak{hm}(_{\max})$ for all reduced c. But $\mathfrak{hm}(c_{\max}) \leq (\mathfrak{hm}(c_{\min}))^+$ by (2); so $\mathfrak{hm}(c) > \mathfrak{hm}(c_{\min})$ implies $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$. $\square$

Note however that it is conceivable that there is a simply defined reduced pair coloring c on a Polish space such that neither $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$ nor $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min})$ is provable in ZFC (see Problem 6.1).

We remark that in Theorem 2.23 above, $c_{\min}$ can be replaced by c_{parity} and $c_{\max}$ can be replaced by c_{random}, since

$$c_{\text{parity}} \leq c_{\min} \leq c_{\max} \leq c_{\text{random}}.$$

2.2.5. *Why $c_{\max}$ is more complicated than $c_{\min}$: Random versus perfect graphs.* In the second part of the paper we shall prove the consistency of $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$. The consistency proof relies on the different finite patterns that appear in each of those two colorings.

Clearly, every finite graph occurs as an induced subgraph of $(2^\omega, c_{\max})$.

A finite graph is called *perfect* if in each of its induced subgraphs the chromatic number is equal to the clique number. A perfect graph with n vertices contains either a clique or an independent set of size $\lfloor \sqrt{n} \rfloor$. This stands in strong contrast to a randomly chosen graph: in a random graph on n vertices there is almost certainly no clique and no independent set of size $2 \log n$ (see [5]).

Fact 2.24 (N. Alon)**.** *Every finite (induced) subgraph H of $(\omega^\omega, c_{\text{parity}})$ satisfies that the chromatic number of H is equal to the maximal size of a clique in H.*

Proof. Two proofs of this fact are in [4]. The proof we include here was suggested to us by Stevo Todorčević. Define a partial order on ω^ω by $\eta_1 \leq \eta_2$ iff $\eta_1 = \eta_2$ or $\Delta(\eta_1, \eta_2)$ is odd and η_1 precedes η_2 in the lexicographic ordering on ω^ω. A finite induced subgraph of ω^ω is a clique iff its elements form a chain in the poset just defined and is an independent set iff its elements form an anti-chain in the same poset. Now recall that a finite partially ordered set with no chain of length $k + 1$ is a union of k antichains. $\square$

Thus only perfect graphs occur as finite induced subgraphs of $c_{\min}$.

In particular:

$$c_{\max} \not\leq c_{\min}. \tag{6}$$

3. Covering a square by functions

The problem of covering a Euclidean space by smaller geometric objects is well investigated. Klee [25] proved that no separable Banach space can be covered by fewer than $2^{\aleph_0}$ hyperplanes. Steprāns [31] proved the consistency of covering $\mathbb{R}^{n+1}$ by fewer than continuum *smooth manifolds of dimension n*.

We recall that a point $(x, y) \in X^2$ is *covered* by a function $f : X \to X$ if $f(x) = y$ or $f(y) = x$. By f^{-1} we mean the set $\{(y, x) : f(x) = y\}$, the *inverse of f*. Thus (x, y) is covered by f iff $(x, y) \in f \cup f^{-1}$. For a metric space X denote by $\text{Cov}(\mathcal{C}ont(X))$ the minimal number of continuous functions from X to X needed to cover X^2 and by $\text{Cov}(\mathcal{L}ip(X))$ denote the analogous number for Lipschitz functions.

Hart and van der Steeg showed the consistency of covering $(2^\omega)^2$ by fewer than continuum continuous functions [23], a result that actually follows from Steprāns'

result mentioned above using some easy arguments from the present article. Ciesielski and Pawlikowski observed that Steprans' result also implies that $\mathbb{R}^2$ is consistently covered by fewer than continuum continuously differentiable functions [13]. This is optimal in the following sense:

It was pointed out by Márton Elekes that $\mathbb{R}^2$ is not the union of less than continuum twice differentiable functions and inverses of differentiable functions. This is because there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and an infinite perfect set $P \subseteq \mathbb{R}$ such that the derivative of f is constantly 0 on P and no function which is twice differentiable intersects $f \restriction P$ in infinitely many points [3]. Since the derivative of f is 0 on P, no inverse of a differentiable function intersects $f \restriction P$ in more than finitely many points. It follows that already the graph of f is not included in the union of less than continuum twice differentiable functions and inverses of differentiable functions.

In [20] it was shown that $(2^\omega)^2$ can consistently be covered by fewer than continuum Lipschitz functions, which also follows from the result by Ciesielski and Pawlikowski on covering $\mathbb{R}^2$ by continuously differentiable functions. Hart asked whether $\mathrm{Cov}(\mathcal{L}ip(2^\omega))$ can be different from $\mathrm{Cov}(\mathcal{C}ont(2^\omega))$. Recently, Abraham and Geschke [1] proved that it is consistent to cover $\mathbb{R}^{n+1}$ by κ n-ary continuous functions with $2^{\aleph_0} = \kappa^{+n}$.

Let us state the following folklore result that was brought to the authors' attention by Ireneusz Recław (and which should be well-known):

Theorem 3.1. *Let κ be an infinite cardinal. Then the least number of functions from κ^+ to κ^+ needed to cover $\kappa^+ \times \kappa^+$ is κ.*

Proof. For every $\alpha < \kappa^+$ fix a surjection $f_\alpha : \kappa \to (\alpha + 1)$. Now define, for $\beta < \kappa$, $g_\beta(\alpha) := f_\alpha(\beta)$. The functions $\{g_\beta : \beta < \kappa\}$ cover $\kappa^+ \times \kappa^+$.

To show that $\kappa^+ \times \kappa^+$ is not covered by less than κ functions, let $\mathcal{F}$ be a family of functions from κ^+ to κ^+ of size $< \kappa$. We shall find a pair $(x, y) \in \kappa^+ \times \kappa^+$ that is not covered by $\mathcal{F}$.

Let $X \subseteq \kappa^+$ be a set of size κ which is closed under all functions in $\mathcal{F}$. Choose $y \in \kappa^+ \setminus X$. Now for all $f \in \mathcal{F}$ and all $x \in X$, $f(x) \neq y$. Since $|\mathcal{F}| < |X|$, there is $x \in X$ such that for all $f \in \mathcal{F}$, $x \neq f(y)$. Now (x, y) is not covered by any function in $\mathcal{F}$. $\square$

This theorem implies that if the continuum is a successor cardinal, then fewer than continuum functions suffice to cover the square of the continuum. In [1] a generalization of Theorem 3.1 to higher dimension is proved.

In the rest of this section the connection between $c_{\min}$-homogeneous sets and covering $(2^\omega)^2$ by Lipschitz functions will be explored, and used to prove the inequalities (2) and (3) which were used in the previous Section. Inequality (2) was already proved in [20]. Inequality (3) follows from Theorem 3.9 below.

After proving the crucial Theorem 3.9 we investigate covering by *continuous* functions.

3.1. ♭m and covering a square by Lipschitz functions.

Definition 3.2. *For $a, b > 0$ let $\mathcal{L}ip_{a,b}$ denote the σ-ideal on $(2^\omega)^2$ generated by the Lipschitz functions of constant a and the inverses of Lipschitz functions of constant b. The covering number of this ideal, $\mathrm{Cov}(\mathcal{L}ip_{a,b})$, is the least number of sets in the ideal needed to cover $(2^\omega)^2$.*

Clearly, as the graph of every continuous function is a nowhere-dense subset of $(2^\omega)^2$, $\mathrm{Cov}(\mathcal{L}ip_{a,b}) > \aleph_0$ for every choice of positive a, b. By Theorem 3.1 we know that $(\mathrm{Cov}(\mathcal{L}ip_{a,b}))^+ \geq 2^{\aleph_0}$.

Lemma 3.3. $\mathfrak{hm} = \mathrm{Cov}(\mathcal{L}ip_{1,\frac{1}{2}})$

Proof. For $x, y \in \omega^\omega$ let $x \otimes y := (x(0), y(0), x(1), y(1), \dots)$. It is easily seen that $\otimes : (\omega^\omega)^2 \to \omega^\omega$ and $\otimes : (2^\omega)^2 \to 2^\omega$ are uniformly continuous homeomorphisms.

Suppose $H_0 \subseteq 2^\omega$ is a maximal $c_{\min}$-homogeneous of color 0. Then $T := T(H_0)$ is a tree with the property that $t \in T$ has two immediate successors in T if and only if $|t|$ is even and has one immediate successor in T otherwise. Let $x \in 2^\omega$ and define $y(n)$ inductively as follows:
Suppose $y(i)$ is defined for all $m < n$, and we have

$$t = (x(0), y(0), x(1), y(1), \dots, x(n-1), y(n-1)) \in T.$$

Let $y(n) \in \{0, 1\}$ be the unique i such that $t^\frown x(n)^\frown i \in T$. Let $f_{H_0}(x)$ denote y, which we have just defined from x and H_0. We then have $(x \otimes f_{H_0}(x)) \in H_0$.

Since the first n digits of y are determined by the first n digits of x, $f_{H_0} : 2^\omega \to 2^\omega$ is a Lipschitz function with constant 1 (with respect to dist).

Similarly, if H_1 is maximal $c_{\min}$-homogeneous of color 1, then for every $x \in 2^\omega$ there is a unique $f_{H_1}(x) \in 2^\omega$ for which $f_{H_1}(x) \otimes x \in H_1$. This time, the function f_{H_1} is of Lipschitz of constant $\frac{1}{2}$.

Conversely, from every 1-Lipschitz function $f : 2^\omega \to 2^\omega$ a maximal $c_{\min}$-homogeneous set H_f of color 0 is defined so that for all x, $y = f(x)$ is the unique such that $x \otimes y \in H_f$ and from every 1/2-Lipschitz function $f : 2^\omega \to 2^\omega$ a maximal $c_{\min}$-homogeneous set $_fH$ of color 1 is defined such that $y = f(x)$ is the unique such that $y \otimes x \in {_fH}$.

Suppose $\mathcal{H}_0$ is a family of maximal $c_{\min}$-homogeneous subsets of 2^ω of color 0 and $\mathcal{H}_1$ is a family of maximal $c_{\min}$-homogeneous subsets of color 1. For $(x, y) \in (2^\omega)^2$, if $x \otimes y \in H$ for some $H \in \mathcal{H}_0$ then $y = f_H(x)$, and if $x \otimes y \in H$ for $H \in \mathcal{H}_1$ then $x = f_H(y)$. Thus $\bigcup \mathcal{H}_0 \cup \bigcup \mathcal{H}_1 = 2^\omega$ implies that for all $(x, y) \in (2^\omega)^2$ there is some $H \in \mathcal{H}$ for which $f_H(x) = y$ or $f_H(y) = x$.

Conversely, suppose that $\mathcal{F}_0$ is a family of 1-Lipschitz functions from 2^ω to itself and that $\mathcal{F}_1$ is a family of $\frac{1}{2}$-Lipschitz functions from 2^ω to itself. Let $z \in 2^\omega$ and write $z = x \otimes y$. If there is $f \in \mathcal{F}_0$ such that $f(x) = y$ then $z \in H_f$ and if there is $f \in \mathcal{F}_1$ such that $f(y) = x$ then $z \in {_fH}$. $\square$

3.1.1. *Varying the Lipschitz constants.* Let $\mathcal{L}ip$ be the σ-ideal on $(2^\omega)^2$ generated by $\bigcup_{a>0} \mathcal{L}ip_{a,a}$, i.e., the σ-ideal generated by all Lipschitz functions and their inverses.

Theorem 3.4. $\mathfrak{hm} = \mathrm{Cov}(\mathcal{L}ip)$

Proof. Clearly, $\mathrm{Cov}(\mathcal{L}ip) \leq \mathrm{Cov}(\mathcal{L}ip_{1,\frac{1}{2}})$. Thus, it follows from Lemma 3.3 that $\mathrm{Cov}(\mathcal{L}ip) \leq \mathfrak{hm}$.

Now we prove the converse inequality $\mathfrak{hm} \leq \mathrm{Cov}(\mathcal{L}ip)$. We define a coloring $c : [2^\omega \times 2^\omega]^2 \to \mathcal{P}(2)$ as follows.

Let $0 \in c((x_0, y_0), (x_1, y_1))$ iff there is a Lipschitz function of constant 1 containing both (x_0, y_0) and (x_1, y_1), i.e., if the slope determined by (x_0, y_0) and (x_1, y_1) is ≤ 1 or equivalently, if x_0 and x_1 do not split after y_0 and y_1.

Let $1 \in c((x_0, y_0), (x_1, y_1))$ iff there is a Lipschitz function of constant 1 containing both (y_0, x_0) and (y_1, x_1), i.e., if y_0 and y_1 do not split after x_0 and x_1.

It is clear that c is continuous and the color $\emptyset$ does not occur. We construct a (nonempty) perfect set $X \subseteq (2^\omega)^2$ with the following properties:

   (i) $c \restriction X$ only takes the values $\{0\}$ and $\{1\}$.
   (ii) $c \restriction X$ is reduced.
   (iii) For every Lipschitz function $f : 2^\omega \to 2^\omega$, $f \cap X$ and $f^{-1} \cap X$ are the unions of finitely many c-homogeneous sets.

If we can construct X, we are done. This is because by (iii), every family $\mathcal{F}$ of Lipschitz functions that covers $(2^\omega)^2$ induces a family $\mathcal{H}$ of size at most $|\mathcal{F}|$ that covers X and consists of c-homogeneous sets. By (i) and (ii), we have $\mathfrak{hm} \leq |\mathcal{H}|$ and thus $\mathfrak{hm} \leq |\mathcal{F}|$.

The required X will be chosen to be (the graph of) a homeomorphism between two perfect subsets of 2^ω. For its construction, partition ω into countably many intervals I_i, $i \in \omega$, such that the length of every I_i is at least i and the elements of I_i are below the elements of I_j for $i < j$. For every $i \in \omega$ let n_i denote the first element of I_i.

Let T_0 be a perfect subtree of $2^{<\omega}$ that fully splits at all the levels of height n_i for $i \equiv 0, 3 \mod 4$ and does not split at any other level. Let T_1 be a perfect subtree of $2^{<\omega}$ that fully splits at every level of height n_i for $i \equiv 1, 2 \mod 4$ and does not split anywhere else.

Let X be the (graph of the) natural (order preserving) homeomorphism between $[T_0]$ and $[T_1]$. Clearly X is closed and satisfies (i) and (ii). It remains to show (iii).

Let $f : 2^\omega \to 2^\omega$ be a Lipschitz function. Choose $i \in \omega$ so that the Lipschitz constant of f is below 2^i and $i \equiv 0 \mod 4$. T_0 is the union of finitely many perfect subtrees $T_0^1, \ldots, T_0^m$ that have no splittings below level n_i. Since $i \equiv 0 \mod 4$, the corresponding subtrees $T_1^1, \ldots, T_1^m$ of T_1 do not split below level n_i either. For $k \in \{1, \ldots, m\}$ let $X_k := X \cap ([T_0^k] \times 2^\omega)$. Now for every $k \in \{1, \ldots, m\}$ and all $(x_0, y_0), (x_1, y_1) \in X_k$ the slope determined by the pair $\{(x_0, y_0), (x_1, y_1)\}$ is either $\leq 2^{-i}$ or $\geq 2^i$ since the length of I_i is at least i and by the definition of X. Now by the choice of i, for all $k \in \{1, \ldots, m\}$, $f \cap X_k$ is c-homogeneous of color $\{0\}$.

Similarly, the intersection of every inverse of a Lipschitz function with X is the union of finitely many c-homogeneous sets of color $\{1\}$. This shows (iii) and therefore finishes the proof of the theorem. $\qquad \square$

Lemma 3.5. *Let $a, b > 0$. Then $\mathrm{Cov}(\mathcal{L}ip_{a,b}) = \mathrm{Cov}(\mathcal{L}ip_{2 \cdot a, \frac{b}{2}})$.*

Proof. For $i \in 2$ let X_i be the set of all sequences in 2^ω starting with i. Let $h_i : 2^\omega \to X_i$ be the homeomorphism mapping x to $(i, x(0), x(1), \ldots)$.

Let $f : 2^\omega \to 2^\omega$ be a Lipschitz function of constant a. For $i \in 2$ let $f^i : 2^\omega \to 2^\omega$ be a function which is equal to $f \circ h_i^{-1}$ on X_i and constant on X_{1-i} such that f^i is Lipschitz of constant $2 \cdot a$. (For example, we can choose the constant value of f^i on X_{1-i} to be $f((\overline{1-i}))$ where $(\overline{1-i})$ denotes the constant sequence with value $1-i$.)

If $f : 2^\omega \to 2^\omega$ is a Lipschitz function of constant b, then for $i \in 2$ let $f_i := h_i \circ f$. f_i is a Lipschitz function of constant $\frac{b}{2}$.

Now let $\mathcal{F}$ be a family of Lipschitz functions of constant a and $\mathcal{G}$ a family of Lipschitz functions of constant b. If $(2^\omega)^2 = \bigcup\{f \cup g^{-1} : f \in \mathcal{F} \wedge g \in \mathcal{G}\}$, then $(2^\omega)^2 = \bigcup\{f^i \cup g_i^{-1} : i \in 2 \wedge f \in \mathcal{F} \wedge g \in \mathcal{G}\}$.

It follows that $\mathrm{Cov}(\mathcal{L}ip_{2 \cdot a, \frac{b}{2}}) \leq \mathrm{Cov}(\mathcal{L}ip_{a,b})$. Now the lemma follows from the fact that $\mathrm{Cov}(\mathcal{L}ip_{a,b})$ is symmetric in a and b. $\qquad \square$

Lemma 3.6. *Let $a, b > 0$. If there is $c \in \mathbb{Z}$ such that $2^{c-1} \leq a$ and $2^{-c} \leq b$, then $\mathrm{Cov}(\mathcal{L}ip_{a,b}) = \mathfrak{hm}$. Otherwise $\mathrm{Cov}(\mathcal{L}ip_{a,b}) = 2^{\aleph_0}$.*

Proof. Let $a, b > 0$ and assume that there is no $c \in \mathbb{Z}$ such that $2^{c-1} \leq a$ and $2^{-c} \leq b$. Let $c \in \mathbb{Z}$ be maximal with $2^{c-1} \leq a$. Then $2^{-c} > b$ and therefore $b \cdot 2^c < 1$. $2^{c-1} \leq a$ is equivalent to $a \cdot 2^{-c} \geq \frac{1}{2}$, and since c is maximal, we have $a \cdot 2^{-c} < 1$. By Lemma 3.5, $\mathrm{Cov}(\mathcal{L}ip_{a,b}) = \mathrm{Cov}(\mathcal{L}ip_{2^{-c} \cdot a, 2^c \cdot b})$. But even the diagonal in $(2^\omega)^2$ cannot be covered by less than $2^{\aleph_0}$ Lipschitz functions of constant < 1.

Now suppose there is $c \in \mathbb{Z}$ such that $2^{c-1} \leq a$ and $2^c \leq b$. By Lemma 3.5 we may assume $a \geq 1$ and $b \geq \frac{1}{2}$, hence $\mathrm{Cov}(\mathcal{L}ip_{a,b}) \leq \mathrm{Cov}(\mathcal{L}ip_{1,\frac{1}{2}}) = \mathfrak{hm}$. The inequality $\mathrm{Cov}(\mathcal{L}ip_{a,b}) \geq \mathfrak{hm}$ follows from Theorem 3.4. $\qquad \square$

3.1.2. *Covering $(\omega^\omega)^2$ and $\mathbb{R}^2$ by Lipschitz functions.* We generalize our notation $\mathcal{L}ip_{a,b}$ to metric spaces X. For a metric space X let $\mathcal{L}ip_{a,b}(X)$ be the σ-ideal on $X \times X$ generated by the Lipschitz functions of constant a and the reflections of Lipschitz functions of constant b. $\mathcal{L}ip(X)$ denotes the σ-ideal generated by the union of all the ideals $\mathcal{L}ip_{a,b}(X)$.

Recall that $\mathfrak{hm}(c_{\text{parity}}) = \mathfrak{hm}$. It is easily checked that the main arguments for the correspondence between Lipschitz functions on 2^ω and $c_{\min}$-homogeneous sets also go through for ω^ω and c_{parity}. This shows

Corollary 3.7. *For $X = 2^\omega$ and $X = \omega^\omega$ we have*

$$\text{Cov}(\mathcal{L}ip(X)) = \text{Cov}(\mathcal{L}ip_{1,\frac{1}{2}}(X)) = \mathfrak{hm}.$$

At the very moment we do not know the exact relation between the cardinal invariants mentioned above and $\text{Cov}(\mathcal{L}ip(\mathbb{R}))$. However, we can say something:

Remark 3.8. $\mathfrak{hm} \leq \text{Cov}(\mathcal{L}ip(\mathbb{R}))$

Proof. The argument is similar to the argument in the proof of Theorem 3.4.

It is not difficult to construct a topological embedding $e : 2^\omega \to \mathbb{R}^2$ such that for any two distinct points $x, y \in 2^\omega$ the slope determined by $e(x)$ and $e(y)$ is positive and $\geq \Delta(x,y)$ if $\Delta(x,y)$ is even and $\leq \frac{1}{\Delta(x,y)}$ if $\Delta(x,y)$ is odd.

If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $e^{-1}[f]$ is a finite union of $c_{\min}$-homogeneous sets of color 1 and $e^{-1}[f^{-1}]$ is a finite union of $c_{\min}$-homogeneous sets of color 0. A covering family of Lipschitz real functions induces a covering family of no greater size of $c_{\min}$-homogeneous subsets of 2^ω. This implies $\mathfrak{hm} \leq \text{Cov}(\mathcal{L}ip(\mathbb{R}))$.   □

3.2. **Covering squares by continuous functions.** After having established the equality $\mathfrak{hm} = \text{Cov}(\mathcal{L}ip_{1,\frac{1}{2}}) = \text{Cov}(\mathcal{L}ip)$ and the fact that the Lipschitz constants can be varied to some extent without changing $\text{Cov}(\mathcal{L}ip_{a,b})$, it is natural to ask what happens if we replace the Lipschitz functions by continuous functions.

For a topological space X let $\mathcal{C}ont(X)$ denote the σ-ideal on $X \times X$ generated by the continuous functions from X to X and their inverses. $\mathcal{C}ont$ is $\mathcal{C}ont(2^\omega)$. Obviously, $\mathcal{L}ip_{a,b} \subseteq \mathcal{C}ont$ for all $a, b > 0$. Theorem 3.1 implies that $\text{Cov}(\mathcal{C}ont)^+ \geq 2^{\aleph_0}$. The same is of course true for $\mathfrak{hm}$. The question is whether $\text{Cov}(\mathcal{C}ont)$ can be smaller than $\mathfrak{hm}$. This will be answered in Section 5.

Very often cardinal invariants of σ-ideals on Polish spaces do not depend on the particular space the ideal is defined on. This is not true for $\text{Cov}(\mathcal{C}ont(X))$. While $\text{Cov}(\mathcal{C}ont(2^\omega))$ is consistently smaller than $2^{\aleph_0}$, the fact that every continuous function from a connected space to a zero-dimensional space is constant implies easily that if X is the disjoint union of $\mathbb{R}$ and 2^ω, then $\text{Cov}(\mathcal{C}ont(X)) = 2^{\aleph_0}$.

3.2.1. *The crucial inequality.* We show that $\text{Cov}(\mathcal{C}ont(X))$ is the same for $X = 2^\omega$, $X = \omega^\omega$, and $X = \mathbb{R}$. The proof of this fact depends on the following perhaps surprising Theorem.

Theorem 3.9. $\text{Cov}(\mathcal{C}ont(2^\omega)) \geq \mathfrak{d}$

Let $e : \omega^\omega \to 2^\omega$ be the natural embedding, i.e., the one induced by the mapping that maps $n \in \omega$ to the sequence of zeros of length n followed by a single one. Clearly, $2^\omega \setminus e[\omega^\omega]$ is countable. Now Theorem 3.9 immediately follows from the next lemma, letting $K := 2^\omega$ and $X := e[\omega^\omega]$.

Lemma 3.10. *Let K be an infinite compact space and let $X \subseteq K$ be such that $|K \setminus X| < |X|$. If $\mathcal{F}$ is a family of continuous functions from K to K covering K^2, then X can be covered by $|\mathcal{F}|$ compact subsets of X.*

Proof. Note that $|X| = |K|$. We may assume $|\mathcal{F}| < |K|$. By $|K \setminus X| < |X|$, there is $x \in K \setminus \bigcup_{f \in \mathcal{F}} f[K \setminus X]$.

Let $f \in \mathcal{F}$. By the choice of x, $f^{-1}(x) \subseteq X$ for every $f \in \mathcal{F}$. By the continuity of f, $f^{-1}(x)$ is closed in K and thus compact.

Since $\mathcal{F}$ covers K^2, for all $y \in X$ there is $f \in \mathcal{F}$ such that $f(x) = y$ or $f(y) = x$. In other words,

$$\{f^{-1}(x) : f \in \mathcal{F}\} \cup \{\{f(x)\} : f \in \mathcal{F} \wedge f(x) \in X\}$$

is a family of compact subsets of X sets that covers X. Clearly, this family is of size $\leq |\mathcal{F}|$. $\qquad\square$

Alternative proof of Theorem 3.9. Let $\mathcal{F}$ be an infinite family of continuous functions from 2^ω to 2^ω. Let M be an elementary submodel of a sufficiently large initial segment of the universe with Skolem functions such that $\mathcal{F} \subseteq M$ and $|M| = |\mathcal{F}|$.

Suppose $|\mathcal{F}| < \mathfrak{d}$. Then there is a function $x \in 2^\omega \setminus M$. Let $M[x]$ denote the Skolem hull of $M \cup \{x\}$. Since $|M[x]| = |\mathcal{F}| < \mathfrak{d}$, there is $y : \omega \to \omega$ such that y is not eventually dominated by any function in $\omega^\omega \cap M[x]$.

Clearly, the embedding $e : \omega^\omega \to 2^\omega$ is an element of M. No $f \in \mathcal{F}$ maps x to $e(y)$, since such an f would be an element of $M[x]$ and therefore $y = e^{-1}(f(x))$ would be an element of $M[x]$.

We show that no $f \in \mathcal{F}$ maps $e(y)$ to x.

Let $f \in \mathcal{F}$. Since f is continuous, $f^{-1}(x)$ is a compact subset of 2^ω. Since $2^\omega \setminus e[\omega^\omega]$ is countable, $2^\omega \setminus e[\omega^\omega] \subseteq M$. Since $x \notin M$, $f^{-1}(x)$ is disjoint from $2^\omega \setminus e[\omega^\omega]$. In other words, $f^{-1}(x)$ is a compact subset of $e[\omega^\omega]$. It follows that the set $(f \circ e)^{-1}(x)$ is bounded, i.e., there is a function in ω^ω that eventually dominates every element of $(f \circ e)^{-1}(x)$. By elementarity, $(f \circ e)^{-1}(x)$ is bounded by a function in $\omega^\omega \cap M[x]$.

Since y is not eventually dominated by a function in $\omega^\omega \cap M[x]$, $y \notin (f \circ e)^{-1}(x)$. That is, $f(e(y)) \neq x$.

This shows that no element of $\mathcal{F}$ covers the pair $(x, e(y)) \in (2^\omega)^2$. $\qquad\square$

The main reason for including the alternative proof of Theorem 3.9 is the curious use of two new reals over M. This detail is somewhat hidden in our proof of Lemma 3.10, but still present there.

In many cases, the standard proof of an inequality of the form $\mathrm{Cov}(I) \leq \mathrm{Cov}(J)$, for reasonably nice ideals I and J on the reals, shows that if one adds by forcing over a ground model M a new real r that avoids all elements of I which are coded (by a Borel code) in M, then the generic extension $M[r]$ contains also a new real which avoids all elements of J which are coded in M.

This is not the case with the inequality $\mathfrak{d} \leq \mathrm{Cov}(\mathcal{C}ont(2^\omega))$. It can be shown that after adding a single Miller real m to M (which is unbounded over M, i.e., escapes every bounded subset of ω^ω which is coded in M), in the generic extension $M[m]$, 2^ω is still covered by the $c_{\min}$-homogeneous sets coded in M. In particular, $(2^\omega)^2$ is still covered by the continuous functions coded in M.

The alternative proof of Theorem 3.9 shows that this is not the case after adding *two* Miller reals, or even after adding first any new real r and then a Miller real m over $M[r]$. In other words, the "standard" forcing proof of this inequality involves adding two reals, rather than one.

From Theorem 3.9 we get

Theorem 3.11. $\mathrm{Cov}(\mathcal{C}ont(\omega^\omega)) = \mathrm{Cov}(\mathcal{C}ont(\mathbb{R})) = \mathrm{Cov}(\mathcal{C}ont(2^\omega))$

The proof of this theorem uses the following easy observation.

Lemma 3.12. *Let C be a closed subset of ω^ω and let $f : C \to \omega^\omega$ be continuous. Then f can be continuously extended to all of ω^ω.*

Proof. It is easily checked that every closed subspace C of ω^ω is a retract of ω^ω (by inductively dropping every element of ω^ω into C). That is, there is a continuous map $q : \omega^\omega \to C$ such that $q \restriction C = \mathrm{id}_C$. If f is as above, then $f \circ q$ is a continuous extension of f to all of ω^ω. □

Proof of Theorem 3.11. We first show $\mathrm{Cov}(\mathcal{C}ont(2^\omega)) \leq \mathrm{Cov}(\mathcal{C}ont(\omega^\omega))$. Let $f : \omega^\omega \to \omega^\omega$ be continuous. Then $f^{-1}[2^\omega]$ is closed and thus $A := f^{-1}[2^\omega] \cap 2^\omega$ is a closed subset of 2^ω. We can now extend $f \restriction A$ to a continuous function $\overline{f} : 2^\omega \to 2^\omega$ by the same argument as in the proof of Lemma 3.12.

This shows that a family $\mathcal{F} \subseteq \mathcal{C}ont(\omega^\omega))$ covering $(\omega^\omega)^2$ gives rise to a covering family of no greater size in $\mathcal{C}ont(2^\omega)$ and thus, $\mathrm{Cov}(\mathcal{C}ont(2^\omega)) \leq \mathrm{Cov}(\mathcal{C}ont(\omega^\omega))$.

The same argument goes through for $\mathbb{R}$ instead of ω^ω, using the Tietze-Urysohn theorem.

Now observe that ω^ω can be covered by $\mathfrak{d}$ copies of 2^ω since $\mathfrak{d}$ is the covering number of the ideal of bounded subsets of ω^ω. Let $\mathcal{C}$ be a collection of size $\mathfrak{d}$ of copies of 2^ω covering ω^ω.

To each pair $(C, D) \in \mathcal{C} \times \mathcal{C}$ assign a family $\mathcal{F}_{C,D}$ of size $\mathrm{Cov}(\mathcal{C}ont(2^\omega))$ of continuous functions on ω^ω such that $C \times D \subseteq \bigcup \{f \cup f^{-1} : f \in \mathcal{F}_{C,D}\}$. This is possible by Lemma 3.12. Let $\mathcal{F} := \bigcup\{\mathcal{F}_{C,D} : C, D \in \mathcal{C}\}$. Now $(\omega^\omega)^2 = \bigcup\{f \cup f^{-1} : f \in \mathcal{F}\}$ and the size of $\mathcal{F}$ is $\max(\mathrm{Cov}(\mathcal{C}ont(2^\omega)), \mathfrak{d}) = \mathrm{Cov}(\mathcal{C}ont(2^\omega))$. The last equality is Theorem 3.9.

Again, same argument works for $\mathrm{Cov}(\mathcal{C}ont(\mathbb{R}))$ as well since $\mathbb{R}$ is just ω^ω (the irrationals) together with countably many additional points (the rationals) and therefore also can be covered by $\mathfrak{d}$ copies of 2^ω. We again use the Tietze-Urysohn theorem to extend continuous mappings defined on closed subspaces of $\mathbb{R}$. □

PART II: INDEPENDENCE RESULTS

In the second part of the paper we show that any two rows in Diagram 1 can be separated, where, as mentioned before, we have to exclude $\mathrm{Cov}(\mathcal{L}ip(\mathbb{R}))$ from row (3). We shall prove that every assignment of $\aleph_1$-s and $\aleph_2$-s to the diagram which is consistent with the arrows is realized in a model of set theory.

We provide two new forcing notions. One for separating $\mathfrak{hm}(c_{\min})$ from $\mathfrak{hm}(c_{\max})$ and the other for separating $\mathrm{Cov}(\mathcal{C}ont(2^\omega))$ from $\mathrm{Cov}(\mathcal{L}ip(2^\omega))$. We force over models of CH with countable support iterations of Axiom A forcing notions (see [9]) of size $\aleph_1$ which add new reals. Thus, no cardinals are collapsed and in the resulting models the continuum is $\aleph_2$.

Theorem 3.1 implies that if the continuum is a limit cardinal, all three numbers above are equal to the continuum. In fact, it is very easy to make $\mathrm{Cov}(\mathcal{C}ont(2^\omega))$ equal to the continuum.

Let M be a model of set theory and assume that $\mathcal{F} \in M$ is a family of continuous functions on 2^ω. If $x, y \in 2^\omega$ are generic over M and independent in the sense that $x \notin M[y]$ and $y \notin M[x]$, then no $f \in \mathcal{F}$ can cover (x, y). It follows that after forcing with a large product of some sort in order to increase the continuum one ends up with a model of set theory where $\mathrm{Cov}(\mathcal{C}ont(2^\omega))$ is the continuum. In particular, after forcing with the measure algebra over $2^{\aleph_2}$ over a model of CH, one obtains a model (the Solovay model) in which $\mathfrak{d} = \aleph_1$ (since the ground model elements of ω^ω dominate the new elements) and $\mathrm{Cov}(\mathcal{C}ont(2^\omega)) = \aleph_2$.

In [20] it was shown that in the Sacks model all homogeneity numbers of reduced continuous pair-colorings on Polish spaces are equal to $\aleph_1 < 2^{\aleph_0}$. It follows that $\mathfrak{hm}(c_{\max})$, $\mathfrak{hm}(c_{\min})$, and $\mathrm{Cov}(\mathcal{C}ont(\mathbb{R}))$ are small in the Sacks model.

There is a natural forcing $\mathbb{P}_{c_{\min}}$ for separating $\mathfrak{hm}(c_{\min})$ from the numbers below it: forcing with Borel subsets of 2^ω which are positive with respect to the σ-ideal

$J_{\min}$ generated over 2^ω by all $c_{\min}$-homogeneous sets. We show that the countable support iteration of this forcing of length ω_2 produces a model of $\mathfrak{hm}(c_{\min}) = \aleph_2$ and $\mathrm{Cov}(\mathcal{C}ont(2^\omega)) = \aleph_1$. In this model it holds that covering $\mathbb{R}^2$ by Lipschitz functions is strictly more difficult than covering $\mathbb{R}^2$ by continuous functions.

By a new (and yet unpublished) theorem of Zapletal, the existence of large cardinals implies that $\mathbb{P}_{c_{\min}}$ is optimal for enlarging $\mathfrak{hm}(c_{\min})$ in the sense that it does not enlarge numbers which are consistently smaller than $\mathfrak{hm}(c_{\min})$. Assuming large cardinals, Shelah and Zapletal proved recently that for every reasonably defined σ-ideal I on the reals whose covering number is provably $\geq \mathfrak{hm}(c_{\min})$, the uniformity of I (i.e., the smallest size of a set not in I) is at most $\aleph_3$. (The uniformity of $J_{\min}$ is at most $\aleph_2$.)

The analogous natural forcing for increasing $\mathfrak{hm}(c_{\max})$ is, however, not only not optimal, but actually *increases* the smaller $\mathfrak{hm}(c_{\min})$. So another forcing has to be used for separating $\mathfrak{hm}(c_{\min})$ from $\mathfrak{hm}(c_{\max})$.

We design a new tree-forcing notion $\mathbb{P}_{c_{\max}}$ for increasing $\mathfrak{hm}(c_{\max})$ while leaving $\mathfrak{hm}(c_{\min})$ small. The tree-combinatorics required for this forcing stems from a new result of Noga Alon about a Ramsey connection between perfect graphs and random graphs [4] (which Alon proved for this purpose). The countable support iteration of length ω_2 of $P_{c_{\max}}$ produces a model of set theory in which $\mathfrak{hm}(c_{\min}) = \aleph_1$ and $\mathfrak{hm}(c_{\max}) = \aleph_2$.

4. CONSISTENCY OF $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$

4.1. **The $c_{\max}$-forcing.** We are looking for a notion of forcing which adds a real that avoids all the $c_{\max}$-homogeneous sets in the ground model but does not increase $\mathfrak{hm}$ when iterated.

Definition 4.1. *For a pair-coloring c of a finite set with two colors let* $\mathrm{norm}(c)$ *denote the greatest $n \in \omega$ for which $\chi_{\mathrm{random}} \upharpoonright n \leq c$.*

For a subtree $p \subseteq T(\prod_{n \in \omega}(n+1))$ and $t \in p$ let $c_{t,p} := \chi_{\mathrm{random}} \upharpoonright \{i \in \omega : t^\frown i \in \mathrm{succ}_p(t)\}$. (See 2.16 for notation.)

Let

$$\mathbb{P}_{c_{\max}} := \Big\{ p \subseteq T(\prod_{n \in \omega}(n+1)) : \quad p \text{ is a tree and}$$

$$\forall t \in p \, \forall n \in \omega \, \exists s \in p(s \supseteq t \wedge \mathrm{norm}(c_{s,p}) \geq n) \Big\}$$

The order on $\mathbb{P}_{c_{\max}}$ is set-inclusion.

In the following we write just $\mathbb{P}$ for $\mathbb{P}_{c_{\max}}$. For a condition $p \in \mathbb{P}$ and $t \in p$, let $p_t = \{s \in p : s \subseteq t \vee t \subseteq s\}$, and call p_t the condition p *below* t. It is clear that $p_t \in \mathbb{P}$ for $p \in \mathbb{P}$ and $t \in p$. If $G \subseteq \mathbb{P}$ is a generic filter over a ground model M, the *generic real added by* $\mathbb{P}$ is the unique element of $\bigcap\{[p] : p \in G\}$.

Claim 4.2. *The generic real added by $\mathbb{P}$ avoids all $c_{\max}$-homogeneous sets in the ground model.*

Proof. Suppose that $A \subseteq 2^\omega$ is $c_{\max}$-homogeneous, say with color 0, and $A \in M$. Let $p \in \mathbb{P}$ be arbitrary. Choose $s \in p$ with $t, t' \in \mathrm{succ}_p(s)$ satisfying $\mathrm{random}(t, t') = 1$. Since at least one of $[p_t]$, $[p_{t'}]$ has empty intersection with A, assume without loss of generality that $[p_t] \cap A = \emptyset$. Now $p_t \leq p$ is a condition in $\mathbb{P}$ which forces that the generic real is not in A. Thus, the set of conditions forcing that the generic real is not in A is dense and belongs to M, hence the generic real is not in A. $\qquad \square$

Lemma 4.3. *Let G be $\mathbb{P}$-generic over the ground model M. Then for each $x \in (2^\omega)^{M[G]}$ there is a tree $T \in M$ such that $[T]$ is* parity-*homogeneous and $x \in [T]$.*

For the proof of this lemma we use the following result of Noga Alon [4] that was proved especially for this purpose.

Lemma 4.4. *Let $n \in \omega$ and $c : [n]^2 \to 2$. Then there is $N \in \omega$ and $C : [N]^2 \to 2$ such that whenever $e : N \to 2^\omega$ is 1-1, then there is a $c_{\min}$-homogeneous set $H \subseteq 2^\omega$ such that $c \leq C \upharpoonright e^{-1}[H]$.*

Proof of Lemma 4.3. Let $\dot{x}$ be a name for a new element of 2^ω and let $p \in \mathbb{P}$. Since $\dot{x}$ is a name for a new real, we may assume, by passing to stronger condition if necessary, that for each splitting node $s \in p$ and all $t, t' \in \mathrm{succ}_p(s)$ with $t \neq t'$, the initial segments of $\dot{x}$ decided by p_t and $p_{t'}$ are incompatible.

We may assume that for some $k_s \in \omega$, each p_t, $t \in \mathrm{succ}_p(s)$, decides an initial segment of $\dot{x}$ of length k_s and that the decisions of the p_t's on $\dot{x}$ are pairwise incompatible when restricted to k_s. In other words, for each splitting node s of p we have an embedding $e_s : \mathrm{succ}_p(s) \to 2^{k_s}$ with $p_t \Vdash \dot{x} \supseteq e_s(t)$.

Now Lemma 4.4 implies that we can thin out p to a condition q such that for each splitting node s of q, $e_s[\mathrm{succ}_q(s)]$ is a $c_{\min}$-homogeneous subset of 2^{k_s} of some color $i_s \in 2$.

Thinning out q further if necessary, we may assume that

($*$) whenever s and t are splitting nodes of q and $s \subsetneqq t$, then $\mathrm{norm}(c_{s,q}) < \mathrm{norm}(c_{t,q})$.

Now either q has a cofinal set of splitting nodes s with $i_s = 0$, or there is a node $s \in q$ such that for all splitting nodes $t \in q$ with $s \subseteq t$, $i_t = 1$. In the first case, we can thin out q to a condition r such that for all splitting nodes s of r, $i_s = 0$. The property ($*$) makes sure that r will be a condition. In the second case we can put $r := q_s$ and get a condition such that for all splitting nodes s we have $i_s = 1$.

Finally let $T_r := \{ s \in 2^{<\omega} : \exists r' \leq r (r' \Vdash s \subseteq \dot{x}) \}$ be the *tree of r-possibilities for $\dot{x}$*. Clearly r forces $\dot{x}$ to be a branch of T_r. By the construction of r, $[T_r]$ is $c_{\min}$-homogeneous. □

4.2. Iteration. In this section we show that after forcing with a countable support iteration of the $c_{\max}$-forcing, all the new reals ($\in 2^\omega$) are covered by $c_{\min}$-homogeneous sets in the ground model. This implies that after forcing with a countable support iteration of $\mathbb{P}$ of length ω_2 over a model of CH, we obtain a model of set theory in which $\mathfrak{hm} = \aleph_1$ but $\mathfrak{hm}(c_{\max}) = \aleph_2$. The latter statement follows from Claim 4.2.

4.2.1. *A preliminary lemma.* Our strategy is the following: For an ordinal α let $\mathbb{P}_\alpha$ denote the countable support iteration of $\mathbb{P}$ of length α. Let $\dot{x}$ be a $\mathbb{P}_{\omega_2}$-name for a new element of 2^ω. We may assume that there is $\alpha < \omega_2$ such that $\dot{x}$ is a $\mathbb{P}_\alpha$-name for a real not added at any proper initial stage of the iteration $\mathbb{P}_\alpha$. Let q be a condition in $\mathbb{P}_\alpha$. Recall the definition of $T_q \subseteq 2^{<\omega}$ from the proof of Lemma 4.3:

$$T_q = \{ s \in 2^{<\omega} : \exists q' \leq q (q' \Vdash s \subseteq \dot{x}) \}.$$

For each $p \in \mathbb{P}_\alpha$ we will construct a condition $q \leq p$ such that $[T_q]$ is $c_{\min}$-homogeneous. The next lemma tells us how to choose the color of $[T_q]$ if $\dot{x}$ is added in a limit step. That is, we can decrease p such that it becomes an element of one of the sets E_i, $i \in 2$, defined below. If $p \in E_i$, we can build $q \leq p$ such that $[T_q]$ is $c_{\min}$-homogeneous of color i.

Let us fix some notation. If $\mathbb{Q}$ is any forcing notion and $\dot{y}$ is a $\mathbb{Q}$-name for a new element of 2^ω let $y[p]$ be the maximal element of $2^{<\omega}$ such that $p \Vdash y[p] \subseteq \dot{y}$. $y[p]$ exists since $\dot{y}$ is a name for a new real.

For $i \in 2$ let

$$E_i := \{p \in \mathbb{P}_\alpha : \forall \beta < \alpha \forall q \leq p \exists q' \leq q \exists q_0, q_1 \in \mathbb{P}_{\beta,\alpha}$$
$$(q' \restriction \beta \Vdash q_0, q_1 \leq q' \restriction [\beta, \alpha) \wedge \operatorname{parity}(x[q_0], x[q_1]) = i)\}.$$

Recall that $\operatorname{parity}(s, t) \in 2$ implies that s and t are incompatible, i.e., $s \perp t$.

Lemma 4.5. *E_0 and E_1 are open and $E_0 \cup E_1$ is dense in $\mathbb{P}_\alpha$.*

This lemma is true for all forcing iterations, not only of variations of Sacks forcing. We do not even use the countable supports.

Proof of Lemma 4.5. Let us start with

Claim 4.6. *Let $\beta < \alpha$ and let $q \in \mathbb{P}_\alpha$ be such that for some $i \in 2$ there are q_0 and q_1 such that*

$$q \restriction \beta \Vdash q_0, q_1 \leq q \restriction [\beta, \alpha) \wedge \operatorname{parity}(x[q_0], x[q_1]) = i.$$

Let $\gamma < \beta$. Then there are $q' \leq q$ and q_0' and q_1' such that

$$q' \restriction \gamma \Vdash q_0', q_1' \leq q' \restriction [\gamma, \alpha) \wedge \operatorname{parity}(x[q_0'], x[q_1']) = i.$$

To see this, let $q' \leq q$ be such that $q' \restriction [\beta, \alpha) = q \restriction [\beta, \alpha)$ and $q' \restriction \beta$ decides $x[q_0]$ and $x[q_1]$. For $j \in 2$ let $q_j' := (q' \restriction [\gamma, \beta))^\frown q_j$. Now q', q_0', and q_1' work for the claim.

For the proof of the lemma let $p \in \mathbb{P}_\alpha$. Suppose $p \notin E_0$. We show that p has an extension in E_1. Since $p \notin E_0$, there are $\gamma < \alpha$ and $q \leq p$ such that for all $q' \leq q$ and any two sequences q_0 and q_1 for names of conditions, if $q' \restriction \gamma \Vdash q_0, q_1 \leq q' \restriction [\gamma, \alpha)$, then $q' \restriction \gamma \nVdash \operatorname{parity}(x[q_0], x[q_1]) = 0$. We are done if we can show

Claim 4.7. *$q \in E_1$.*

Let $r \leq q$ and $\beta < \alpha$. Note that by Claim 4.6, the sets E_i are not changed if in the definition we replace "$\forall \beta < \alpha$" by "for cofinally many $\beta < \alpha$". Thus we may assume $\beta \geq \gamma$.

Since we assumed that $\dot{x}$ is not added in a proper initial stage of the iteration (before α), there are q_0 and q_1 such that

$$r \restriction \beta \Vdash q_0, q_1 \leq r \restriction [\beta, \alpha) \wedge x[q_0] \perp x[q_1].$$

Decreasing $r \restriction \beta$ if necessary, we may assume that $r \restriction \beta$ decides $\operatorname{parity}(x[q_0], x[q_1])$ to be $i \in 2$.

By Claim 4.6, there are $r' \leq r$ and r_0 and r_1 such that

$$r' \restriction \gamma \Vdash r_0, r_1 \leq r' \restriction [\gamma, \alpha) \wedge \operatorname{parity}(x[r_0], x[r_1]) = i.$$

By the choice of q, $i \neq 0$. Thus $i = 1$. This shows $q \in E_1$. $\qquad\square$

4.2.2. *Some forcing notation.* For $n \in \omega$ and $p \in \mathbb{P}$ let p^n be the set of all minimal $t \in p$ such that $\operatorname{norm}(c_{t,p}) > n$. For $p, q \in \mathbb{P}$ we write $q \leq_n p$ if $q \leq p$ and $p^n = q^n$.

A sequence $(p_n)_{n \in \omega}$ in $\mathbb{P}$ is a *fusion sequence* if there is a nondecreasing unbounded function $f : \omega \to \omega$ such that for all $n \in \omega$, $p_{n+1} \leq_{f(n)} p_n$. If $(p_n)_{n \in \omega}$ is a fusion sequence, then $p_\omega = \bigcap_{n \in \omega} p_n$ is a condition in $\mathbb{P}$, the *fusion* of the sequence. In this definition, the function f is only added for technical convenience. If we only talk about the identity function instead of arbitrary f, we arrive at an essentially equivalent notion.

The idea behind fusion is that in $\mathbb{P}$, even though it is not countably closed, lower bounds exist for suitably chosen countable sequences. All we have to do while inductively thinning out a condition, is to leave splitting nodes with more and more complicated colorings on their successors untouched. This is exactly what

we did, although less formally, in the proof of Lemma 4.3. The method can be extended to countable support iterations.

Let α be an ordinal. For $F \in [\alpha]^{<\aleph_0}$, $\eta : F \to \omega$, and $p, q \in \mathbb{P}_\alpha$ let $q \leq_{F,\eta} p$ if $q \leq p$ and for all $\beta \in F$, $q \restriction \beta \Vdash q(\beta) \leq_{\eta(\beta)} p(\beta)$. Roughly speaking, $q \leq_{F,\eta} p$ means that on each coordinate from F, q is $\leq_n$-below p where n is given by η.

A sequence $(p_n)_{n\in\omega}$ of conditions in $\mathbb{P}_\alpha$ is a *fusion sequence* if there is an increasing sequence $(F_n)_{n\in\omega}$ of finite subsets of α and a sequence $(\eta_n)_{n\in\omega}$ such that for all $n \in \omega$, $\eta_n : F_n \to \omega$, $p_{n+1} \leq_{F_n,\eta_n} p_n$, for all $\gamma \in F_n$ we have $\eta_n(\gamma) \leq \eta_{n+1}(\gamma)$, and for all $\gamma \in \text{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$.

This notion is precisely what is needed in countable support iterations to get suitable fusions. It essentially means that once we have touched (i.e., decreased) a coordinate of p_0, we have to build a fusion sequence in that coordinate.

If $(p_n)_{n\in\omega}$ is a fusion sequence in $\mathbb{P}_\alpha$, its fusion p_ω is defined inductively. Let $F_\omega := \bigcup F_n$.

Suppose $p_\omega(\gamma)$ has been defined for all $\gamma < \beta$ for some $\beta < \alpha$. If $\beta \notin F_\omega$, let $p_\omega(\beta)$ be a name for $1_\mathbb{P}$. If $\beta \in F_\omega$, then $p_\omega \restriction \beta$ forces $(p_n(\beta))_{n\in\omega}$ to be a fusion sequence in $\mathbb{P}$. Let $p_\omega(\beta)$ be a name for the fusion of the $p_n(\beta)$'s.

4.2.3. *Keeping $\mathfrak{hm}$ small.* Let $\dot{x}$ and α be as before. The way to build a condition q for which T_q is $c_{\min}$-homogeneous is the following: q will be the fusion of a fusion sequence $(p_n)_{n\in\omega}$ with witness $(F_n, \eta_n)_{n\in\omega}$. For each n, (p_n, F_n, η_n) will determine a finite initial segment T_n of T_q. We have to make sure that T_q is the union of the T_n and that the T_n are good enough to guarantee the $c_{\min}$-homogeneity of $[T_q]$. The latter will be ensured by the (F_n, η_n)-faithfulness of each p_n, which is defined below.

First we introduce some tools that help us to carry out the necessary fusion arguments.

We call a condition $p \in \mathbb{P}$ *normal* if for every $s \in p$ with $n := |\text{succ}_p(s)| > 1$, $c_{s,p}$ is isomorphic to $c_{\text{random}} \restriction n$ and moreover, if $t \in p$ is a minimal proper extension of s with more than one successor in p, then $|\text{succ}_p(t)| = |\text{succ}_p(s)| + 1$. Thus, $s \in p^n$ iff $|\text{succ}_p(s)| = n + 1$.

Let $I := T(\prod_{i\in\omega}(i+1)) = \bigcup\{\prod_{i<n}(i+1) : n \in \omega\}$ and $I_n := \{\rho \in I : \text{dom}(\rho) = n\}$. If $p \in \mathbb{P}$ is a normal condition, then each $\rho \in I_n$ determines an element s_ρ of p^n. Let $p * \rho := p_{s_\rho} = \{t \in p : s_\rho \subseteq t \vee t \subseteq s_\rho\}$.

A condition $q \in \mathbb{P}_\alpha$ is normal if for all $\beta < \alpha$, $q \restriction \beta$ forces that $q(\beta)$ is normal. Suppose $q \in \mathbb{P}_\alpha$ is a normal condition. For $F \in [\alpha]^{<\aleph_0}$, $\eta : F \to \omega$, and $\sigma \in \prod_{\gamma\in F} I_{\eta(\gamma)}$ let $q * \sigma$ be defined as follows:

For $\gamma \in F$ let $(q * \sigma)(\gamma)$ be a name for a condition in $\mathbb{P}$ such that $\Vdash_{\mathbb{P}_\gamma} (q * \sigma)(\gamma) = q(\gamma) * \sigma(\gamma)$. For $\gamma \in \alpha \setminus F$ let $(q * \sigma)(\gamma) := q(\gamma)$.

Now $(q * \sigma)_{\sigma \in \prod_{\gamma\in F} I_{\eta(\gamma)}}$ is a finite maximal antichain below q. Consider the tree T generated by $\{x[q*\sigma] : \sigma \in \prod_{\gamma\in F} I_{\eta(\gamma)}\}$. If $q' \leq_{F,\eta} q$, then $T_{q'}$ is an end-extension of T.

It is clear that the normal conditions in $\mathbb{P}$ form a dense subset and the same is true for $\mathbb{P}_\alpha$. Therefore, from now on all the conditions we consider are assumed to be normal. We have to be careful at one point, however. Suppose $p \in \mathbb{P}$ is a normal condition and we have constructed some $q \leq_n p$. q is not necessarily normal. But it is easy to see that there is some $q' \leq_n q$ which is normal. We call the process of passing from q to q' *normalization at n*. Normalization at n will be applied automatically without being mentioned whenever we construct some $q \leq_n p$.

Definition 4.8. *Let $i \in 2$ and $\dot{x}$ be fixed. For F and η as before, a condition $q \in \mathbb{P}_\alpha$ is (F, η)-faithful if for all $\sigma, \tau \in \prod_{\gamma\in F} I_{\eta(\gamma)}$ with $\sigma \neq \tau$, $\text{parity}(x[q*\sigma], x[q*\tau]) = i$.*

Now we are ready to formulate the lemma that will allow us to handle the case where $\dot{x}$ is added at a limit step of the iteration.

Lemma 4.9. *Let α be a limit ordinal and let $\dot{x}$ be a $\mathbb{P}_\alpha$-name for an element of 2^ω which is not added by an initial stage of the iteration. Let F, η, and i be as in Definition 4.8 and suppose that $q \in \mathbb{P}_\alpha$ is (F, η)-faithful.*

a) Let $\beta \in \alpha \setminus F$ and let $F' := F \cup \{\beta\}$ and $\eta' := \eta \cup \{(\beta, 0)\}$. Then q is (F', η')-faithful.

b) Suppose $q \in E_i$. Let $\beta \in F$ and let $\eta' := \big(\eta \restriction (F \setminus \{\beta\})\big) \cup \{(\beta, \eta(\beta) + 1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

Proof. a) follows immediately from the definitions.

For b) let $\delta := \max(F) + 1$ and $n := \eta(\beta)$.

Claim 4.10. *There is a condition $q' \leq_{F,\eta} q$ such that for each $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ there are sequences $q_{\sigma,0}, \ldots, q_{\sigma,n}$ of names for conditions such that for all $k \leq n$,*

$$q' * \sigma \restriction \delta \Vdash q_{\sigma,k} \leq q \restriction [\delta, \alpha),$$

$q' * \sigma \restriction \delta$ decides $x[q_{\sigma,k}]$, and for all $l \leq n$ with $k \neq l$,

$$q' * \sigma \restriction \delta \Vdash \mathrm{parity}(x[q_{\sigma,k}], x[q_{\sigma,l}]) = i.$$

For the proof of the claim, let $\{\sigma_1, \ldots, \sigma_m\}$ be an enumeration of $\prod_{\gamma \in F} I_{\eta(\gamma)}$. We build a $\leq_{F,\eta}$-decreasing sequence $(q_j)_{j \leq m}$ such that $q_0 := q$ and $q' := q_m$ works for the claim. As we construct q_j, we find suitable $q_{\sigma_j,k}$ for all $k < n$.

Let $j \in \{1, \ldots, m\}$ and assume that q_{j-1} has already been constructed. Since $q \in E_i$ and E_i is open, there are $q_j^* \leq q_{j-1} * \sigma_j$ and sequences $q_{\sigma_j,0}$ and $q_{\sigma_j,1}^*$ of names of conditions such that

$$q_j^* \restriction \delta \Vdash q_{\sigma_j,0}, q_{\sigma_j,1}' \leq q \restriction [\delta, \alpha) \wedge \mathrm{parity}(x[q_{\sigma_j,0}], x[q_{\sigma_j,1}^*]) = i.$$

Iterating this process by splitting $q_{\sigma_j,1}^*$ into $q_{\sigma_j,1}$ and $q_{\sigma_j,2}^*$ and so on and decreasing q_j^*, we finally obtain $q_j^* \leq q_{j-1} * \sigma_j$ and $q_{\sigma_j,k}$, $k \leq n$, such that for all $k \leq n$.

$$q_j^* \restriction \delta \Vdash q_{\sigma_j,k} \leq q \restriction [\delta, \alpha)$$

and for all $l \leq n$ with $l \neq k$,

$$q_j^* \restriction \delta \Vdash \mathrm{parity}(x[q_{\sigma_j,k}], x[q_{\sigma_j,l}]) = i.$$

We may assume that $q_j^* \restriction \delta$ decides $x[q_{\sigma_j,k}]$ for all $k \leq n$. Let $q_j \leq_{F,\eta} q_{j-1}$ be such that $q_j * \sigma_j \restriction \delta = q_j^* \restriction \delta$ and $q_j \restriction [\delta, \alpha) = q \restriction [\delta, \alpha)$. This finishes the construction, and it is easy to check that it works.

Continuing the proof of lemma 4.9, let $q_{\sigma,k}$ and q' be as in the claim. For $\rho \in I_{\eta(\beta)}$ let $r^{\rho \frown 0}, \ldots, r^{\rho \frown n}$ be sequences of names for conditions such that for all $k \leq n$ and all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ with $\sigma(\beta) = \rho$,

$$q' * \sigma \restriction \delta \Vdash r^{\rho \frown k} = q_{\sigma,k}.$$

Let r be a sequence of names for conditions such that $r \restriction \delta = q' \restriction \delta$ and for all $\sigma \in \prod_{\gamma \in F} I_{\eta'(\gamma)}$,

$$q' * \sigma \restriction \delta \Vdash r \restriction [\delta, \alpha) = r^{\sigma(\beta)}.$$

With this choice of r we have $r \leq_{F,\eta} q$. It follows from the construction that r is (F, η')-faithful. $\qquad\square$

A similar lemma is true if the new real is added in a successor step.

Lemma 4.11. *Let α be a successor ordinal, say $\alpha = \delta + 1$ and let $\dot{x}$ be a $\mathbb{P}_\alpha$-name for an element of 2^ω which is not added by an initial stage of the iteration. Let F, η, and i be as in Definition 4.8 and suppose that $q \in \mathbb{P}_\alpha$ is (F, η)-faithful.*

  a) Let $\beta \in \alpha \setminus F$ and let $F' := F \cup \{\beta\}$ and $\eta' := \eta \cup \{(\beta, 0)\}$. Then q is (F', η')-faithful.

  b) Suppose

$$q \upharpoonright \delta \Vdash \text{``}[T_{q(\delta)}] \text{ is } c_{\min}\text{-homogeneous of color } i\text{''}.$$

Let $\beta \in F$ and let $\eta' := \eta \upharpoonright F \setminus \{\beta\} \cup \{(\beta, \eta(\beta) + 1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

Proof. As in Lemma 4.9, a) follows directly from the definitions.

For the proof of b) we have to consider two cases. First suppose $\beta = \delta$. In this case let q' be a name for a condition in $\mathbb{P}$ such that for all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ and all $k, l \leq \eta(\beta)$ with $k \neq l$,

$$q * \sigma \upharpoonright \delta \Vdash q' \leq_{\eta(\delta)} q(\delta) \wedge x[q' * (\sigma(\delta)^\frown k)] \perp x[q' * (\sigma(\delta)^\frown l)].$$

Let $r \leq_{F,\eta} q$ be such that $r \upharpoonright \delta \Vdash r(\delta) = q'$ and for all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ and all $k \leq \eta(\beta)$, $r * \sigma \upharpoonright \delta$ decides $x[r(\delta) * (\sigma(\delta)^\frown k)]$.

Note that r is indeed (F, η')-faithful since we assumed $q \upharpoonright \delta$ to force that $T_{q(\delta)}$ is $c_{\min}$-homogeneous of color i.

If $\beta \neq \delta$, the argument will be similar to the one used for Lemma 4.9. Let $n := \eta(\beta)$.

For all $k \leq n$ and all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ let $q_{\sigma,k}$ be a name for a condition such that

$$q * \sigma \upharpoonright \delta \Vdash q_{\sigma,k} \leq q(\delta) * \sigma(\delta)$$

and for all $l \leq n$ with $l \neq k$

$$q * \sigma \upharpoonright \delta \Vdash x[q_{\sigma,k}(\delta)] \perp x[q_{\sigma,l}(\delta)].$$

Now fix $q' \leq_{F,\eta} q$ such that for all $\sigma \in \prod_{\gamma \in F} I_{\eta(\gamma)}$ and all $k \leq n$, $q' * \sigma \upharpoonright \delta$ decides $x[q_{\sigma,k}]$. Note that for all $k, l \leq n$ with $k \neq l$ we have that

$$q' * \sigma \upharpoonright \delta \Vdash \text{parity}(x[q_{\sigma,k}], x[q_{\sigma,l}]) = i$$

since $[T_{q(\delta)}]$ was forced to be $c_{\min}$-homogeneous.

Choose r such that $r \upharpoonright \delta = q' \upharpoonright \delta$ and for all $\sigma \in \prod_{\gamma \in F} I_{\eta'(\gamma)}$

$$r * \sigma \upharpoonright \delta \Vdash r(\delta) * \sigma(\delta) = q_{\sigma,k}$$

where $k = \sigma(\beta)(n)$ (i.e., k is the last digit of $\sigma(\beta)$).

It follows from the definition of r that $r \leq_{F,\eta} q$. It is easily checked that r is (F, η')-faithful. □

Combining the last two lemmas, we can show

Lemma 4.12. *Let G be $\mathbb{P}_{\omega_2}$-generic over the ground model M. Then in $M[G]$, 2^ω is covered by $c_{\min}$-homogeneous sets coded in the ground model. In particular, in $M[G]$, 2^ω is covered by $\aleph_1$ $c_{\min}$-homogeneous sets.*

Proof. We work in M. Let $\dot{x}$ be a name for an element of 2^ω. We show that $\dot{x}$ is forced to be a branch through a parity-homogeneous tree in M. We may assume that for some $\alpha < \omega_2$, $\dot{x}$ is an $\mathbb{P}_\alpha$-name for a real not added in a proper initial stage of the iteration $\mathbb{P}_\alpha$. Clearly, $\text{cf}(\alpha) \leq \aleph_0$. Let $p \in \mathbb{P}_\alpha$. If α is a limit ordinal, using Lemma 4.5, we can decrease p such that for some $i \in 2$, $p \in E_i$. If α is a successor ordinal, say $\alpha = \delta + 1$, we can use Lemma 4.3 to decrease p such that for some $i \in 2$

$$p \upharpoonright \delta \Vdash \text{``}[T_{p(\delta)}] \text{ is } c_{\min}\text{-homogeneous of color } i\text{''}.$$

By induction, we define a sequence $(p_n, F_n, \eta_n)_{n \in \omega}$ such that

(1) for all $n \in \omega$, $p_n \in \mathbb{P}_\alpha$, $p_n \leq p$, $F_n \in [\alpha]^{<\aleph_0}$, $\eta_n : F_n \to \omega$, and p_n is (F_n, η_n)-faithful,

(2) for all $n \in \omega$, $F_n \subseteq F_{n+1}$, $p_{n+1} \leq_{F_n,\eta_n} p_n$, and for all $\gamma \in F_n$ we have $\eta_n(\gamma) \leq \eta_{n+1}(\gamma)$, and

(3) for all $n \in \omega$ and all $\gamma \in \operatorname{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$.

This construction can be done using parts a) and b) of Lemma 4.9 and Lemma 4.11 respectively, depending on whether α is a limit ordinal or not, to extend F_n or to make η_n bigger, together with some bookkeeping to ensure 3. Now $(p_n)_{n \in \omega}$ is a fusion sequence. Let q be the fusion of this sequence. For each $n \in \omega$ let T_n be the tree generated by $\{x[p_n * \sigma] : \sigma \in \prod_{\gamma \in F_n} I_{\eta(\gamma)}\}$. It is easily seen that $T_q = \bigcup_{n \in \omega} T_n$.

It now follows from the faithfulness of the p_n that $[T_q]$ is $c_{\min}$-homogeneous of color i. Clearly, q forces $\dot{x}$ to be a branch through T_q. It follows that the set of conditions in $\mathbb{P}_\alpha$ forcing $\dot{x}$ to be an element of a $c_{\min}$-homogeneous set coded in M is dense in $\mathbb{P}_\alpha$. Since $\mathbb{P}_\alpha$ is completely embedded in $\mathbb{P}_{\omega_2}$, this finishes the proof of the lemma. $\square$

Corollary 4.13. *It is consistent with ZFC that $2^{\aleph_0} = \aleph_2$ and 2^ω is covered by $\aleph_1$ $c_{\min}$-homogeneous sets, but it is not covered by less than $2^{\aleph_0}$ $c_{\max}$-homogeneous sets.*

4.3. **Why forcing with $\mathbb{P}_{c_{\max}}$?** One may ask whether there is an essentially simpler way of increasing $\mathfrak{hm}(c_{\max})$ while keeping $\mathfrak{hm}$ small other than iterating our basic forcing notion $\mathbb{P}$. Zapletal [34] showed that in certain cases there is an optimal way of increasing a covering number of a σ-ideal. He observed that there is an optimal way of increasing $\mathfrak{hm}$ in the sense that all cardinal invariants which are not bigger than $\mathfrak{hm}$ in ZFC are kept small (assuming the existence of some large cardinals). The natural forcing to do this is the following:

Definition 4.14. *The $c_{\min}$-forcing $\mathbb{P}_{c_{\min}}$ is the partial order consisting of all perfect subtrees p of $2^{<\omega}$ with the property that for all $s \in p$ there are splitting nodes t_0 and t_1 of p which extend s such that the length of t_0 is even and the length of t_1 is odd.*

It is easy to see that the $\mathbb{P}_{c_{\min}}$-generic real avoids all the $c_{\min}$-homogeneous sets in the ground model. Therefore iterating $\mathbb{P}_{c_{\min}}$ increases $\mathfrak{hm}$.

The natural approach for increasing $\mathfrak{hm}(c_{\max})$ would be forcing with an iteration of the Borel subsets of 2^ω modulo the σ-ideal generated by the $c_{\max}$-homogeneous subsets. However, this attempt must fail. Zapletal observed that this forcing notion is not homogeneous, that is, the forcing notion does not stay the same when restricted to some Borel set not covered by countably many $c_{\max}$-homogeneous sets. We show that in fact, this forcing notion increases $\mathfrak{hm}$.

Theorem 4.15. *Let X be any Polish space with some nontrivial continuous pair-coloring $c : [X]^2 \to 2$. Then the Boolean algebra of Borel subsets of X modulo the σ-ideal generated by the c-homogeneous sets is forcing equivalent to $\mathbb{P}_{c_{\min}}$.*

The theorem easily follows from the next lemma, which is a strengthening of Lemma 2.9.

Lemma 4.16. *Assume that $A \subseteq X$ is analytic. If A is not covered by countably many c-homogeneous sets, then $c_{\min} \leq c \upharpoonright A$, i.e., A has a perfect subset on which c is isomorphic to $c_{\min}$.*

Proof. Since A is analytic, there is a continuous map $f : \omega^\omega \to A$ which is onto. For $s \in \omega^{<\omega}$ let $O_s := \{x \in \omega^\omega : s \subseteq x\}$. For $B \subseteq \omega^\omega$ let

$$B' := B \setminus \bigcup \{O_s : s \in \omega^{<\omega} \wedge f[B \cap O_s] \text{ is not covered}$$
$$\text{by countably many } c\text{-homogeneous sets}\}.$$

Note that B' is closed if B is.

Let $B_0 := \omega^\omega$, $B_{\alpha+1} := B'_\alpha$ for $\alpha < \omega_1$ and $B_\delta := \bigcap_{\alpha<\delta} B_\alpha$ for limit ordinals $\delta < \omega_1$. Since there are only countably many O_s, there is $\alpha < \omega_1$ such that $B_\alpha = B_{\alpha+1}$. Let $B := B_\alpha$.

Since A is not covered by countably many c-homogeneous sets, B is not empty. Clearly, for every open set $O \subseteq \omega^\omega$, $O \cap B$ is empty or $f[O \cap B]$ is not covered by countably many c-homogeneous sets and therefore is not homogeneous. It now follows from the continuity of f and c that for all $s \in T(B)$ and all $i \in 2$ there are $s_0, s_1 \in T(B)$ extending s such that c is constant on $f[O_{s_0} \cap B] \times f[O_{s_1} \cap B]$ with value i.

This is sufficient to construct inductively a perfect binary subtree T of $T(B)$ such that $f \restriction [T]$ is 1-1 and $f[[T]]$ has the desired properties. □

5. CONSISTENCY OF $\mathrm{Cov}(\mathcal{C}ont(\mathbb{R})) < \mathrm{Cov}(\mathcal{L}ip(\mathbb{R}))$

This section is devoted to the proof of

Theorem 5.1. $\mathrm{Cov}(\mathcal{C}ont) < \mathfrak{hm}$ *is consistent.*

In Definition 4.14 we have already introduced the forcing notion $\mathbb{P}_{c_{\min}}$ as the right tool to increase $\mathfrak{hm}$.

In this section we write $\mathbb{P}$ for $\mathbb{P}_{c_{\min}}$. As usual, for every ordinal α, $\mathbb{P}_\alpha$ denotes the countable support iteration of $\mathbb{P}$ of length α. We have to show

Lemma 5.2. *After forcing with $\mathbb{P}_{\omega_2}$ over a model of CH the continuous functions coded in the ground model cover $(2^\omega)^2$ (in the extension).*

How do we construct the required continuous mappings in the ground model?

Let $\dot{x}$ and $\dot{y}$ be $\mathbb{P}_{\omega_2}$-names for elements of 2^ω. Assume that both, $\dot{x}$ and $\dot{y}$, are forced to be new reals. We may do so because the constant functions take care about covering pairs $(x, y) \in (2^\omega)^2$ where x or y are in the ground model.

We may also assume that there are $\alpha, \beta < \omega_2$ such that $\dot{x}$ is in fact a $\mathbb{P}_\alpha$-name forced not to be added in a proper initial stage of the iteration $\mathbb{P}_\alpha$ and the same is true for $\dot{y}$ with respect to β. Finally, we may assume $\beta \leq \alpha$. In particular, $\dot{y}$ can be considered as a $\mathbb{P}_\alpha$-name.

Now let $p \in \mathbb{P}_\alpha$. We find $q \leq p$ with the following property:

$(*)_{q,\dot{x},\dot{y}}$ Let $T_q(\dot{x})$ be the tree of q-possibilities for $\dot{x}$ defined as in the proof of Lemma 4.3. Then in the ground model there is a continuous map $f : [T_q(\dot{x})] \to 2^\omega$ such that if G is $\mathbb{P}_\alpha$-generic with $q \in G$, then f maps $\dot{x}_G$ to $\dot{y}_G$.

Clearly, q forces $f(\dot{x}) = \dot{y}$. Thus, the set of conditions forcing $f(\dot{x}) = \dot{y}$ for some continuous ground model function f is dense in $\mathbb{P}_\alpha$. This shows that for every $\mathbb{P}_{\omega_2}$-generic filter G, $(\dot{x}_G, \dot{y}_G)$ is indeed covered by some continuous ground model function. Note that the function f in $(*)_{q,\dot{x},\dot{y}}$ is just a partial function. But since f is defined on the closed set $[T_q(\dot{x})]$, it can be extended continuously to all of 2^ω.

This finishes the proof of Lemma 5.2 provided we know

Lemma 5.3. *Let α be an ordinal and $\dot{x}$ a $\mathbb{P}_\alpha$-name for an element of 2^ω which is not added in a proper initial stage of the iteration. Then for every $p \in \mathbb{P}_\alpha$ and every $\mathbb{P}_\alpha$-name $\dot{y}$ for an element of 2^ω there is $q \leq p$ such that $(*)_{q,\dot{x},\dot{y}}$ holds.*

Proof. We follow closely the proof of Lemma 4.12. We fix $\dot{x}$ throughout the following proof.

For $p \in \mathbb{P}$ and $n \in \omega$ let p^n denote the set of those splitting nodes of p that have exactly n splitting nodes among their proper initial segments. For $q \leq p$ we write $q \leq_n p$ if $q^n = p^n$. Every $\rho \in 2^n$ determines an element s_ρ of p^n. Let $p * \rho := p_{s_\rho} = \{s \in p : s \subseteq s_\rho \vee s_\rho \subseteq s\}$.

We call a condition $p \in \mathbb{P}$ *normal* if for all splitting nodes $s, t \in p$ such that $s \subsetneqq t$ and t is a minimal splitting node above s, $\mathrm{dom}(t) \setminus \mathrm{dom}(s)$ is odd, i.e., $\mathrm{dom}(s)$ and $\mathrm{dom}(t)$ have a different parity.

As in the $\mathbb{P}_{c_{\max}}$-case, if p is a normal condition and $q \leq_n p$, then there is a normal condition $r \leq_n q$. This is normalization at n that from now on will be done automatically, just as in the $\mathbb{P}_{c_{\max}}$-case.

We extend the notion of normality to conditions in $\mathbb{P}_\alpha$ and for $F \in [\alpha]^{<\aleph_0}$ and $\eta : F \to \omega$ we define $\leq_{F,\eta}$ on $\mathbb{P}_\alpha$ as for $\mathbb{P}_{c_{\max}}$ (see section 4.2.2). Fusion sequences are defined as for $\mathbb{P}_{c_{\max}}$ and it should be clear that fusions of fusion sequences in $\mathbb{P}_\alpha$ are again conditions, provided the elements of the fusion sequence are normal.

For f and η as above, $p \in \mathbb{P}_\alpha$, and $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$, $p * \sigma$ is defined as in Section 4.2.3.

We also use the notion of faithfulness, but in the present context the definition is a weaker than in Section 4.2.3.

Definition 5.4. *For F and η as above, $p \in \mathbb{P}_\alpha$ is (F, η)-faithful iff for all $\sigma, \tau \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ with $\sigma \neq \tau$, $x[p * \sigma] \perp x[p * \tau]$.*

The corresponding statement to Lemma 4.9 and Lemma 4.11 is

Claim 5.5. *Let F and η be as before and suppose that $q \in \mathbb{P}_\alpha$ is (F, η)-faithful.*
a) Let $\beta \in \alpha \setminus F$ and let $F' := F \cup \{\beta\}$ and $\eta' := \eta \cup \{(\beta, 0)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F', η')-faithful.
b) Let $\beta \in F$ and let $\eta' := \eta \upharpoonright F \setminus \{\beta\} \cup \{(\beta, \eta(\beta) + 1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

Proof. In contrast to the $\mathbb{P}_{c_{\max}}$-case, a) is not trivial here. This is because $\leq_0$ is not equivalent to $\leq$. But this is rather a notational issue. a) clearly follows from the proof of b).

For b) let $\delta := \max F$ and let $\{\sigma_0, \ldots, \sigma_m\}$ be an enumeration of $\prod_{\gamma \in F} 2^{\eta(\gamma)}$. We define a $\leq_{F,\eta}$-decreasing sequence $(q_j)_{j \leq m}$ along with names $q_{\sigma,0}$ and $q_{\sigma,1}$, $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$, for conditions.

Let $j \in \{1, \ldots m\}$ and assume that q_{j-1} has been constructed already. Since $\dot{x}$ is not added in a proper initial stage of the iteration, there are $q_{\sigma_j,0}$ and $q_{\sigma_j,1}$ such that for all $i \in 2$

$$q_{j-1} * \sigma_j \upharpoonright \delta \Vdash q_{\sigma_j,i} \leq (q(\delta) * (\sigma_j(\delta){}^\frown i)){}^\frown q \upharpoonright (\delta, \alpha)$$

and

$$q_{j-1} * \sigma_j \upharpoonright \delta \Vdash x[q_{\sigma_j,0}] \perp x[q_{\sigma_j,1}].$$

Let $q_j \leq_{F,\eta} q_{j-1}$ be such that $q_j * \sigma_j \upharpoonright \delta$ decides $x[q_{\sigma_j,0}]$ and $x[q_{\sigma_j,1}]$. This finishes the inductive construction of the q_j.

Now let $r \leq_{F,\eta} q_m$ be such that $r \upharpoonright \delta = q_m \upharpoonright \delta$ and for all $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ and all coordinatewise extensions $\tau \in \prod_{\gamma \in F} 2^{\eta'(\gamma)}$ of σ,

$$r * \tau \upharpoonright \delta \Vdash r * \tau \upharpoonright [\delta, \alpha) = q_{\sigma,\tau(\eta(\beta))}.$$

It is easy to check that r works for the claim. $\qquad\square$

To conclude the proof of Lemma 5.3, let $p \in \mathbb{P}_\alpha$. Using some bookkeeping and parts a) and b) of Claim 5.5 we construct a sequence $(p_n)_{n \in \omega}$ and a sequence

$(F_n, \eta_n)_{n \in \omega}$ witnessing that $(p_n)_{n \in \omega}$ is a fusion sequence such that $p = p_0$ and for all $n \in \omega$, p_n is (F_n, η_n)-faithful. We can construct the sequences $(p_n)_{n \in \omega}$ and $(F_n, \eta_n)_{n \in \omega}$ with the additional property that for every $n \in \omega$ and every $\sigma \in \prod_{\gamma \in F_n} 2^{\eta(\gamma)}$, $p_n * \sigma$ decides $\dot{y} \restriction n$. Let q be the fusion of the sequence $(p_n)_{n \in \omega}$. We have to check that $(*)_{q, \dot{x}, \dot{y}}$ holds.

Let $a \in [T_q(\dot{x})]$ and $n \in \omega$. Now $q \leq_{F_n, \eta_n} p_n$ and p_n is (F_n, η_n)-faithful. It follows that there is exactly one $\sigma_{a,n} \in \prod_{\gamma \in F_n} 2^{\eta_n(\gamma)}$ such that $x[q * \sigma_{a,n}] \subseteq a$.

Let $f(a) := \bigcup_{n \in \omega} y[q * \sigma_{a,n}]$. Since for all $n \in \omega$, $q * \sigma_{a,n} \leq p_n * \sigma_{a,n}$ and $p_n * \sigma_{a,n}$ decides $\dot{y} \restriction n$, $f(a) \in 2^\omega$. It is easily checked that $f : [T_q(\dot{x})] \to 2^\omega$ is a continuous map witnessing $(*)_{q, \dot{x}, \dot{y}}$.   □

6. Concluding remarks and open problems

The numbers $\mathfrak{hm}(c_{\min})$, $\mathfrak{hm}(c_{\max})$, $\mathrm{Cov}(\mathcal{C}ont(\mathbb{R}))$ and $\mathrm{Cov}(\mathcal{L}ip(\mathbb{R}))$ are examples of covering numbers of meager ideals. Although the hope expressed by Blass in [10] to find a classification of all "simple" cardinal invariants of the continuum was shattered by the construction in [21] of uncountably many different covering numbers of simply defined meager ideals, there is still hope to find the "largest" nontrivial covering number of a meager ideal. By "nontrivial" it is meant that the number can consistently be smaller than the continuum.

At the moment $\mathfrak{hm}(c_{\max})$ is a leading candidate for such a number. The numbers $\mathrm{Cov}(\mathcal{C}ont(\mathbb{R}))$ and $\mathrm{Cov}(\mathcal{L}ip(\mathbb{R}))$ are also very large, and the largest known nontrivial covering number of an ideal on $\mathbb{R}^2$ generated by functions and inverses of functions is the covering number of the ideal generated by continuously differentiable functions and their inverses. So it would be natural to compare $\mathfrak{hm}(c_{\max})$ to covering by continuously differentiable functions. As mentioned before, the covering number of the ideal on $\mathbb{R}^2$ generated by twice differentiable functions and inverses of differentiable functions is already continuum.

The meager ideals which historically led to the study of homogeneity numbers are the *convexity ideals* of closed subsets of $\mathbb{R}^2$. If a closed subset of a Euclidean space is not covered by countably many convex subsets (namely, its convex subsets generate a proper σ-ideal), it has a closed subset on which the convex subsets of the whole set generate a meager ideal (see [24] or [19]). For some closed subsets of the plane, this meager ideal coincides with the homogeneity ideal of some continuous pair coloring [20].

Saharon Shelah remarked recently to the authors that he came close to discovering the properties of $\mathfrak{hm}$ in his investigations of monadic theory of order [29]. In an attempt to remove GCH from the proof in the last section of [28] Shelah found a proof from the assumption $\mathfrak{hm} = 2^{\aleph_0}$. He was able to prove that $\mathfrak{hm} = 2^{\aleph_0}$ if the continuum is a limit cardinal, but did not prove more about $\mathfrak{hm}$ and eventually found a way to eliminate GCH which did not involve $\mathfrak{hm}$, which was consequently published in [22].

It is not clear why homogeneity numbers of continuous pair-colorings on Polish spaces were not studied earlier. We can only speculate about that. In the very short time since their study was begun, these numbers were related to quite a few subjects. Apart from the relation to planar convex geometry and to finite random graphs, which were mentioned above, there are relations to large cardinals, determinacy and pcf theory. Quite recently, Shelah and Zapletal [30] defined n-dimensional generalizations of $\mathfrak{hm}(c_{\min})$ and integrated forcing, pcf theory and determinacy theory to prove several duality theorem for those numbers.

We do not know at the moment if $\aleph_1 < \mathfrak{hm} < 2^{\aleph_0}$ is consistent or not. We do not know if there is a closed planar set whose convexity number is equal to $\mathfrak{hm}(c_{\max})$. We also find the following intriguing:

Problem 6.1. *Are the equalities $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min})$ and $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$ which hold in $V^{P_{c_{\max}}}$ absolute for a reduced coloring c on a Polish space X? In other words, does a reduced coloring c that satisfies $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$ in some model of set theory which separates $\mathfrak{hm}(c_{\min})$ and $\mathfrak{hm}(c_{\max})$ satisfy this in every model that separates $\mathfrak{hm}(c_{\min})$ and $\mathfrak{hm}(c_{\max})$?*

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