THE COINITIALITIES OF EFIMOV SPACES

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Abstract. We use $\diamondsuit$ to construct Efimov spaces of countable and uncountable coinitiality, showing that at least consistently there are compact spaces of uncountable cofinality without uncountable dyadic families.

0. Introduction

In [9] Koppelberg defined the *cofinality* $\mathrm{cof}(B)$ of an infinite Boolean algebra B to be the least limit ordinal δ such that there is a strictly increasing sequence $(B_\alpha)_{\alpha<\delta}$ of subalgebras of B such that $B = \bigcup_{\alpha<\delta} B_\alpha$. (See also [5].)

Clearly, the cofinality of an infinite Boolean algebra B is an infinite regular cardinal bounded by the size of B. If C is an infinite quotient of B, then $\mathrm{cof}(B) \leq \mathrm{cof}(C)$. Koppelberg showed that $\mathcal{P}(\omega)$, and in fact every infinite complete Boolean algebra, has cofinality $\aleph_1$. Moreover, since every infinite Boolean algebra has an infinite quotient of size $\leq 2^{\aleph_0}$, there is no Boolean algebra whose cofinality exceeds $2^{\aleph_0}$. Koppelberg asked whether there can be any Boolean algebra with a cofinality $> \aleph_1$.

Let us call a Boolean algebra of cofinality $> \aleph_1$ a *Koppelberg algebra.* If B is Koppelberg, then it cannot have an infinite countable quotient or an infinite complete quotient. Now by Stone duality, the Stone space $\mathrm{Ult}(B)$ cannot contain a nontrivial converging sequence or a copy of $\beta\omega$. In other words, $\mathrm{Ult}(B)$ has to be an Efimov space. The consistency of the existence of an Efimov space was shown by Fedorchuk [4]. Nowadays, various constructions of Efimov spaces are available. However, none of these constructions seems to have the potential of producing the Stone space of a Koppelberg algebra.

On the other hand, CH implies that there are no Koppelberg algebras. It is also known that there are models of set theory in which $2^{\aleph_0}$ is large but every Boolean algebra has cofinality $\leq \aleph_1$. (See for example [10], [8] and [7]).

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1. Coinitialities of compact spaces

By Stone duality, the cofinality of a Boolean algebra can be expressed in terms of inverse limit representations of the Stone space of the algebra. This yields the notion of *coinitiality* of a topological space. We use the notation in [6] for inverse systems.

Definition 1.1. Let X be a topological space. The *coinitiality* $\mathrm{ci}(X)$ of X is the least limit ordinal δ such X is the limit of an inverse system $\{X_\alpha, \pi^\beta_\alpha, \delta\}$ whose bonding maps π^β_α are onto and not 1-1, provided such an inverse system exists.

For every Boolean space X, i.e., if X is the Stone space of a Boolean algebra A, one can show that $\mathrm{ci}(X) = \mathrm{cof}(A)$. The fundamental facts about the cardinal invariant $\mathrm{ci}(X)$ are summarized in the following lemma. The easy proofs can be found in [7].

Lemma 1.2. *(a) If X is an infinite closed subspace of a compact space Y, then* $\mathrm{ci}(Y) \leq \mathrm{ci}(X)$.

(b) For every infinite compact space X, $\mathrm{ci}(X) \leq \mathrm{cf}(\mathrm{w}(X))$ *where* $\mathrm{w}(X)$ *denotes the weight of X.*

(c) For every infinite compact space X, $\mathrm{ci}(X) \leq 2^{\aleph_0}$.

Countable coinitiality can be characterized in terms of *double sequences.*

Definition 1.3. Let X be a compact space and let $(x_n)_{n\in\omega}$ be a discrete sequence in X. Then $(x_n)_{n\in\omega}$ is a double sequence if for all free ultrafilters p over ω, the p-limits of $(x_{2n})_{n\in\omega}$ and $(x_{2n+1})_{n\in\omega}$ are the same.

Theorem 1.4 (See [7]). *Let X be an infinite compact space. Then* $\mathrm{ci}(X) = \aleph_0$ *if and only if X contains a double sequence.*

Since direct limits are often easier to handle than inverse limits, it is sometimes convenient to compute instead of the coinitiality of a space X the *cofinality* of the C^*-algebra $C(X)$ of continuous functions from X to $\mathbb{C}$.

Definition 1.5. Let A be an infinite dimensional C^*-algebra. Then the *cofinality* $\mathrm{cof}(A)$ of A is the least infinite cardinal κ such that there is a strictly increasing chain $(A_\alpha)_{\alpha<\kappa}$ of closed $*$-subalgebras of A such that $\bigcup_{\alpha<\kappa} A_\alpha$ is dense in A.

Observe that since a C^*-algebra A is a metric space, if $(A_\alpha)_{\alpha<\delta}$ is an increasing chain of closed $*$-subalgebras of A and δ is an ordinal of uncountable cofinality, then $\bigcup_{\alpha<\delta} A_\alpha$ is a closed $*$-subalgebra of A. Thus, if $\bigcup_{\alpha<\delta} A_\alpha$ is dense in A, then it is actually equal to A.

Theorem 1.6 (See [7]). *For every infinite compact space X,* $\mathrm{ci}(X) = \mathrm{cof}(C(X))$.

2. Dyadic families

Definition 2.1. Let X be a topological space. A family $(S^i_\alpha)_{\alpha\in J, i\in 2}$ of closed subsets of X is *dyadic* if for each $\alpha \in J$, S^0_α and S^1_α are disjoint and for all disjoint finite sets $E, F \subseteq J$,

$$\bigcap\{S^0_\alpha : \alpha \in E\} \cap \bigcap\{S^1_\alpha : \alpha \in F\} \neq \emptyset.$$

Shapirovskiĭ showed that a compact space X maps onto I^{ω_1} iff it has an uncountable dyadic family [11]. It is easy to show directly that a compact space has an uncountable dyadic family iff it has a closed subspace that maps onto 2^{ω_1}.

Adapting Koppelberg's proof of $\mathrm{cof}(\mathcal{P}(\omega)) \leq \aleph_1$ [9], we can show

Theorem 2.2. *If a compact space X has an uncountable dyadic family, then* $\mathrm{ci}(X) \leq \aleph_1$.

Proof. Suppose X is compact and has an uncountable dyadic family. By the previous remark, X has a closed subspace that maps onto 2^{ω_1}. Since the coinitiality of X is bounded from above by the coinitialities of the infinite closed subspaces of X, we may assume that X itself maps onto 2^{ω_1}. Let $f : X \to 2^{\omega_1}$ be a continuous map witnessing this. Using Zorn's lemma we may actually assume that f is irreducible, i.e., no proper closed subspace of X is mapped onto 2^{ω_1} by f.

Let $G(2^{\omega_1})$ be the Gleason cover of 2^{ω_1}, i.e., the Stone space of the completion $\mathrm{ro}(\mathrm{Fr}\,\omega_1)$ of the free Boolean algebra with $\aleph_1$ generators. Let $g : G(2^{\omega_1}) \to 2^{\omega_1}$ be the Stone dual of the embedding from $\mathrm{Fr}\,\omega_1$ into its completion. $G(2^{\omega_1})$ is projective in the category of compact spaces and hence there is a continuous map $h : G(2^{\omega_1}) \to X$ such that

$$\begin{tikzcd} G(2^{\omega_1}) \arrow[r, "h"] \arrow[dr, "g"'] & X \arrow[d, two heads, "f"] \\ & 2^{\omega_1} \end{tikzcd}$$

commutes. Since f is irreducible, h is onto.

Let $C = C(G(2^{\omega_1}))$,

$$B = \{c \in C : c = b \circ h \text{ for some } b \in C(X)\}$$

and

$$A = \{c \in C : c = a \circ g \text{ for some } a \in C(2^{\omega_1})\}.$$

Then A and B are closed $*$-subalgebras of C that are isomorphic to $C(2^{\omega_1})$ and $C(X)$, respectively. We say that B is the algebra of elements of C that *factor through* h and A is the algebra of elements of C that factor through g.

Now, whenever $\alpha < \omega_1$, there are natural quotient maps $\pi_\alpha : 2^{\omega_1} \to 2^\alpha$ and $\rho_\alpha : G(2^{\omega_1}) \to G(2^\alpha)$. Here ρ_α is the Stone dual of the canonical embedding from $\mathrm{ro}(\mathrm{Fr}\,\alpha)$ into $\mathrm{ro}(\mathrm{Fr}\,\omega_1)$. For $\alpha<\omega_1$ let $g_\alpha : G(2^\alpha)\to 2^\alpha$ be the dual of the embedding of $\mathrm{Fr}\,\alpha$ into $\mathrm{ro}(\mathrm{Fr}\,\alpha)$. The diagram

$$\begin{tikzcd} G(2^{\omega_1}) \arrow[r, "\rho_\alpha"] \arrow[d, "g"'] & G(2^\alpha) \arrow[d, "g_\alpha"] \\ 2^{\omega_1} \arrow[r, "\pi_\alpha"'] & 2^\alpha \end{tikzcd}$$

commutes.

For each $\alpha<\omega_1$ let C_α be the algebra of elements of C that factor through ρ_α and let A_α be the algebra of elements of C that factor through $\pi_\alpha\circ g$. Note that $A_\alpha = C_\alpha\cap A$. Since $\mathrm{ro}(\mathrm{Fr}\,\omega_1) = \bigcup_{\alpha<\omega_1}\mathrm{ro}(\mathrm{Fr}\,\alpha)$, $G(2^{\omega_1})$ is the inverse limit of the $G(2^\alpha)$, $\alpha<\omega_1$. It follows that $\bigcup_{\alpha<\omega_1} C_\alpha$ is dense in C and thus, by the remark after Definition 1.5, $\bigcup_{\alpha<\omega_1} C_\alpha = C$.

For every $\alpha<\omega_1$ let $B_\alpha = B\cap C_\alpha$. Since C is the union of the C_α, $\bigcup_{\alpha<\omega_1} B_\alpha = B$. Since $A_\alpha = A\cap C_\alpha = A\cap B_\alpha$ and the sequence $(A_\alpha)_{\alpha<\omega_1}$ is strictly increasing, the sequence $(B_\alpha)_{\alpha<\omega_1}$ is strictly increasing. This shows that $\mathrm{cof}(B)\le\aleph_1$. Hence $\mathrm{ci}(X)\le\aleph_1$ □

3. Simple extensions

In this section we assume $\diamondsuit$ to construct two different Efimov spaces, one of coinitiality $\aleph_1$ and one of coinitiality $\aleph_0$. This in particular shows that for compact spaces X being of uncountable coinitiality does not imply that X contains a copy of $\beta\omega$ or an uncountable dyadic family.

Remark 3.1. It is well known that a compact space X contains a copy of $\beta\omega$ if and only if it has a closed subspace that maps onto $2^{2^{\aleph_0}}$. Under CH this reduces to a subspace that maps onto $2^{\aleph_1}$. Hence under CH we have that a compact space X contains a copy of $\beta\omega$ if and only if it has an uncountable dyadic family.

Definition 3.2. Let X be a compact space. Then $p: Y\to X$ is a *simple extension* of X if Y is compact, f is continuous and onto, and there is at most one point $x_0\in X$ whose preimage with respect to p is not a singleton.

X is *simplistic* [1] if X is the limit of a continuous inverse system $\{X_\alpha,\pi^\beta_\alpha,\delta\}$ where each $\pi^{\alpha+1}_\alpha : X_{\alpha+1}\to X_\alpha$ is a simple extension and X_0 is a singleton.

In other words, exactly one point is split when passing from X to Y, and it is split into exactly 2 points. The Stone dual of a simple extension is a *minimal extension*, where a Boolean algebra A is a minimal extension of a proper subalgebra B if there is no subalgebra of A that is strictly between B and A, i.e., if A is a minimal proper superalgebra of B.

Lemma 3.3 (Koppelberg [9])**.** *If X is simplistic, then X has no uncountable dyadic family.*

For a topological proof of this lemma see [2].

Theorem 3.4. *Assume $\diamondsuit$. Then there is a zero-dimensional Efimov space X of coinitiality $\aleph_1$ without isolated points.*

Proof. For concreteness, we construct X as a subspace of 2^{ω_1}. X will be the limit of a continuous inverse system $\{X_\alpha, \pi^\beta_\alpha, \omega_1\}$ where each X_α is a subspace of 2^α and each π^β_α, $\alpha < \beta < \omega_1$, is the restriction of the projection from 2^β to 2^α to X_β. For limit ordinals $\delta < \omega$ we let X_δ be the limit of the inverse system $\{X_\alpha, \pi^\beta_\alpha, \delta\}$. Note that X_δ can be naturally considered as a subset of 2^δ. For $\alpha < \omega_1$, we will choose $X_{\alpha+1} \subseteq 2^{\alpha+1}$ so that $\pi^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ is a simple extension. By Lemma 3.3 this implies that X has no uncountable dyadic family and therefore contains no copy of $\beta\omega$.

For all $\alpha < \omega$ we choose X_α as a closed subspace of 2^α such that for all $\alpha < \omega$, $p^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ is a simple extension and the inverse limit $X_\omega \subseteq 2^\omega$ of the X_α, $\alpha < \omega$, has no isolated points. Since we will never introduce isolated points during the remaining construction, all the X_α, $\omega \le \alpha < \omega_1$ will be homeomorphic to 2^ω and and the target space X will have no isolated points.

By $\diamondsuit$, CH holds and thus $(2^{<\omega_1})^\omega$ is of size $\aleph_1$. Hence, using some suitable coding, from the $\diamondsuit$-sequence we can obtain a sequence $(x^\alpha_n)_{n<\omega,\alpha<\omega_1}$ such that for $\alpha < \omega_1$, $(x^\alpha_n)_{n\in\omega}$ is a sequence in 2^α and for every sequence $(y_n)_{n\in\omega}$ in 2^{ω_1},

$$\{\alpha < \omega_1 : (y_n \restriction \alpha)_{n\in\omega} = (x^\alpha_n)_{n\in\omega}\}$$

is stationary in ω_1.

Suppose for some $\alpha < \omega_1$ we have defined X_α. We define $X_{\alpha+1} \subseteq 2^{\alpha+1}$.

First assume that $(x^\alpha_n)_{n\in\omega}$ is not a double sequence in X_α. Choose any point $x_0 \in X_\alpha$ and a strictly descreasing sequence $(A_n)_{n\in\omega}$ of clopen subsets of X_α such that $A_0 = X_\alpha$ and $\bigcap_{n\in\omega} A_n = \{x_0\}$. Let

$$A = \{x_0\} \cup \bigcup_{n\in\omega} A_{2n} \setminus A_{2n+1}$$

and

$$B = \{x_0\} \cup \bigcup_{n\in\omega} A_{2n+1} \setminus A_{2n+2}.$$

The crucial property of A and B is that both sets are closed, union up to X_α and have intersection $\{x_0\}$.

If $(x^\alpha_n)_{n\in\omega}$ is a double sequence in X_α, we choose x_0 to be an accumulation point of $(x^\alpha_n)_{n\in\omega}$. This is possible since X_α is compact. Since $(x^\alpha_n)_{n\in\omega}$ is a double sequence, x_0 is an accumulation point of $(x^\alpha_{2n})_{n\in\omega}$. Since X_α is first countable, there is a strictly increasing sequence $(n_i)_{i\in\omega}$

of natural numbers such that $(x^\alpha_{2n_i})_{i\in\omega}$ converges to x_0. Since $(x^\alpha_n)_{n\in\omega}$ is a double sequence, $(x^\alpha_{2n_i+1})_{i\in\omega}$ converges to x_0 as well.

Now let $(A_n)_{n\in\omega}$ be a strictly descreasing sequence of clopen subsets of X_α such that $A_0 = X_\alpha$ and $\bigcap_{n\in\omega} A_n = \{x_0\}$. We can thin out the sequence $(n_i)_{i\in\omega}$ to a strictly increasing sequence $(m_i)_{i\in\omega}$ such that for some strictly increasing sequence $(k_i)_{i\in\omega}$ of natural numbers we have that for all $i \in \omega$, $x_{2m_i}, x_{2m_i+1} \in A_{k_i}$ and for all $j < i$, $x_{2m_i}, x_{2m_i+1} \notin A_{k_j}$. We can assume that $k_0 = 0$.

Now for each $i \in \omega$ we choose a clopen set $B_i \subseteq A_{k_i} \setminus A_{k_i+1}$ such that $x_{2m_i} \in B_i$ and $x_{2m_i+1} \notin B_i$. Let

$$A = X_\alpha \setminus \bigcup_{i\in\omega} B_i$$

and

$$B = \{x_0\} \cup \bigcup_{n\in\omega} B_i.$$

A and B are nonempty closed sets that union up to X_α and have intersection $\{x_0\}$.

In either case let $X_{\alpha+1}$ consist of all functions $x \in 2^\alpha$ such that $(x \restriction \alpha \in B$ and $x(\alpha) = 0)$ or $(x \restriction \alpha \in A$ and $x(\alpha) = 1)$.

Note that $X_{\alpha+1}$ is a simple extension of X_α since only the point x_0 has two preimages in $X_{\alpha+1}$. It is easily checked that $X_{\alpha+1}$ is a closed subspace of $2^{\alpha+1}$ without isolated points. This finishes the definition of the inverse system whose limit is our target space X.

As already indicated previously, X has no isolated points. By Lemma 3.3, the space has no uncountable dyadic family and therefore contains no copy of $\beta\omega$. We show that X also has no double sequences, from which it follows that X is of coinitiality $\aleph_1$.

Suppose $(y_n)_{n\in\omega}$ is a double sequence in X. Then for some $\alpha < \omega_1$, $(y_n \restriction \alpha)_{n\in\omega}$ is discrete and hence a double sequence. For every $\beta < \omega_1$ with $\beta \geq \alpha$ it holds that $(y_n \restriction \beta)_{n\in\omega}$ is a double sequence. By the choice of $(x^\gamma_n)_{n\in\omega,\gamma\in\omega_1}$, there is $\beta \geq \alpha$ such that $(y_n \restriction \beta)_{n\in\omega} = (x^\beta_n)_{n\in\omega}$. When we constructed $X_{\beta+1}$, a strictly increasing sequence $(n_i)_{i\in\omega}$ was chosen such that $(x^\beta_{2n_i})_{i\in\omega}$ and $(x^\beta_{2n_i+1})_{i\in\omega}$ converge to a point $x_0 \in X_\beta$. Since $(y_n \restriction \beta)_{n\in\omega} = (x^\beta_n)_{n\in\omega}$ is discrete, x_0 is not one of the elements of the sequence $y_n \restriction \beta$.

The point x_0 has two preimages in $X_{\beta+1}$, namely $x_0{}^\frown 0$ and $x_0{}^\frown 1$. All the other points of X_β only have a single preimage. In particular, each $y_n \restriction \beta$ has only a single preimage in $X_{\beta+1}$, namely $y_n \restriction \beta+1$. $X_{\beta+1}$ is the union of two disjoint clopen sets $C_0 = \{x \in X_{\beta+1} : x(\beta) = 0\}$ and $C_1 = \{x \in X_{\beta+1} : x(\beta) = 1\}$. By the construction of $X_{\beta+1}$, for each $i \in \omega$ the preimage of $x^\beta_{2n_i}$ in $X_{\beta+1}$, i.e., $y_{2n_i} \restriction \beta + 1$, is an element of C_0 and the preimage of $x^\beta_{2n_i+1}$, i.e., $y_{2n_i+1} \restriction \beta + 1$, is an element of C_1.

Since C_0 and C_1 are closed, the sets $\{y_{2n_i} \restriction \beta+1 : i \in \omega\}$ and $\{y_{2n_i+1} \restriction \beta+1 : i \in \omega\}$ have disjoint closures. In particular, $(y_n \restriction \beta+1)_{n\in\omega}$ is not a double sequence. A contradiction. $\square$

Theorem 3.5. *Assume $\diamondsuit$. Then there is a zero-dimensional Efimov space X of coinitiality $\aleph_1$ with infinitely many isolated points.*

Proof. For each $\alpha < \omega$ choose X_α as a closed subspace of 2^α such that each $p^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ is a simple extension and such that the inverse limit $X_\omega \subseteq 2^\omega$ of the X_α, $\alpha < \omega$, is homeomorphic to the disjoint union of 2^ω and the one-point compactification of the discrete space ω.

We continue the construction of the sequence $(X_\alpha)_{\alpha<\omega_1}$ as in the proof of Theorem 3.4. We have to make sure that we never split any of the original isolated points of X_ω. This is possible since no discrete sequence accumulates at an isolated point and since we always have non-isolated points at our disposal that can be split at stage $\alpha \geq \omega$. This construction yields a zero-dimensional Efimov space with an infinite set of isolated points. $\square$

4. Collapsing coinitialities

If a compact space X has an infinite set of isolated points, we can easily turn it into a space of countable coinitiality by splitting isolated points.

Let X be a compact space with infinitely many isolated points. Let $(x_n)_{n\in\omega}$ be a 1-1 sequence of isolated points in X. Let Y be the space obtained from X by splitting each x_n into two distinct points y_{2n} and y_{2n+1}. In other words, $Y = X \setminus \{x_n : n \in \omega\} \cup \{y_n : n \in \omega\}$ where $\{y_n : n \in \omega\}$ is disjoint from X. The topology on Y is generated by the sets $\{y_n\}$, $n \in \omega$, and

$$O \setminus \{x_n : n \in \omega\} \cup \{y_m : m = 2n \text{ or } m = 2n+1 \text{ for some } n \in \omega \text{ with } x_n \in O\},$$

O an open subset of X.

Theorem 4.1. *The space Y defined above is compact and* $\operatorname{cof}(Y) = \aleph_0$.

Proof. It is easily checked that $(y_n)_{n\in\omega}$ is a double sequence in Y. $\square$

Corollary 4.2. *Assume $\diamondsuit$. Then there is a zero-dimensional Efimov space X of countable coinitiality.*

Proof. By Theorem 3.5 there is a zero-dimensional Efimov space X with an infinite set of isolated points. From X we construct a compact space Y of countable cofinality as in Theorem 4.1. It is easily checked that Y is zero-dimensional and does not contain a copy of $\beta\omega$. Now assume that Y contains a nontrivial convergent sequence. In this case one of the three sets $\{y_{2n} : n \in \omega\}$, $\{y_{2n+1} : n \in \omega\}$ and $Y \setminus \{y_n : n \in \omega\}$ contains a nontrivial convergent sequence, possibly converging to a point in Y that lies outside the set itself. If one of the first two sets contains such a sequence, then so does $\{x_n : n \in \omega\}$, a contradiction. If $Y \setminus \{y_n : n \in \omega\}$ contains such a sequence, then so does $X \setminus \{x_n : n \in \omega\}$, again conradicting the choice of X. $\square$

5. Discussion

The author had conjectured previously that the absence of double sequences in an infinite compact space implies the existence of an uncountable dyadic family. This conjecture is refuted by Theorem 3.4. It is very likely that Efimov spaces of countable and uncountable cofinality can be constructed assuming just CH, using the method developed in [3].

The main question, whether it is consistent that there is a compact space of coinitiality $> \aleph_1$ remains wide open. We conclude with two less ambitious problems.

Problem 5.1. Characterize compact spaces of coinitiality $\leq \aleph_1$.

Problem 5.2. Is it consistent that every compact space of uncountable coinitiality contains an uncountable dyadic family? What happens under PFA?

Observe that the last problem is a weakening of Efimov's problem whether it is consistent that every infinite compact space space has a convergent sequence or contains a copy of $\beta\omega$.

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