AN UPPER BOUND FOR THE CELLULARITY OF THE PHASE SPACE OF A MINIMAL DYNAMICAL SYSTEM

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ABSTRACT. Let G be a topological group acting continuously on an infinite compact space X. Suppose the dynamical system (X, G) is minimal. If G is κ-bounded for some infinite cardinal κ, then the cellularity of X is at most κ.

1. INTRODUCTION

The purpose of this note is to point out a relation between cardinal invariants of the phase space and the group of a minimal dynamical system.

Generalizing a theorem of Balcar and Błaszczyk [1], it was shown in [5] that whenever (G, X) is a minimal dynamical system and G is $\aleph_0$-bounded, then the Boolean algebra $\mathrm{ro}(X)$ of regular open subsets of X is the completion of a free Boolean algebra. In particular, X is of countable cellularity. This result is clearly related to an older result of Uspenskiĭ [8], who showed that if an $\aleph_0$-bounded group acts continuously and transitively on a compact space X, then X is Dugundji and hence of countable cellularity.

Using some of the ideas from [5], we show that whenever G is a κ-bounded group and (G, X) is a minimal system, then the cellularity of X is at most κ.

This result might be interesting for compact homogeneous spaces. A well-known open question by van Douwen (see [7]) about compact homogeneous spaces is whether the cellularity of such a space can be larger than $2^{\aleph_0}$. One feasible approach to show that it cannot, is to try to construct, for a given compact homogeneous space X, a $2^{\aleph_0}$-bounded group acting sufficiently transitively on X, i.e., in such a way that that (G, X) is a minimal system.

2. PRELIMINARIES

Let G be a topological group and X a compact space. An *action* of G on X is a homomorphism φ from G to the group $\mathrm{Aut}(X)$ of autohomeomorphisms of X. The action φ is *continuous* if the map

$$G \times X \to X; (g, x) \mapsto \varphi(g)(x)$$

is continuous. Typically we will not mention φ and write gx instead of $\varphi(g)(x)$.

A topological group G together with a topological space X and a continuous action of G on X is a *dynamical system*. X is the *phase space* of the system. For every $x \in X$ the set $Gx = \{gx : g \in G\}$ is the G-*orbit* of x. The dynamical system (G, X) is *minimal* if every G-orbit is dense in X.

For an infinite cardinal κ, the group G is κ-bounded if for every non-empty open subset O of G there is a set $S \subseteq G$ of size κ such that $SO = G$. Here SO denotes the set $\{gh : g \in S \wedge h \in O\}$.

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The cellularity of X is the least cardinal κ such that every family $\mathcal{O}$ of size $> \kappa$ of non-empty open subsets of X contains two distinct sets with a non-empty intersection.

3. Proof of the main result

Let X be a compact space. $C(X)$ denotes the space of continuous real valued functions on X equipped with the sup-norm $||\cdot||_\infty$. If G acts on X via φ, then the natural action of G on $C(X)$ is defined by letting $gf = f \circ \varphi(g)$. It is easily checked that G acts on $C(X)$ by isometries and that the action of G on $C(X)$ is continuous if the action on X is continuous.

The action of G on $C(X)$ provides us with a simple way of constructing G-equivariant quotients of X, i.e., quotients for which the quotient map commutes with the group actions. Let B be a closed subalgebra of $C(X)$ which is closed under the action of G on $C(X)$. Define an equivalence relation $\sim_B$ on X as follows:

For all $x, y \in X$ let $x \sim_B y$ iff for all $b \in B$, $b(x) = b(y)$. It is well-known that $X/\sim_B$ is Hausdorff. Since B is closed under the action of G, the action of G on X is compatible with $\sim_B$. Hence, there is a natural action of G on $X/\sim_B$. This action is continuous. $X/\sim_B$ is a G-equivariant quotient of X.

Definition 3.1. A continuous map $f : X \to Y$ between topological spaces is *semi-open* if for every non-empty open set $O \subseteq X$, $f[O]$ has a non-empty interior.

The following is well known.

Lemma 3.2. *Let (G, X) and (G, Y) be dynamical systems. Assume that $\pi : X \to Y$ is continuous, onto and G-equivariant, i.e., assume that π commutes with the actions. Suppose that (G, X) is a minimal system. Then π is semi-open.*

For the convenience of the reader we include a proof of this lemma.

Proof. Suppose $O \subseteq X$ is a non-empty open set. Let $U \subseteq O$ be a non-empty open set with $\mathrm{cl}_X U \subseteq O$. Since (G, X) is minimal, every G-orbit in X meets the set U. It follows that $GU = X$. Since X is compact, a finite number of translates of U covers X. It follows that a finite number of translates of $\pi[U]$ and hence of $\pi[\mathrm{cl}_X U]$ cover Y. Since the translates of $\pi[\mathrm{cl}_X U]$ are closed sets, one of them has a non-empty interior, by the Baire Category Theorem. It follows that $\pi[\mathrm{cl}_X U]$, and therefore $\pi[O]$, has a non-empty interior. $\square$

Lemma 3.3. *Let κ be an infinite cardinal. Suppose G is a κ-bounded group acting continuously on a metric space Z. Then every G-orbit in Z has a dense subset of size $\leq \kappa$.*

Proof. Let $z \in Z$. For every $n \in \omega$ let U_n be the open ball of radius $\frac{1}{2^n}$ around z. Since G acts continuously on Z, the map $G \to Z; g \mapsto gz$ is continuous. Thus, there is an open neighborhood V_n of the neutral element of G such that $V_n z \subseteq U_n$. Since G is κ-bounded, there is a set $S_n \subseteq G$ of size $\leq \kappa$ such that $S_n V_n = G$. Now $Gz = S_n V_n z \subseteq SU_n$. It is easily checked that $\bigcup_{n\in\omega} S_n z$ is dense in Gz. $\square$

In the following, we use elementary submodels of $\mathcal{H}_\chi = (\mathcal{H}_\chi, \in)$ for some infinite cardinal χ. Here $\mathcal{H}_\chi$ denotes the set of all sets whose transitive closure is of size $< \chi$. The readers not familiar with the method of elementary submodels might consult [3], [4] or [6] for an introduction.

Fix a sufficiently large cardinal χ. Note that, for every cardinal κ, if M is an elementary submodel of $\mathcal{H}_\chi$ and $\kappa \subseteq M$, then for every set $S \in M$ which is of size κ, $S \subseteq M$ since M contains a bijection between κ and S.

Lemma 3.4. *Let Z be a metric space and suppose that a κ-bounded group acts continuously on Z. If M is an elementary submodel of $\mathcal{H}_\chi$ such that $\kappa \cup \{\kappa, Z, G\} \subseteq M$, then $\mathrm{cl}_Z(Z \cap M)$ is closed under the action of G.*

Proof. Let $z \in Z \cap M$. By Lemma 3.3, Gz has a dense subset D of size κ. M knows about this and hence we may assume $D \in M$. Since $\kappa \subseteq M$, $D \subseteq M$. It follows that $Gz \subseteq \mathrm{cl}_Z(Z \cap M)$.

Now let $z \in \mathrm{cl}_Z(Z \cap M)$. By the first part of the proof, $G(Z \cap M) \subseteq \mathrm{cl}_Z(Z \cap M)$. Hence

$$Gz \subseteq G\,\mathrm{cl}_Z(Z \cap M) = \mathrm{cl}_Z(G(Z \cap M)) \subseteq \mathrm{cl}_Z(Z \cap M).$$

□

Corollary 3.5. *Let (G, X) be a dynamical system such that G is κ-bounded. If M is an elementary submodel of size κ of $\mathcal{H}_\chi$ such that $\kappa \cup \{\kappa, X, G\} \subseteq M$, then $B = \mathrm{cl}_{C(X)}(C(X) \cap M)$ is a closed subalgebra of $C(X)$, which is closed under the action of G. In particular, $X/\sim_B$ is a G-equivariant quotient of X of weight $\leq \kappa$.*

Proof. By Lemma 3.4, B is closed under the action of G. It is easily checked that $C(X) \cap M$ is a subalgebra of $C(X)$. It follows that $B = \mathrm{cl}_{C(X)}(C(X) \cap M)$ is a closed subalgebra of $C(X)$.

Now $X/\sim_B$ is a G-equivariant quotient of X. $C(X/\sim_B)$ is isometrically isomorphic to B and therefore has a dense subset of size $\leq \kappa$. It follows that $X/\sim_B$ is of weight $\leq \kappa$. □

Theorem 3.6. *Let (G, X) be a minimal system and suppose that G is κ-bounded. Then the cellularity of X is at most κ.*

Proof. Let $\mathcal{A}$ be a maximal family of pairwise disjoint non-empty open subsets of X. We may assume that each $A \in \mathcal{A}$ is of the form $f_A^{-1}[\mathbb{R} \setminus \{0\}]$ for some continuous $f_A : X \to \mathbb{R}$. Let M be an elementary submodel of $\mathcal{H}_\chi$ of size κ such that $\kappa \cup \{\kappa, X, G, \mathcal{A}\} \subseteq M$. Let $B = \mathrm{cl}_{C(X)}(C(X) \cap M)$. By Corollary 3.5, $X/\sim_B$ is a G-equivariant quotient of X of weight $\leq \kappa$. Let $\pi : X \to X/\sim_B$ be the quotient map. By Lemma 3.2, π is semi-open. Note that $C(X/\sim_B)$ is isometrically isomorphic to B via the map

$$\cdot \circ \pi : C(X/\sim_B) \to B; f \mapsto f \circ \pi.$$

Claim 3.7. $\mathcal{A} \cap M$ is a maximal family of pairwise disjoint open sets in X.

Let $O \subseteq X$ be non-empty and open. Choose a non-empty open set $U \subseteq \pi[O]$. We may assume that U is of the form $f^{-1}[\mathbb{R} \setminus \{0\}]$ for some continuous $f : X/\sim_B \to \mathbb{R}$ with $f \circ \pi \in \mathrm{cl}_{C(X)}(C(X) \cap M)$.

Choose $n \in \omega$ so that $||f||_\infty - \frac{1}{n} > \frac{1}{n}$. Let $f_M : X/\sim_B \to \mathbb{R}$ be such that $f_M \circ \pi \in C(X) \cap M$ and $||f - f_M||_\infty < \frac{1}{n}$. Now

$$U_M = f_M^{-1}\left[\mathbb{R} \setminus \left(-\frac{1}{n}, \frac{1}{n}\right)\right] \subseteq U.$$

Note that $\pi^{-1}[U_M] = (f_M \circ \pi)^{-1}\left[\mathbb{R} \setminus \left(-\frac{1}{n}, \frac{1}{n}\right)\right]$ is an element of M.

Since M knows that A is a maximal family of disjoint open sets, there is $A \in \mathcal{A} \cap M$ such that $A \cap \pi^{-1}[U_M]$ is non-empty. Now $\pi[A] \cap \pi[O] \neq \emptyset$. By our assumption on $\mathcal{A}$, $A = f_A^{-1}[\mathbb{R} \setminus \{0\}]$ for some continuous function f. M knows about this and hence we can choose $f_A \in M$. The function f_A witnesses that for all $x, y \in X$ with $x \in A$ and $y \notin A$, $x \not\sim_B y$. Hence $\pi^{-1}[\pi[A]] = A$.

The set $\pi[A \cap \pi^{-1}[U_M]]$ is a nonempty subset of $\pi[O]$. It follows that there are $x \in A$ and $y \in O$ such that $\pi(x) = \pi(y)$. But $\pi^{-1}[\pi[A]] = A$ and therefore $y \in A$. It follows that $A \cap O \neq \emptyset$. This proves the claim.

Since $\mathcal{A} \cap M$ is already a maximal family of pairwise disjoint open subsets of X, $\mathcal{A} = \mathcal{A} \cap M$ and therefore $\mathcal{A} \subseteq M$. Since $|M| \leq \kappa$, $|\mathcal{A}| \leq \kappa$. □

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