INCOMPLETE METRIC SPACES AND BANACH'S FIXED POINT THEOREM

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Abstract. We show that the real line can be partitioned into two sets such that each contractive map from one of the sets to itself is constant. This answers a question of Behrends.

1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is *Lipschitz of constant* $c \geq 0$ if for all $x_0, x_1 \in X$, $|f(x_0) - f(x_1)| \leq c \cdot |x_0 - x_1|$. The mapping f is *contractive* if it is Lipschitz of some constant < 1. Banach's fixed point theorem says that in a complete metric space X, every contractive mapping from X to X has a fixed point.

It is natural to ask whether every metric space X with the property that every contractive mapping $f : X \to X$ has a fixed point is complete. It is well known that this is not the case. We even construct a subspace of $\mathbb{R}$ on which every contractive selfmap is constant.

2. Diagonalizing continuous mappings

Definition 2.1. Let $\mathcal{F}$ be the family of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which one of the following holds:

(1) f is the identity;
(2) there is $a \in \mathbb{R}$ such that $f \restriction (-\infty, a)$ is constant and $f \restriction (a, \infty) = \mathrm{id}_{(a,\infty)}$;
(3) there is $b \in \mathbb{R}$ such that $f \restriction (b, \infty)$ is constant and $f \restriction (-\infty, b) = \mathrm{id}_{(-\infty,b)}$;
(4) there are $a, b \in \mathbb{R}$ such that $a < b$, $f \restriction (-\infty, a)$ and $f \restriction (b, \infty)$ are constant, and $f \restriction (a, b) = \mathrm{id}_{(a,b)}$.

Lemma 2.2. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and not in $\mathcal{F}$. Then there is an open interval $I \subseteq \mathbb{R}$ such that $f \restriction I$ is not constant and for all $x \in I$, $f(x) \neq x$.*

Proof. The set $D = \{x \in \mathbb{R} : f(x) = x\}$ is closed. Suppose there exists a finite maximal open interval I contained in the complement of D. Then by continuity, f is not constant on I.

Now suppose every maximal open interval contained in the complement of D is infinite. Then, since $f \notin \mathcal{F}$, there is one such interval I such that f is not constant on I. $\square$

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Theorem 2.3. *$\mathbb{R}$ can be partitioned into two sets X_0 and X_1 such that for every $i \in 2$ and every continuous function $f : \mathbb{R} \to \mathbb{R}$, if $f[X_i] \subseteq X_i$, then $f \in \mathcal{F}$.*

Proof. Let $(i_\alpha, f_\alpha)_{\alpha<2^{\aleph_0}}$ be an enumeration of all pairs (i, f) where $i \in 2$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function not in $\mathcal{F}$. Inductively we construct sequences $(x_0^\alpha)_{\alpha<2^{\aleph_0}}$ and $(x_1^\alpha)_{\alpha<2^{\aleph_0}}$ of real numbers such that for all $\alpha < 2^{\aleph_0}$ the following hold:

(1) x_0^α and x_1^α are distinct and not contained in $\{x_i^\beta : \beta < \alpha \wedge i \in 2\}$;
(2) $f_\alpha(x_i^\alpha) = x_{1-i}^\alpha$.

Suppose we can find $(x_0^\alpha)_{\alpha<2^{\aleph_0}}$ and $(x_1^\alpha)_{\alpha<2^{\aleph_0}}$ as above. Let $X_0 = \{x_0^\alpha : \alpha < 2^{\aleph_0}\}$ and $X_1 = \mathbb{R} \setminus X_0$. By (1), $\{x_1^\alpha : \alpha < 2^{\aleph_0}\} \subseteq X_1$.

If a continuous function $f : \mathbb{R} \to \mathbb{R}$ is not in $\mathcal{F}$ and if $i \in 2$, then there is $\alpha < 2^{\aleph_0}$ such that $f = f_\alpha$ and $i = i_\alpha$. Now $x_i^\alpha \in X_i$, $x_{1-i}^\alpha \in X_{1-i}$, and

$$f(x_i^\alpha) = f_\alpha(x_{i_\alpha}^\alpha) = x_{1-i_\alpha}^\alpha = x_{1-i}^\alpha.$$

In particular, $f[X_i] \not\subseteq X_i$.

It remains to construct the sequences $(x_0^\alpha)_{\alpha<2^{\aleph_0}}$ and $(x_1^\alpha)_{\alpha<2^{\aleph_0}}$. Supposed we have constructed $(x_0^\beta)_{\beta<\alpha}$ and $(x_1^\beta)_{\beta<\alpha}$ for some $\alpha < 2^{\aleph_0}$. Let $i = i_\alpha$ and $f = f_\alpha$. Since $f \notin F$, by Lemma 2.2, there is a nonempty open interval I such that $f \restriction I$ is not constant $f(x) \neq x$ for all $x \in I$.

Since $f \restriction I$ is not constant, there is a nonempty open interval $J \subseteq f[I]$. The set

$$A = (I \cap f^{-1}[J \setminus \{x_j^\beta : \beta < \alpha \wedge j \in 2\}]) \setminus \{x_j^\beta : \beta < \alpha \wedge j \in 2\}$$

is of size $2^{\aleph_0}$. Pick $x_i^\alpha \in A$ and let $x_{1-i}^\alpha = f(x_i^\alpha)$. It is easily checked that x_i^α and x_{1-i}^α have the desired properties. $\square$

3. A set without nontrivial contractions

We show that the sets X_0 and X_1 have no non trivial contractive selfmaps. In order to prove this, we need some facts about the extendibility of Lipschitz functions.

Lemma 3.1. *Let X be a nonempty closed subset of $\mathbb{R}$, $f : X \to \mathbb{R}$, and $c \geq 0$ such that for all $x, y \in X$ with $x \neq y$,*

$$|f(x) - f(y)| \leq c \cdot |x - y|.$$

In other words, suppose that f is Lipschitz of constant c. Then there is $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ with $x \neq y$,

$$|\overline{f}(x) - \overline{f}(y)| \leq c \cdot |x - y|.$$

Proof. Since X is closed, every point in the complement of X is contained in an unique open interval which is maximal with the property that it is disjoint from X. Let I be such an interval.

I can be infinite, but since X is nonempty, I has at least one endpoint a. Note that $a \in X$ by the maximality of I. If I is infinite, then for each $x \in I$ let

$\overline{f}(x) = f(a)$. If I is finite, then I has another endpoint $b \in X$. We may assume $a < b$. For $x \in I$ let

$$\overline{f}(x) = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

It is easily checked that $\overline{f}$ has the desired properties. □

Lemma 3.2. *Let $X \subseteq \mathbb{R}$ be a set. Let $f : X \to \mathbb{R}$ a Lipschitz function of constant c. Then f can be continuously extended to the closure of X, with the same Lipschitz constant.*

Proof. Since f is Lipschitz, it maps every Cauchy sequence in X to a Cauchy sequence in $\mathbb{R}$. This implies that f extends to a continuous function $\overline{f} : \mathrm{cl}(X) \to \mathbb{R}$. The computation of the Lipschtz constant of $\overline{f}$ is straight forward. □

Combining these two lemmas we get

Corollary 3.3. *Let $X \subseteq \mathbb{R}$ be a set. If $f : X \to \mathbb{R}$ is Lipschitz of constant $c \geq 0$, then f can be extended to a function $\overline{f} : \mathbb{R} \to \mathbb{R}$ which is Lipschitz of constant c.*

Using Theorem 2.3 we obtain

Theorem 3.4. *$\mathbb{R}$ can be partitioned into two sets X_0 and X_1 such that for every $i \in 2$ every contraction $f : X_i \to X_i$ is constant.*

Proof. Let X_0 and X_1 be as in Theorem 2.3. Let $i \in 2$. Let $f : X_i \to X_i$ be a contraction. Then f can be extended to a contraction $\overline{f} : \mathbb{R} \to \mathbb{R}$ by Corollary 3.3. Since $\overline{f}[X_i] \subseteq X_i$, by the choice of X_i, $\overline{f} \in \mathcal{F}$. But a contraction is in $\mathcal{F}$ only if it is constant. □

It it worth pointing out that X_0 and X_1 actually have the following stronger property:

For every $i \in 2$, if $f : X_i \to X_i$ is such that for all $x, y \in X_i$ with $x \neq y$ we have

$$|f(x) - f(y)| < |x - y|,$$

then f is constant.

For suppose $f : X_i \to X_i$ is such that for all $x, y \in X_i$ with $x \neq y$,

$$|f(x) - f(y)| \leq |x - y|.$$

Then f extends to a function $\overline{f} : \mathbb{R} \to \mathbb{R}$ which is Lipschitz of constant 1. Since $\overline{f}[X_i] \subseteq X_i$, $f \in \mathcal{F}$. If f is not constant, then there are $a, b \in f[X_i] \subseteq X_i$ with $a \neq b$. Since $f \in \mathcal{F}$, $f(a) = a$ and $f(b) = b$. In particular

$$|a - b| = |f(a) - f(b)|.$$

This shows the strengthening of Theorem 3.4

Moreover, it is unneccesary to use Lemma 3.2 in the proof of Theorem 3.4 since it is easily checked that the sets X_0 and X_1 are dense.

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