Derived Deformation Theory: An Introduction

Severin Bunk

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Abstract

This is the introduction and overview talk for a research seminar on derived deformation theory. We give a very quick tour of some of the main principles of derived deformation theory, starting from very basic notions. We introduce the idea of a deformation problem, observe that it is possible to treat infinitesimal deformations of algebraic and geometric structures on the same footing, and motivate the concept of the tangent space of a deformation problem. After very briefly invoking some examples (to be covered in detail in the seminar) in which differential graded Lie algebras (dglas) play a prominent role, we change perspective and associate deformation problems to a dgla. This naturally leads us to considering space-valued functors on Artinian dg algebras as the most general deformation problems – these are Lurie's (derived) formal moduli problems. We can then state and appreciate the classification theorem for formal $\mathbb{E}_\infty$-moduli problems, which is the main objective of this seminar.

Contents

1 Moduli problems and deformation theory

Deformation theory is the study of families of mathematical objects of a given type. What this actually means, though, will depend on what we mean by a 'family'.

Example 1.1 As a very simple example, consider a manifold[1] M, and let us try to study families of points $x \in M$. First, one can just view all points $x \in M$ individually, merely acknowledging their existence. From this point of view, the moduli object of points in M would be the underlying *set* UM of M.

However, since M has got much more structure than its underlying set, we can say when elements of UM belong to a particularly good type of family: a family of points of M, parameterised by a manifold P, is simply a smooth map $f\colon P \to M$. These families have yet more structure when taken together: they assemble into a functor

$$\underline{M}\colon \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}\,, \qquad P \mapsto \mathcal{M}\mathrm{fd}(P, M)\,.$$

[1] All our manifolds will be smooth.

That is, they form a presheaf on the category of manifolds. In fact, M can be reconstructed from this presheaf by an application of the Yoneda Lemma. Thus, the manifold M and, equivalently, the presheaf $\underline{M}$ contain all information about smooth families of points in M. $\triangleleft$

From the point of view of moduli problems, this is a (if not 'the') perfect-world example and even almost tautological, but it illustrates well the idea of families over parameter spaces and of their interplay as we change the parameter space.

However, we would like a more robust and general theory of deformations; for instance, in Example 1.1 we cannot treat deformations of structures that make no reference to smoothness or a topology.

Example 1.2 How should we treat deformations of an algebra (A, μ) over a field k, without assuming that the underlying vector space A is topological and that μ is continuous? Here one understands a deformation of the algebra structure as a polynomial (or power series) $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots$ with coefficients in $\mathcal{M}\mathrm{od}_k(A \otimes A, A)$ such that, for each $\lambda \in k$, the map μ_λ makes A into an algebra. $\triangleleft$

Observe that here we have the following features:

(1) an arbitrary field k (in Example 1.1 we are forced to use $k = \mathbb{R}$ or $k = \mathbb{C}$) and

(2) a degree of deformation given by the highest power of t.

Question 1.3 *Is there a way to treat the deformations/families of Example 1.1 and of Example 1.2 on the same footing?* $\triangleleft$

We cannot describe the algebraic deformations in Example 1.2 in a smooth way, but we can describe the geometric deformations in Example 1.1 in an algebraic way: to that end, we enlarge our category of manifolds to a category of (locally) ringed spaces. These are pairs $M = (|M|, \mathcal{O}_M)$ of a topological space $|M|$ and a sheaf $\mathcal{O}_M$ of (local) rings on M. A morphism $(|M|, \mathcal{O}_M) \to (|N|, \mathcal{O}_N)$ is a pair (f, ϕ) of a continuous map $f\colon |M| \to |N|$ and a morphism of sheaves of (local) rings $\phi\colon \mathcal{O}_N \to f_*\mathcal{O}_M$. Any smooth manifold M is a locally ringed space via its underlying topological space and its sheaf of smooth $\mathbb{R}$-valued functions.

Example 1.4 Consider the locally ringed space $\mathbb{D}^1 = (*, \mathbb{R}[\epsilon]/\epsilon^2)$. It can be though of as the point $* \in \mathcal{T}\mathrm{op}$ with an additional, *infinitesimal* direction. This is justified by the observation that, for a manifold M, a map $\mathbb{D}^1 \to M$ is equivalently a pair (x, v), where $x\colon * \to M$ is a point in M and where $v\colon C^\infty(M, \mathbb{R}) \to \mathbb{R}[\epsilon]/\epsilon^2$ is a morphism of rings. For $f \in C^\infty(M, \mathbb{R})$, we have that v factors through the germ of $\mathcal{O}_M$ at x and that

$$v(f) = v_0(f) + \epsilon\, v_1(f)\,.$$

The fact that v is a morphism of rings implies that

(1) v_0 is a ring homomorphism, and hence[2] given by evaluation at $x \in M$, and

(2) v_1 is an $\mathbb{R}$-valued derivation, and hence equivalently a tangent vector at $x \in M$.

In other words, the data $(x, v)\colon \mathbb{D}^1 \to M$ corresponds uniquely to a tangent vector $V \in T_xM$. $\triangleleft$

Thus, a morphism $\mathbb{D}^1 \to M$ is a first-order deformation of a point $x \in M$. Morphisms $(*, \mathbb{R}[\epsilon]/\epsilon^n) \to M$ can be understood as higher-order deformations. Generally, morphisms of this type can be understood as *infinitesimal deformations* of points in M. In fact, this interpretation makes sense for all ringed spaces $(*, A)$, where $A \in \mathcal{A}\mathrm{rt}_\mathbb{R}$ is a finite-dimensional local $\mathbb{R}$-algebra with maximal ideal $\mathfrak{m}_A$

[2] $\ker(v_0)$ is a prime ideal since $\mathbb{R}$ is a field, and prime ideals in $C^\infty(M, \mathbb{R})$ correspond to points in M. This point must then be x since v_0 factors through the germ at x.

and residue field $A/\mathfrak{m}_A \cong \mathbb{R}$. These algebras satisfy[3] $(\mathfrak{m}_A)^n = \{0\}$ for $n \gg 0$, so that we can view the space $(*, A)$ as a point with infinitesimal directions controlled by $\mathfrak{m}_A$.

Morphisms $(*, A) \to (*, B)$ are in bijection with ring morphisms $B \to A$ in the opposite direction. Thus, the functoriality of our presheaf $\underline{M}$ from Example 1.1 changes: when we first extend the functor to ringed spaces and then restrict to spaces of the form $(*, A)$, where $A \in \mathcal{A}\mathrm{rt}_{\mathbb{R}}$, we are left with a functor

$$\underline{M}\colon \mathcal{A}\mathrm{rt}_{\mathbb{R}} \to \mathcal{S}\mathrm{et}\ .$$

Definition 1.5 *A functor $X\colon \mathcal{A}\mathrm{rt}_k \to \mathcal{S}\mathrm{et}$ is called a* $\mathcal{S}\mathrm{et}$-valued deformation functor (over k)*, or a* classical moduli problem over k *if it satisfies*

(1) for every surjective morphisms $A' \to A \leftarrow A''$ the resulting morphism

$$X\big(A' \underset{A}{\times} A''\big) \longrightarrow X(A') \underset{X(A)}{\times} X(A'')$$

is surjective, and

(2) if $A' = k[\epsilon]/\epsilon^2$ and $A = k$, the morphism is bijective.

Deformation functors allow to study (infinitesimal) deformations of a wide variety of objects; for instance, this approach unifies Example 1.1 and Example 1.2. We will see motivating examples of such functors in detail in both a purely algebraic framework in Talk 2 and in a geometric setting in Talk 3. In both cases, we will see that deformations are related to cohomology theories (Hochschild and Kodaira-Spencer). The (classical) formal study of deformation functors will be the subject of Talk 4.

2 The tangent space and the presence of dglas

Definition 2.1 *Let $F\colon \mathcal{A}\mathrm{rt}_k \to \mathcal{S}\mathrm{et}$ be a classical moduli problem over k. Motivated by Example 1.4 we call the value $T_F := F(k[\epsilon]/\epsilon^2) \in \mathcal{S}\mathrm{et}$ the* tangent space *of the moduli problem F.*

Example 2.2 An example with both algebraic and geometric flavour is the following. Consider the functor

$$SL_2\colon \mathcal{R}\mathrm{ing} \to \mathcal{S}\mathrm{et}\,, \qquad SL_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2{\times}2, R) \;\middle|\; ad - bc = 1 \right\}.$$

This is not yet a deformation functor; heuristically, the functor SL_2 is defined on all rings (hence on schemes, i.e. 'extended/non-infinitesimal' parameter spaces), rather than just Artinian algebras over a field k (which we view as infinitesimal test spaces, see Example 1.4). Deformations are taken around a chosen point, according to the ideas from Section 1. A *point* in SL_2 is a pair $x = (k, \eta)$ of a field k and an element $\eta \in SL_2(k)$, which we take to be the unit $2{\times}2$-matrix $\mathbb{1}$ over k. We obtain a functor

$$SL_{2,x}\colon \mathcal{A}\mathrm{rt}_k \to \mathcal{S}\mathrm{et}\,, \qquad SL_{2,x} = \{\mathbb{1}\} \underset{SL_2(k)}{\times} SL_2(-)\,,$$

$$SL_{2,x}(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2{\times}2, A) \;\middle|\; ad - bc = 1,\ p(a) = p(d) = 1,\ p(b) = p(c) = 0 \right\}.$$

Its tangent space reads as

$$T_{SL_{2,x}} = SL_{2,x}(\mathbb{D}^1) = \left\{ \begin{pmatrix} 1 + r\epsilon & s\epsilon \\ u\epsilon & 1 + v\epsilon \end{pmatrix} \in M(2{\times}2, \mathbb{D}^1) \;\middle|\; 1 = (1 + r\epsilon)(1 + v\epsilon) = 1 + (r + v)\epsilon \right\}$$

[3]This is a consequence of the Nakayama Lemma.

$$
\begin{aligned}
&= \left\{ \begin{pmatrix} r & s \\ u & v \end{pmatrix} \in M(2{\times}2, k) \;\middle|\; r + v = 0 \right\} \\
&\cong \mathfrak{sl}_2(k)\,.
\end{aligned}
$$

This is the vector space we expect to obtain as the tangent space to SL_2 at the unit element; for instance, for $k = \mathbb{R}$ it is the vector space underlying the Lie algebra of the Lie group $SL_2(\mathbb{R})$. $\triangleleft$

We will see that by the axioms of a moduli problem, T_X comes endowed with the structure of a k-vector space. For deformations of a k-algebra A it turns out that T_X is related to the Hochschild cohomology of A. For deformations of complex structures on a manifold/variety M, the tangent space T_X is related to the cohomology of the Gerstenhaber complex of polyvector fields on M. This will be shown in Talk 2 and Talk 3. An important observation is that the cochain complexes underlying both these cohomologies carry additional structure: they form differential graded Lie algebras (dglas). This structure turns out to be abundant in commutative deformation problems; why will become clear when we follow Lurie's treatment of those deformation problems.

3 Moduli problems from differential graded Lie algebras

Motivated by the above examples, we start to investigate dglas in more detail; this will be the main subject of Talk 5.

Definition 3.1 *Let k be a field of characteristic zero. A* differential graded Lie algebra *over k is a graded k-vector space $L = \bigoplus_{i\in\mathbb{Z}} L^i$ together with a linear map $\mathrm{d}\colon L \to L[-1]$ and $[-,-]\colon L \otimes L \to L$ such that*

(1) (L, d) is a cochain complex,

(2) the bracket $[-,-]$ is graded antisymmetric, $[a, b] = -(-1)^{|a||b|}[b, a]$,

(3) $[-,-]$ satisfies the (graded) Jacobi identity $[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[c,a]]$,

(4) d is a (graded) derivation for $[-,-]$: $\mathrm{d}[a,b] = [\mathrm{d}a, b] + (-1)^{|b|}[a, \mathrm{d}b]$.

Definition 3.2 *Let $(L, \mathrm{d}, [-,-])$ be a dgla over k. An element $a \in L^1$ is called a* Maurer-Cartan element *of L if it satisfies*

$$\mathrm{d}a = \frac{1}{2}[a, a] = 0\,.$$

We denote the set of Maurer-Cartan elements of L by $\mathrm{MC}(L)$.

Lemma 3.3 *Let $(L, \mathrm{d}_L, [-,-])$ be a dgla over k, and let (C, d_C) be a (not necessarily unital) commutative differential graded algebra (cdga) over k. Then, there is a new dgla $L \otimes C$, with underlying graded vector space given by*

$$(L \otimes C)^n = \bigoplus_{i,j\in\mathbb{Z},\, i+j=n} L^i \otimes C^j\,,$$

differential given by

$$\mathrm{d}(\ell \otimes c) = \mathrm{d}_L\ell \otimes c + (-1)^{|\ell|}\ell \otimes \mathrm{d}_C c$$

on pure vectors, and with bracket defined by

$$[\ell \otimes c, \ell' \otimes c'] = [\ell, \ell'] \otimes cc'\,.$$

As it turns out, every dgla gives rise to a deformation functor via its theory of Maurer-Cartan elements. One idea to obtain a functor $\mathcal{A}\mathrm{rt}_k \to \mathcal{S}\mathrm{et}$ from $(L, \mathrm{d}, [-,-])$ is to set

$$\widetilde{F}(L)(A) \coloneqq \mathrm{MC}(L \otimes \mathfrak{m}_A) = \mathrm{Span}_k\{\ell \otimes r \,|\, \ell \in L^1, r \in \mathfrak{m}_A,\, \mathrm{d}\ell \otimes r + \frac{1}{2}[\ell,\ell] \otimes r^2 = 0\}\,.$$

We would like to see whether this gives a sensible tangent space; thus, we compute

$$T_{\widetilde{F}(L)} = \widetilde{F}(L)(\mathbb{D}^1) = \mathrm{MC}(L \otimes k\epsilon) \cong \{\ell \in L^1 \,|\, \mathrm{d}\ell = 0\} = Z^1(L)\,.$$

This does not quite match our expectations, since in the examples before, the tangent space was always the first cohomology group, rather than the vector space of cocycles. In order to obtain $H^1(L)$ as the tangent space of a moduli problem $F(L)$ associated to L, we need to have elements $\ell \in L^0$ of degree zero appear in the computation of $T_{F(L)}$.

Now consider the cdga $\Omega^\bullet(\Delta^1)$, whose underlying vector space is $k[t] \oplus k[t]\mathrm{d}t[1]$, i.e. the element $\mathrm{d}t$ has degree 1. Its differential reads as

$$\mathrm{d}_\Omega(p_0(t) + p_1(t)\mathrm{d}t) = \partial_t p_0(t)\mathrm{d}t\,.$$

Consider the dgla

$$L \otimes \Omega^\bullet(\Delta^1) \cong \{\ell_0(t) \otimes \ell_1(t)\mathrm{d}t \,|\, \ell_0, \ell_1 \in L[t] = L \otimes_k k[t]\}\,.$$

We may write the differential as

$$\mathrm{d}(\ell_0(t) \otimes \ell_1(t)\mathrm{d}t) = \mathrm{d}_L\ell_0(t) + \big(-\partial_t\ell_0(t) + \mathrm{d}_L\ell_1(t)\big)\mathrm{d}t\,.$$

Here, d_L acts only on the coefficients of polynomials, and the minus sign stems from the fact that we need to move the differential past the L-factors in order to differentiate the polynomial factors.

Observe each $\lambda \in k$ induces a morphism

$$e_\lambda\colon \widetilde{F}(L \otimes \Omega^\bullet(\Delta^1)) \longrightarrow \widetilde{F}(L)\,,$$

given by evaluating the polynomial ℓ_0 at λ.

Let us have a look at the tangent space of $\widetilde{F}(L \otimes \Omega^\bullet(\Delta^1))$: we have

$$\begin{aligned}
T_{\widetilde{F}(L\otimes\Omega^\bullet(\Delta^1))} &= \mathrm{MC}\big(L \otimes \Omega^\bullet(\Delta^1) \otimes k\epsilon\big)\\
&= \big\{\ell_0(t) + \ell_1(t)\mathrm{d}t \,\big|\, \ell_0 \in L^1[t], \ell_1 \in L^0[t],\, 0 = \mathrm{d}\big(\ell_0(t) + \ell_1(t)\mathrm{d}t\big)\big\}\\
&= \big\{\ell_0(t) + \ell_1(t)\mathrm{d}t \,\big|\, \ell_0 \in L^1[t], \ell_1 \in L^0[t],\, \mathrm{d}_L\ell_0(t) = 0,\, \mathrm{d}_L\ell_1(t) = \partial_t\ell_0(t)\big\}\,.
\end{aligned}$$

Proposition 3.4 *We have*

$$\frac{T_{\widetilde{F}(L)}}{(e_1 - e_0)(T_{\widetilde{F}(L\otimes\Omega^\bullet(\Delta^1))})} = H^1(L)\,.$$

Proof. We already know that $T_{\widetilde{F}(L)} = Z^1(L)$. We thus need to show that $\mathrm{im}(e_1 - e_0) = \mathrm{im}(\mathrm{d}) = B^1(L)$. First, we prove the inclusion $\mathrm{im}(e_1 - e_0) \supset \mathrm{im}(\mathrm{d})$. To see this, let $\ell \in L^0$, and consider the element $\mathrm{d}_L\ell\, t + \ell\,\mathrm{d}t \in L \otimes \Omega^\bullet(\Delta^1)$. By construction, it gives rise to an element in $T_{\widetilde{F}(L\otimes\Omega^\bullet(\Delta^1))}$, and it satisfies

$$(e_1 - e_0)(\mathrm{d}_L\ell\, t + \ell\,\mathrm{d}t) = \mathrm{d}_L\ell\,.$$

To show that $\mathrm{im}(e_1 - e_0) \subset \mathrm{im}(\mathrm{d})$, we use a Poincaré Lemma argument: let $\ell(t) = \ell_0(t) + \ell_1(t)\,\mathrm{d}t$ lie in the tangent space of $\widetilde{F}(L \otimes \Omega^\bullet(\Delta^1))$. We compute

$$\begin{aligned}
\mathrm{d}_L\Big(\int_0^1 \ell_1(t)\mathrm{d}t\Big) &= \int_0^1 \mathrm{d}_L \ell_1(t)\mathrm{d}t \\
&= \int_0^1 \partial_t \ell_0(t)\mathrm{d}t \\
&= \ell_0(1) - \ell_0(0)\,.
\end{aligned}$$

This proves the claim. □

A good candidate for a deformation problem associated to L would thus be the quotient

$$F(L) = \widetilde{F}(L)/\sim\,,$$

where $\sim$ is the equivalence relation generated by $\ell \sim \ell'$ in $\widetilde{F}(L)(A)$ if there exists an element $\widehat{\ell} \in \widetilde{F}(L \otimes \Omega^\bullet(\Delta^1))(A)$ such that $\ell' = e_1(\widehat{\ell})$ and $\ell = e_0(\widehat{\ell})$. For more details, we refer to [Man99]. This leads us to two crucial observations:

Remark 3.5 By construction, the values of $F(L)$ look like the connected components $\pi_0(X_L)$ of a simplicial set X_L. As we know, the standard simplices $\Delta^\bullet$ form a cosimplicial object in topological spaces, or even in manifolds with corners (or one can even consider extended affine versions to obtain nice varieties or manifolds). Taking cdgas $\Omega^\bullet$ on them, one would hope to obtain a simplicial cdga. This is indeed true, and it implies that *one can promote $\widetilde{F}(L)$ to a functor valued in $\mathcal{S}\mathrm{et}_\Delta$* [Hin01, Get09]. ◁

Remark 3.6 Despite all this work, we are currently only using elements of $L^{\leq 1}$ in the formation of then tangent space associated to L, even after extending $\Omega^\bullet(\Delta^1)$ to $\Omega^\bullet(\Delta^\bullet)$. The reason is that the Maurer-Cartan equation makes sense only for elements of degree -1, and the elements of degree n of the relevant dgla $L \otimes \Omega^\bullet(M) \otimes k\epsilon$ (for any M) are spanned by elements of the form

$$\ell = \sum_{i \in \mathbb{N}_0} \ell_i \otimes \omega_i \otimes \epsilon\,,$$

where $|\omega_i| = i$ and, consequently, $|\ell_i| = n - i$.

The solution to this is as simple as it is powerful: we observe that everything changes if we give ϵ itself a degree. Then elements of degree n suddenly become

$$\ell = \sum_{i \in \mathbb{N}_0} \ell_i \otimes \omega_i \otimes \epsilon\,,$$

where $|\omega_i| = i$ and, consequently, $|\ell_i| = n + |\epsilon| - i$. Hence, the highest degree of L-factors is $|\ell_0| = 1 - |\epsilon|$, and the exact same computations as we have done above go through analogously and yield

$$\frac{\widetilde{F}(L)(k \oplus k[\epsilon]/\epsilon^2)}{(e_1 - e_0)\big(\widetilde{F}(L \otimes \Omega^\bullet(\Delta^1))(k \oplus k[\epsilon]/\epsilon^2)\big)} \cong H^{1-|\epsilon|}(L)\,.$$

Thus, allowing differential graded Artin k-algebras as the domain of our deformation functors, we suddenly detect the entire cohomology of L. ◁

4 Derived formal moduli problems

Motivated by Remark 3.5 and Remark 3.5 we extend the domain and codomain of our moduli problems. Following Lurie [Lur10, Lur11], we work in an ∞-categorical framework and enlarge the ∞-category of cochain complexes of k-modules (cdgas over k) to the ∞-category $\mathcal{M}\mathrm{od}_k$ of k-module spectra (the ∞-category $\mathcal{C}\mathcal{A}\mathrm{lg}_k$ of $\mathbb{E}_\infty$-objects in $\mathcal{M}\mathrm{od}_k$). The analogue of $\mathcal{A}rt$ is denoted $\mathcal{C}\mathcal{A}\mathrm{lg}_{k,sm}$, and is developed in detail in [Lur10, Lur11]. We write $k \oplus k[n]$ for the square-zero extension of a field (or ring) k by a parameter ϵ of degree n. Let $\mathcal{S}$ denote the ∞-category of spaces.

Definition 4.1 [Lur10, Def. 4.6] *Let k be a field. A* formal moduli problem over k *is a functor $X\colon \mathcal{C}\mathcal{A}\mathrm{lg}_{k,sm} \to \mathcal{S}$ satisfying the following properties:*

(1) $X(k) \simeq *$.

(2) If $A' \to A \leftarrow A''$ are morphisms in $\mathcal{C}\mathcal{A}\mathrm{lg}_{k,sm}$ such that the induced morphisms $\pi_0 A' \to \pi_0 A \leftarrow \pi_0 A''$ are surjective, then the canonical morphism

$$X\big(A' \underset{A}{\times} A''\big) \longrightarrow X(A') \underset{X(A)}{\times} X(A'')$$

is a homotopy equivalence.

This defines an ∞-category FMP_k.

It follows that if $X \in \mathrm{FMP}_k$, then $\pi_0 X$ is a classical moduli problem (Def. 1.5). One defines the tangent space of $X \in \mathrm{FMP}_k$ as $T_X(0) = X(k \oplus k[0])$. A crucial observation is that $T_X(0)$ is part of a spectrum $T_X \in \mathcal{M}\mathrm{od}_k$ with $T_X(n) = X(k \oplus k[n])$.

The main objective of this seminar – apart from familiarising ourselves with the ideas of deformation theory and learn lots of new mathematics – is to prove the following theorem, which gives a fully precise formulation of our (vague) observation that dglas are strongly related to moduli problems over k (in the commutative case).

Theorem 4.2 [Lurie, Pridham] *Let k be a field of characteristic zero. There is an equivalence of ∞-categories $\Psi\colon \mathrm{FMP}_k \to \mathcal{L}\mathrm{ie}_k^{\mathrm{dg}}$. The underlying k-module spectrum of $\Psi(X)$ is $T_X[-1]$.*

The reason that $T_X[-1]$ should carry a Lie algebra structure can now be seen very elegantly [Lur10]: Since $X(k) \simeq *$ and since every (unital) k-algebra A comes with a canonical morphism $k \to A$, each space $X(A)$ is canonically pointed (up to contractible choices). Hence, it makes sense to form the new moduli problem $\Omega X = \Omega \circ X$, where $\Omega\colon \mathcal{S}_* \to \mathcal{S}_*$ is the based loop space functor. One can now observe that there is an equivalence

$$T_{\Omega X} \simeq T_X[-1]\,.$$

Since ΩX is valued in group-like loop spaces, we should expect its tangent space to carry a Lie algebra structure; compare Example 2.2.

In fact, in [Lur11] Lurie deduces Theorem 4.2 after developing a very general framework for deformation problems. After a transitional Talk 6, the Talks 7–10 will survey this general theory of deformations. (Talk 9 is a small aside and covers the theory of ∞-toposes following [Lur09, Ch. 6], which is partially used in the proof of 4.2.) In Talks 11 and 12 we will specialise to deformation problems in the commutative world and prove Theorem 4.2 as an application of the general formalism we learnt. The key concept here is that of Koszul duality, which relates pairs of algebraic structures – for example, it relates commutative algebras and Lie algebras. The final talks will deal with further applications of the general formalism, for instance in the non-commutative world of $\mathbb{E}_n$-algebras for $n < \infty$.

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