# On $E_n$-operads and loop spaces

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**Abstract**

We will introduce the topological $E_n$-operads and relate them to loop spaces via May's delooping theorem, following [2]. We also briefly state how to obtain the $E_n$ $\infty$-operads.

# Contents



# 1 Prerequisites

This section introduces the loop space and suspension functors. Further the Dold-Kan correspondence is introduced and used to construct infinite loop spaces.

## 1.1 Loop spaces and suspension

Let $(X,x),(Y,y)$ be elements of the category $Top_*$ of based topological spaces. The loop space $\Omega X$ of $X$ is the set of continuous maps

$$\Omega X := Top_*(\mathbb{S}^1, X)$$

equipped with the compact open topology. The (reduced) suspension $\Sigma X$ of $X$ is given

$$\Sigma X := X \times [0,1]/(X \times 0,1 \cup \{x\} \times [0,1])$$

which is homotopy equivalent to the smash product

$$\Sigma X \simeq X \wedge \mathbb{S}^1.$$

The suspension has the nice property, that

$$\Sigma \mathbb{S}^n \simeq \mathbb{S}^{n+1}.$$

To show that there is an adjuntion $\Sigma \dashv \Omega$, consider the inverse maps

$$\phi : T(X, \Omega Y) \to T(\Sigma X, Y),$$

$$\phi(f)[(x,s)] = f(x,s)$$

and

$$\psi : T(\Sigma X, Y) \to T(X, \Omega Y).$$

$$\psi(g)(x,s) = g[(x,s)]$$

The second map $\psi$ is clearly well defined, checking that for the first one is not hard. For example $[x,s] = [x,0]$ for all $s \in [0,1]$ and

$$f(x,0) = y = f(x,s),$$

for $f \in T(X, \Omega Y)$ because the map preserves base points.
Next we want to study the effect of the loop space on the homotopy groups of X. Recall: The n-th homotopy group of X has the underlying set of homotopy classes of based maps $\mathbb{S}^n \to X$. The problem at this point is that the adjunction relates the sets of morphisms and not homotopy classes of morphisms. It turns out is does, by arguments involving the internal hom from the circle, smash product with the circle adjunction in $Top_*$, when X is Hausdorff. For all details, see corollary 2.8 in [3]. One obtains the for the classes

$$[f] \in [(\mathbb{S}^n, \Omega X)] = \pi_n(\Omega X) \sim [\phi(f)] \in [(\mathbb{S}^{n+1}, \Omega X)] = \pi_{n+1}(X).$$

Excercise: Show this isomorphisms respects the group structure.
The unit of the adjunctions gives a map

$$X \to \Omega\Sigma X$$

we will later repeatedly use.
At this point we can ask for examples of loop spaces, iterated loop spaces and also infinite loop spaces. Of course, given any space $(X, x)$, we can form $\Omega^n X$. and even $\Omega^\infty\Sigma^\infty := \lim\limits_{\to k} \Omega^k \Sigma^k X$, where we use the maps $\Omega^n\Sigma^n \to \Omega^{n+1}\Sigma^{n+1}$ obtained from the adjunction unit. Also, recall that the direct limit in Top can be obtained by taking the direct limit in Set with the final topology from the maps into the colimit, i.e. the finest topology so that there are continuous. An easy example, which is already an infinite loop space is the circle $\mathbb{S}^1$, because

$$\mathbb{S}^1 \simeq K(\mathbb{Z}, 1).$$

We call a space X the Eilenberg-MacLane $K(G,n)$ if

$$\pi_k(X) = \begin{cases} G, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases}$$

It turns out these exist if $G$ is group and $n = 1$ or $G$ is an abelian group and $n > 1$ and are unique up to homotopy. We will construct them in the next section. Given $K(G,n)$, its loopspace satisfies $\Omega K(G,n) = K(G, n-1)$. Thus any Eilenberg-MacLane space is the loop space of another Eilenberg-MacLane space and thus an infinite loop space.

## 1.2 Dold-Kan correspondence

The Dold-Kan correspondence is a classical tool from homological algebra, relating the category of chain complexes and category of simplicial objects in an abelian category. The main refrence beeing used is [4].

**Theorem 1.1** ((Dold-Kan correspondence))**.** *For any abelian category $\mathscr{A}$, the normalized chain complex functor $N$ is an equivalence of categories between $S\mathscr{A}$, the category of functors*

$$S\mathscr{A} := Fun(\Delta^{op}, \mathscr{A}),$$

*and $Ch_{\leq 0}(\mathscr{A})$. Under this correspondence, simplicial homotopy corresponds to homology*

$$\pi_*(A_*) \simeq H_*(N(A_*)),$$

*for $A_* \in S\mathscr{A}$. The functor $N$ is defined by*

$$N(A_*)_n = \bigcap_{i=0}^{n-1} ker(\partial_i : A_n \to A_{n-1}), \text{ with differential } d = (-1)^n \partial_n.$$

We will not proof this theorem, but only state the inverse functor K. Let $C \in Ch_{\leq 0}(\mathscr{A})$, define the sets

$$K_n(C) := \bigoplus_{\eta:[n]\twoheadrightarrow[m]} C_m.$$

Let $\alpha : [m] \to [n]$. To define the action $K_n(\alpha)$ on the factor $C_p$ corresponding to $\eta : [n] \to [p]$, take the epi-monic factorization $\eta^{'} \circ \epsilon$ of $\eta \circ \alpha$. Then define $K_n(\alpha) : C_p \to C_{dom(\epsilon)}$, where the summands correspond to $\eta$ and $\eta^{'}$, by

$$K_n(\alpha) := \begin{cases} id_{C_p}, & \text{if } dom(\epsilon) = [n], \\ d_p, & \text{if } dom(\epsilon) = [n-1], \\ 0, & \text{otherwise.} \end{cases}$$

To relate simplicial constructions like the two sided bar construction, see section (2.3) , to known constructions for chain complexes, it can be useful to use a variation of the normalized chain complex functor N, called the alternating face map complex functor C, given by

$$C(A_*)_n = A_n, \text{ with differential } d_n = \sum_{i=0}^{n} (-1)^i \partial_i.$$

It turns out that these are homotopy equivalent.
To construct $K(G, n)$, if G is abelian, take the chain complex with entry G in degree n and 0 otherwise, apply the functor K and take the geometric realization. If $n = 1$ and G is nonabelian, take the geometric realization of the category $BG$ to obtain $K(G, 1)$.

# 2 May's delooping theorem

Our goal will be to find conditions on when a space is an n-fold loop space, following [2]. To state and proof May's delooping theorem we have to develop some tools first.

## 2.1 Monads from operads

We want to study operads in the category of topological spaces. An algebra over a topological operad C is also called a C-space. Such an action corresponds to the data of maps

$$\theta^i : C(i) \times X^i \to X$$

satisfying associativity and unitality properties.
We now want to construct a monad from an operad.

**Definition** A monad $(T, \mu, \eta)$ in $Top_*$ is a monoid in the category of endofunctors $T \in Fun(Top_*, Top_*)$, meaning that for any $X \in Top_*$ the following diagrams commute.

$$\begin{tikzcd} TX \arrow[r, "T\eta"] \arrow[dr, "id"'] & T^2X \arrow[d, "\mu"] & TX \arrow[l, "\eta T"'] \arrow[dl, "id"] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} T^3X \arrow[r, "T\mu"] \arrow[d, "\mu T"] & T^2 \arrow[d, "\mu"] \\ T^2X \arrow[r, "\mu"] & TX \end{tikzcd}$$

An algebra over T is an object $X \in Top$ with a map $\xi : TX \to X$, called the structure map, so that the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "id"'] & TX \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu"] \arrow[d, "T\xi"] & T \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"] & X \end{tikzcd}$$

**Example** Given an adjunction $f : A \leftrightarrow B : g$, one can associate the monad $f \circ g : B \to B$, where $\eta$ and $\mu$ are constructed from the unit and counit respectively.

Given a topological operad $(O, \gamma)$, we can associate a monad $T_O$, definded on objects $X \in Top_*$ by

$$\coprod_{0 \le j} O(j) \times X^j / \sim$$

where $\sim$ is the equivalence relation generated by relating the action of the symmetric group $\Sigma_j$ on $O(j)$ and $X^j$ and factoring out degenerate elements, where the second part means that for $y \in X^{j-1}$ and $c \in O(j)$

$$(\gamma(c; 1^i \times * \times 1^{j-i-1}), y) \sim (c, (y_1, \ldots, y_i, *, y_{i+1}, \ldots, y_{j-1})).$$

On morphisms $f : X \to X'$, define

$$T_O(f) : T_O X \to T_O X'.$$

$$[c, x] \mapsto [c, f(x)]$$

We define the multiplication $\mu : T_O^2 X \to T_O X$ and unit $\eta : X \to T_O X$ pointwise by

$$\mu([c, [d_1, y_1], \ldots, [d_k, y_k]]) = [\gamma(c; d_1, \ldots, d_k), (y_1, \ldots, y_k)],$$

where $c \in O(k)$, $d_s \in C(j_s)$ and $y_s \in X^{j_s}$ and

$$\eta(x) = [1, x],$$

where $x \in X$. The well definedness is more or less obvious. We only check one of the unitality properties. Given $[c, y] \in TX$, one finds that

$$\mu(\eta T([c, y])) = \mu((1, [c, y])) = [\gamma(1; c), y] = [c, y].$$

This construction is also functorial in the input of the operad, where the functor acts pointwise on the first component of the equivalence classes. The point of this construction is the following proposition relating algebras over the operad and monad respectivly.

**Proposition 2.1.** *Let $O$ be an operad and $T_O$ the associated monad. There is an isomorphism between the category of $T_O$ algebras and the category of $O$ algebras. It assigns to a $T_O$-space $X$ with structure map $\xi$ the $O$-algebra action*

$$\theta_i : O(i) \times X^i \xrightarrow{\pi} CX \xrightarrow{\eta} X,$$

*where $\pi$ is the inclusion into the coproduct followed by the canonical projection.*

The proof consists of verifying the different properties and checking that the above functor is fully faithful. A more detailed account can be found in chapter 2 of [2].

## 2.2 $E_n$-operads

Recall the little intervals planar operad, whose n-ary operations are orientation preserving embedding from the n-times unit interval into the unit interval. The $E_1$ operad is similar. The difference is that the embeddings have to be rectilinear but not orientation preserving, which comes from the fact that the little intervals operad was a planar operad, that is without the action of the symmetric group, which we can freely add by dropping the requirement that the embeddinges be orientation preserving. We now generalize this construction to embeddings of higher dimenions.

**Definition** An open little n-cube is a map $f = f_1 \times \cdots \times f_n$ from $J^n$, where $J$ is the open interval $(0,1)$ ,to $J^n$, where $f_i(t) = (y_i - x_i)t + x_i$ for $0 \leq x_i < y_i \leq 1 \in J$. For $n, j \in \mathbb{N}$, define $E_n(j)$ to be the space of j-tuples of little n-cubes with disjoint image. Here we choose the subspace topology induced from the compact-open topology on $Top(\coprod_{i=1}^{j} J^n, J^n)$. Define the toplogical operad $E_n$ with the above data and the composition

$$\gamma(c; d_1, \ldots, d_k) = c \circ (d_1 \amalg \cdots \amalg d_k).$$

The identity $1 \in E_n(1)$ is the identity function and the action of $\Sigma_j$ on $E_n(j)$ is by permuting the order of the little cubes.

All properties required by an operad are easily shown to be satisfied. To construct the operad $E_\infty$ we need maps $\sigma_n : E_n \to E_{n+1}$, given by

$$\sigma_{n,j}(c_1, \ldots, c_j) = (c_1 \times 1, \ldots, c_j \times 1).$$

**Definition** Define the space $E_\infty(j) = \varinjlim E_n(j)$. This data can be extended to an operad $E_\infty$ with composition

$$\gamma_\infty([c]; [d_1], \ldots, [d_k]) = \gamma_N(c', d_1', \ldots, d_k'),$$

where each equivalence class is represented by an element (denoted by a prime) of $E_N$ for a sufficiently large N.

Recall the configuration spaces

$$F(J^n, j) := \{(x_1, \ldots, x_j) | x_i \in J^n, x_i \neq x_j\}.$$

**Lemma 2.2.** *For $1 \leq n \leq \infty$ an $1 \leq j$, $E_n(j)$ is homotopy equivalent to $F(J^n, j)$.*

*Proof.* For $n < \infty$ define $g : E_n(j) \to F(J^n, j)$ by

$$g(c_1, \ldots, c_j) = (c_1(a), \ldots, c_n(a)), \text{ where } a = (\frac{1}{2}, \ldots, \frac{1}{2}).$$

The inverse map is given by associating to points $(x_1, \ldots, x_j)$ the little cubes given by the biggest open cubes centered at these points so that the cubes do not intersect. A more explicit desciption and a verification that these maps are homotopy inverse is given in the proof of theorem 4.8 in [2].
To proof the case $n = \infty$, we consider the embeddings

$$\sigma_n : J^n \to J^{n+1}.$$
$$x \mapsto (x, \frac{1}{2})$$

Then the following diagramm commutes

$$\begin{array}{ccc} E_n(j) & \xrightarrow{\sigma_{n,j}} & E_{n+1}(j) \\ \downarrow{g_n} & & \downarrow{g_{n+1}} \\ F(J^n, j) & \xrightarrow{\sigma_{n,j}} & F(J^{n+1}, j) \end{array}$$

This allows us to obtain an isomorphism

$$g_\infty = \lim_{\to} g_n : E_\infty(j) \to F(J^\infty, j).$$

□

This comparison allows us to learn about the topological properties of the $E_n(j)$, if we know the topology of the configuration spaces. May states the following proposition based on work of Fadell and Neuwirth.

**Proposition 2.3.** *For $n > 3$ the configuration space $F(J^n, j)$, is $n-2$ connected. For $n = \infty$, the configuration spaces $F(J^\infty, j)$ are contractible and for $n = 1$ the space $F(J^1, j)$ is homotopy equivalent to $\Sigma_j$ (with the discrete topology).*

Recall that the associative topolgical operad $Ass$ has the $n$-ary operations $Ass(n) = \Sigma_j$, with the discrete topology. The commutative topological operad $Comm$ has the $n$-ary operations $Comm(n) = *$. As a corollary of the above proposition we obtain, that $E_1 \simeq Ass$ and $E_\infty \simeq Comm$. It is easy to see, that the composition in $E_1$ and $Ass$ agree under the levelwise homotopy equivalence.
Now that we have learned about the $E_n$-operads, where are ready to show that any n-fold loop space comes with equipped with an action of one. Let $Y = \Omega^n X \in Top$ with $X \in Top_*$. Define the maps

$$\theta_{n,j} : E_n(j) \times (Y)^j \to Y$$

for $c = (c_1, \ldots, c_j) \in E_n(j)$, $y = (y_1, \ldots, y_j) \in Y^j$ and $v \in \mathbb{S}^n$ by

$$\theta_{n,j}(c, y)(v) = \begin{cases} y_r(u) & \text{if } c_r(u) = v, \\ * & \text{if } v \notin Im(c). \end{cases}$$

This is however easier understood pictorially. It is easily checked that this gives an action of the operad $E_n$. As before, this action extends in the case of $n = \infty$ because the action is compatible with the inclusions $\sigma_n : E_n \to E_{n+1}$, i.e. the map from $E_n$ into the endomorphism operad factors through $E_{n+1}$. The rest of this section is concerned with the converse. Is every space with an action of the $E_n$ operad an n-fold loop space?
Towards this end, we need a comparison of the monads $T_n := T_{E_n}$ and $\Omega^n \Sigma^n$.

**Theorem 2.4** (approximation theorem)**.** *For $1 \leq n \leq \infty$ and $X \in Top_*$ let $\alpha_n : T_nX \to \Omega^n\Sigma^nX$ defined by*

$$\alpha_n : T_nX \xrightarrow{T_n\eta} T_n\Omega^n\Sigma^nX \xrightarrow{\theta_n} \Omega^n\Sigma^nX.$$

*This gives a map of monads and $\alpha_n$ is a weak homotopy equivalence for all connected X.*

The proof is long and complicated. To see that $\alpha_n$ gives a map of monads can be done by evaluating explicitly commutative diagrams associated with the various properties and morphisms involved. To see that $\alpha_n$ is also a weak homotopy equivalence is more difficult. It is done inductively, by finding $C_nX$ and $\Omega^n\Sigma^nX$ as the fibers of quasi fibrations from two contratible spaces into $C_{n-1}SX$ and $\Omega^{n-1}\Sigma^nX$.

## 2.3 Two-sided bar constuction and the delooping theorem

We want to use the approxiamtion theorem from the last section to prove that any $E_n$-space is an n-fold loop space. The problem is that the map $T_nX \to X$ is in general not a homotopy equivalence. We will replace $T_nX$ by a space, so that X is a strong deformation retract of that space, satisfying some nice properties that allows us to apply the approximation theorem. In homologial algebra, this would be called 'taking a resolution'. To do this we must first pass to the category $STop$ of simplicial spaces, i.e. functors

$$\Delta^{op} \to Top.$$

We will then get back to Top by applying geometric realization.
We can extend objects and endofunctors from $Top$ to $STop$. Given $X \in Top$, let $X_* \in STop$ be the degenerate simplicial space $X_* = X$ with face and degeneracy maps the identity. Given an endofunctor T in Top, we can obtain an endofunctor $T_*$ in $STop$, by $(T_*(X_*))n = T(X_n)$.

**Definition** Let T be a monad in $Top_*$. A T-functor $(F, \lambda)$ in a category $C$ is a functor $F : Top_* \to C$ with $\lambda : FT \to F$ such that the following diagrams commute.

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "id"'] & FT \arrow[d, "\lambda"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} FTT \arrow[r, "F\mu"] \arrow[d, "\lambda T"'] & FT \arrow[d, "\lambda"] \\ FT \arrow[r, "\lambda"] & F \end{tikzcd}$$

Note that T is itself a T-functor and that the composition of a T-funtor F with any functor $G : C \to C^{'}$ is a $C^{'}$-functor.

**Example** The functor $\Sigma^n$ is a $T_n$-functor, with transformation

$$\lambda_X : \Sigma^nT_n(X) \simeq \Sigma^n\Omega^n\Sigma^n(X) \xrightarrow{\eta\circ\Sigma^n} \Sigma^n(X),$$

for $X \in Top_*$ and $\eta$ the counit of the adjunction $\Sigma^n \dashv \Omega^n$.

Given a monad $(T, \mu, \eta)$, a T-functor $(F, \lambda)$ and a T-algebra $(X, \xi)$, we define

$$B_q(F, T, X) := FT^qX \in C.$$

We extend this to a simplicial object $B_*$ in $C$, called the two-sided bar construction, by the face and degeneracy maps

$$\partial_i = \begin{cases} \lambda_{T^{q-1}X} & \text{if } i = 0, \\ FT^{i-1}\mu_{T^{q-i+1}X} & \text{if } 0 < i < q, \\ FT^{q-1}\xi & \text{if } i = q, \end{cases}$$

$$s_i = FT^i\eta_{T^{q-i}X}.$$

The functoriality is given levelwise as the composition of the involved functors between the monads and algebras. Explicitly, given a map $\psi : T \to T^{'}$ of monads, a map $\pi : F \to \psi^* F^{'}$ of T-functors and a map of T-algebras $f : X \to \psi^* X^{'}$, where

$$\psi^*(F^{'}, \lambda^{'}) = (F^{'}, \lambda^{'} \circ F^{'}\psi) \tag{1}$$

$$\psi^*(X^{'}, \xi^{'}) = (X^{'}, \xi^{'} \circ \psi),$$

one can check that this gives a map

$$B(\pi, \psi, f) : B(F, T, X) \to B(F^{'}, T^{'}, X^{'}).$$

One can visualise the face maps as follows

$$\begin{array}{ccccc} F & T & T & T & X \end{array}$$

$$\begin{array}{cccc} F & T & T & X \end{array}$$

The face maps are each given by joining two components using the available maps. For those familiar with Hochschild homology or perhaps group homology, this construction will be reminiscent of bar construction there. This is not a coincidence, by applying the alternating face map complex functor from the Dold Kan correspondence, we exactly obtain the aforementioned bar complex. The bar complex is a resolution of the original algebra. In this more general setting, we want to show that $X_*$ is a strong deformation retract of $B(T, T, X)_*$. We need to show that the maps

$$\begin{aligned} &f_* : B_*(T, T, X) \to X_* \\ &f_n = \xi \circ \dots \circ T^n \xi \\ &g_* : X_* \to B_*(T, T, X) \\ &g_n = \eta^{n+1} \end{aligned}$$

are homotopy inverse. It is immediate, that

$$f_* \circ \beta_* = id_{X_*}.$$

To specify a homotopy from $g_* \circ f_*$ to the identity, i.e. a map

$$B_*(T, T, X) \times \Delta^1 \to B_*(T, T, X)$$

with appropriate conditions on the restrictions, we can give maps

$$(h_i)_* : B_*(T, T, X) \to B_{*+1}(T, T, X)$$

with the property that

$$\partial_0 h_0 = g_* \circ f_*,$$

$$\partial_{n+1} h_n = id_{B_*}$$

and satisfying some further compatibility identities with the boundary and face maps. The formulae can be found in chapter 8.3.1 in [4], where the homotopy is also constructed from the above data.

We will not prove that the maps

$$(h_i)_* : B_*(T,T,X) \to B_{*+1}(T,T,X)$$
$$h_i = s_0^i \eta_{T^{q+1-i}X}, \partial_0^i$$

where $s_0^i$ and $\partial_0^i$ are the i-th powers of the degeneracy and face maps of $B_*$, satisfy the required identities.
We also observe that for any functor $G$

$$B_*(GF,T,X) = G_* B_*(F,T,X).$$

We now apply the geometric realization. Given a simplicial space $X_*$, we define the geometric realization as the space

$$\coprod_{n \leq 0} X_n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation generated by

$$(\partial_i x, u) \sim (x, \delta_i u)$$

$$(\sigma_i x, u) \sim (x, s_i u),$$

where $\delta_i$ and $s_i$ are the topological face and boundary maps of the n-simplex. One can show that

$$|X_*| \simeq X$$

and that the geometric realization preserves homotopies. We denote $g := |g_*|$ and obtain a homotopy equivalence

$$X \xrightarrow{g} |B_*(T,T,X)| := B(T,T,X).$$

In total we obtain the recognition principle.

**Theorem 2.5** (May's delooping theorem)**.** *Let $X \in Top$ be connected. $X$ is an algebra over the operad $E_n$, for $1 \leq n \leq \infty$, if and only if $X \simeq \Omega^n Y$ for some $Y \in Top_*$.*

*Proof.* Let $n < \infty$. Suppose X is an algebra over an $E_n$-Operad. Then we obtain a algebra over the induced monad $T_n$ and obtain a homotopy equivalence

$$X \xrightarrow{g} B(T_n,T_n,X) \xrightarrow{B(\alpha_n,1,1)} B(\Omega^n\Sigma^n,T_n,X) = |\Omega_*^n B_*(\Sigma^n,T_n,X)| \simeq \Omega^n B(\Sigma^n,T_n,X).$$

The last homotopy equivalence follows from the nontrivial equivalence

$$|\Omega_* X_*| \simeq \Omega |X_*|. \tag{2}$$

For $n = \infty$, the approximation theorem gives us an equivalence $\alpha_\infty : T_\infty(X) \xrightarrow{\simeq} \Omega^\infty\Sigma^\infty X$. Then as before we obtain a homotopy equivalence

$$X \xrightarrow{g} B(T_\infty,T_\infty,X) \xrightarrow{B(\alpha_\infty,1,1)} B(\Omega^\infty\Sigma^\infty,T_\infty,X).$$

To complete the argument we need to show that

$$B(\Omega^\infty\Sigma^\infty,T_\infty,X) \simeq \Omega^\infty B(\Sigma^\infty,T_\infty,X).$$

We will not show this, but only say what we mean by the right hand side. The direct limit is taken over the maps defined by the commutativity of the following diagram,

$$\begin{array}{ccc} B(\Omega^i\Sigma^i, T_i, X) & \xrightarrow{B(\Omega^i\eta\Sigma^i,\sigma_i,X)} & B(\Omega^{i+1}\Sigma^{i+1}, T_{i+1}, X) \\ \downarrow\simeq & & \downarrow\simeq \\ \Omega^i B(\Sigma^i, T_i, X) & \longrightarrow & \Omega^{i+1}B(\Sigma^{i+1}, T_{i+1}, X) \end{array} \tag{3}$$

where $\sigma_i : T_i \to T_{i+1}$ is a morphism of monads induced from the morphism of the operads $E_i \to E_{i+1}$. Here one also needs to verify that $(\Omega^i\eta\Sigma^i, \sigma_i, id_X)$ is a suitable morphism in the sense that $B$ is functionial. The isomorphisms in the above diagram are constructed using that the geometric realization commutes with colimits and equation (2).
The converse was shown in section (2.2). $\square$

**Remark** To see that $\Omega^\infty B(\Sigma^\infty, T_\infty, X)$ is actually an $n$-fold loop space for all n, we construct the $n$-fold delooping by

$$\Omega^n \varinjlim \Omega^{i-n} B(\Sigma^i, T_i, X) \simeq \Omega^\infty B(\Sigma^\infty, T_\infty, X).$$

The space $\varinjlim \Omega^{i-n}B(\Sigma^i, T_i, X)$ is constructed analogeous to diagram (3). We then obtain the following commutative diagramm, which implies the above equivalence.

$$\begin{array}{ccc} \Omega^n\Omega^{i-n}B(\Sigma^i, T_i, X) & \longrightarrow & \Omega^n\Omega^{i-n+1}B(\Sigma^{i+1}, T_{i+1}, X) \\ \downarrow\simeq & & \downarrow\simeq \\ \Omega^i B(\Sigma^i, T_i, X) & \longrightarrow & \Omega^{i+1}B(\Sigma^{i+1}, T_{i+1}, X) \end{array}$$

**Remark** In the bar construction $B(\Sigma^n, T_n, X)$, we need $T_n^q X$ to be a pointed space. The point is chose to be

$$[T_n(0) \times (T_n^{q-1}X)^0] \in T_n^q X.$$

# 3 $E_n$ $\infty$-operads

In the previous section we have constructed the topological $E_n$ operad. We can obtain a simplicial colored operad $\tilde{E}_n$, by applying the Sing functor to all $E_n(j)$. As in the case of colored operads, we can obtain an $\infty$-operad from this, as follows. There is a simplicial category $\bar{E}_n^{\otimes}$ with objects beeing natural numbers, and the simplicial mapping set

$$\bar{E}_n^{\otimes}(n, m) := \coprod_{\alpha:\langle n\rangle\to\langle m\rangle} \prod_{1\le j\le n} \tilde{E}_n(\alpha^{-1}(j)).$$

Taking the simplicial nerve gives a $\infty$-category

$$E_n^{\otimes} := N_\Delta(\bar{E}_n^{\otimes}).$$

Clearly, there is a forgetful functor

$$E_n^{\otimes} \to \mathcal{F}in_*.$$

As in the case of colored operads, proven in proposition 2.1.1.27. in HA [1], we obtain that $E_n^{\otimes}$ is an $\infty$-operad.
We have seen that the topological $E_1$ operad is equivalent to the topological associative operad and that the topological $E_\infty$ operad is equivalent to the topological commutative operad. It is immediate, that the construction above gives equivalent simplicial categories, which are mapped to equivalent $\infty$-operads by the simplicial nerve.

# References

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