Introduction to Higher Category Theory

These notes are based on the lectures given by Prof. Dr. Tobias Dyckerhoff during the winter semester of 2018 at Universität Hamburg.

Contents

Chapter 1

Category Theory

1.1 Categories

Definition 1.1.1. A *category* $\mathcal{C}$ consists of

(1) a set $\mathrm{ob}(\mathcal{C})$ of *objects*,

(2) for every pair (x, y) of objects, a set $\mathcal{C}(x, y)$ of *morphisms from x and y*,

(3) for every object x, a morphism

$$\mathrm{id}_x \in \mathcal{C}(x, x),$$

called the *identity morphism of x*,

(4) for every triple (x, y, z) of objects, a map

$$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, z), (g, f) \mapsto f \circ g$$

called the *composition law*,

subject to the following conditions:

- *Unitality.* For every morphism $f \in \mathcal{C}(x, y)$, we have

$$\mathrm{id}_y \circ f = f \circ \mathrm{id}_x = f.$$

- *Associativity.* For every triple $h \in x \to y$, $g : y \to z$, $f : z \to w$ of composable morphisms, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Given two objects x, y of a category $\mathcal{C}$, we typically write an arrow $f : x \to y$ to denote a morphism from x to y. Examples of categories arise in a diverse range of contexts. We provide a small selection:

Example 1.1.2. Every partially ordered set $(P, \leq)$ gives rise to a category with set of objects given by P and a unique morphism from p to p' if $p \leq p'$. We simply denote this category by P leaving the distinction from the poset implicit. In particular, for $n \geq 0$, the standard linearly ordered set

$$[n] = \{0 < 1 < \cdots < n\}$$

gives rise to a category. For example, all objects and morphisms in the category [2] may be depicted as

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

where the arrow $0 \to 2$ is the composite of the arrows $0 \to 1$ and $1 \to 2$. Another class of examples arises from the set $\mathcal{P}(X)$ of subsets of a given set X, equipped with the partial order given by inclusion of subsets. For example, $\mathcal{P}(\{0,1\})$ may be depicted as

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr] & \{0\} \arrow[d] \\ \{1\} \arrow[r] & \{0,1\}. \end{tikzcd}$$

Problem 1.1.3. By construction, categories that arise from posets have the following property: between any given pair of objects, there is at most one morphism. Does every category with this property arise from a poset? What additional properties are needed to characterize those categories that arise from posets?

Problem 1.1.4. Let $Y \subset \mathbb{R}^3$ be a simplicial complex, and define $\Delta(Y)$ to be the poset of subsimplices of Y ordered by inclusion.

(1) For $n = 0, 1, 2, 3$, draw the category $\Delta(Y)$, where Y is the standard n-simplex in $\mathbb{R}^3$.

(2) Draw the category $\Delta(Y)$ for your favorite simplicial complex in $\mathbb{R}^3$.

Example 1.1.5. Every group G defines a category BG with a single object $*$ and $BG(*, *) = G$. The composition law is given by the multiplication law of G. More generally, this construction works more generally for a monoid instead of a group. The categories obtained via this construction are precisely the categories with a single object.

Example 1.1.6. Every topological space X gives rise to a category $\pi_{\leq 1}(X)$, called the *fundamental groupoid of* X. Its set of objects is the underlying set of X and a morphism from x to y is defined to be a homotopy class of continuous paths from x to y. The composition law is given by concatenation of paths.

Example 1.1.7. The simplex category Δ has objects given by the set of nonempty finite standard linearly ordered sets $\{[n] \mid n \geq 0\}$. Morphisms from $[m]$ to $[n]$ are given by weakly monotone maps: A map $f : \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$ is called weakly monotone if, for every $0 \leq i \leq i' \leq m$, we have $f(i) \leq f(i')$.

We would further like to organize the collection of all mathematical structures of a certain kind into a category. For example, we would like to form a category whose objects comprise all sets and morphisms are given by maps between these sets, a category whose objects comprise all groups and morhpisms are given by group homomorphisms, etc. However, according to Definition 1.1.1, the collection of objects in a category is required to form a set, so that it is impossible to define such categories in our current context: Due to Russel's famous paradox, the collection of all sets cannot form a set.

Our way to address this issue will be to enhance the framework of set theory that provides our formal language of reasoning: Namely, we will use an axiom system called **ZFCU**, comprising the axioms:

- **ZF**: Zermelo-Frankel axioms of set theory
- **C**: Axiom of choice
- **U**: Universe axiom

In other words, we work within the usual framework of **ZFC** but we allow ourselves to use an additional, logically independent axiom **U**.[1] To explain this axiom, we begin with the following:

Definition 1.1.8. A *universe* is a nonempty set $\mathcal{U}$ (of sets) with the following properties:

(U1) If $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$.

(U2) If $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$.

(U3) If $x \in \mathcal{U}$ then $\mathcal{P}(x) \in \mathcal{U}$.

(U4) If $I \in \mathcal{U}$ and $\{x_i\}_{i \in I}$ is a family of elements $x_i \in \mathcal{U}$ then $\bigcup_{i \in I} x_i \in \mathcal{U}$.

[1] For a brief and cogent summary of universes in the context of first-order logic and ZFC, see Daniel Murfet's Foundations for Category Theory.

Problem 1.1.9. Show that, as a consequence of the axiomatic properties from Definition 1.1.8, any universe $\mathcal{U}$ has the following further properties:

(1) If $x \in \mathcal{U}$ then $x \subset \mathcal{U}$.

(2) If $y \in \mathcal{U}$ and $x \subset y$ then $x \in \mathcal{U}$.

(3) $\emptyset \in \mathcal{U}$.

(4) If $x, y \in \mathcal{U}$ then $(x, y) := \{x, \{x, y\}\} \in \mathcal{U}$.

(5) If $x, y \in \mathcal{U}$ then $x \cup y$ and $x \times y \in \mathcal{U}$.

(6) If $x, y \in \mathcal{U}$ then $\mathrm{Map}(x, y) \in \mathcal{U}$.

(7) If $I \in \mathcal{U}$ and $\{x_i\}_{i \in I}$ is a family of elements $x_i \in \mathcal{U}$ then $\prod_{i \in I} x_i$, $\coprod_{i \in I} x_i$, and $\bigcap_{i \in I} x_i$ are elements of $\mathcal{U}$.

(8) If $x \in \mathcal{U}$ then $x \cup \{x\} \in \mathcal{U}$.

(9) $\mathbb{N} \subset \mathcal{U}$.[2]

Problem 1.1.10. We make the following recursive definition: A set X is called *hereditarily finite* if X is finite and all elements of X are hereditarily finite. More explicitly, define $V_0 = \emptyset$, $V_1 = \mathcal{P}(V_0)$, $V_2 = \mathcal{P}(V_1)$, $\ldots$ so that $V_0 \subset V_1 \subset V_2 \ldots$ and set

$$V_\omega := \bigcup_{k \geq 0} V_k.$$

Then the elements of the set V_ω are precisely the hereditarily finite sets. Show that V_ω is a universe.

The universe V_ω of hereditarily finite sets from Problem 1.1.10 satisfies $\mathbb{N} \subset V_\omega$, as does any universe, but *not* $\mathbb{N} \in V_\omega$. A universe $\mathcal{U}$ is called *infinite* if $\mathbb{N} \in \mathcal{U}$.

Problem 1.1.11. Let $\mathcal{U}$ be an infinite universe. Show that $\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$, and $\mathbb{C}$ are elements of $\mathcal{U}$.

Problems 1.1.9 and 1.1.11 demonstrate that, whenever we produce a new set from sets in a given universe $\mathcal{U}$, by means of a typical set-theoretic construction, then the new set will also be an element of $\mathcal{U}$. This supports the idea that it is feasible to formulate all mathematical structure of interest within a given infinite universe. One question remains: The existence of an infinite universe. It turns out that the existence of such a universe cannot be proved within **ZFC**, which explains why we add the following *universe axiom*:

- **U**. For every set X, there exists a universe $\mathcal{U}$ such that $X \in \mathcal{U}$.

From now on, we will fix an infinite universe $\mathcal{U}$ and introduce the following terminology:

Definition 1.1.12. Let X be a set. We say that X is

(1) a *small set* if $X \in \mathcal{U}$,

(2) a *class* if $X \subset \mathcal{U}$,

(3) a *large set* if $X \not\subset \mathcal{U}$.

Within this context, we can now introduce the following categories:

- The category **Set** with objects given by the small sets ($\mathrm{ob}(\mathbf{Set}) = \mathcal{U}$) and maps between small sets as morphisms.

- The category **Grp** with objects given by those groups whose underlying set is small and group homomorphisms as morphisms.

[2]In ZFC, the standard way to construct the natural numbers is by taking $0 := \emptyset$, $1 := \{0\} = \{\emptyset\}$, $2 := \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, and so on. Note that this construction defines n to be the set V_n from problem 1.1.10. Once $\mathbb{N}$ is defined, the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ can be constructed in the usual way – i.e. via equivalence relations on products, Dedekind cuts, etc.

- The category **Top** with objects given by those topological spaces whose underlying set is small and continuous maps as morphisms.

- ...

In addition to the specific examples listed above, there are some general techniques for obtaining new categories out of old ones.

- For any category $\mathcal{C}$, we introduce the *opposite category* $\mathcal{C}^{\mathrm{op}}$, which has the same objects as $\mathcal{C}$, but
$$\mathcal{C}^{\mathrm{op}}(x,y) := \mathcal{C}(y,x).$$
That is, $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects as $\mathcal{C}$, and all morphisms going in the opposite direction. Unitality and associativity follow directly from the unitality and associativity of $\mathcal{C}$.

- For any categories $\mathcal{C}$ and $\mathcal{D}$, we can define the *product category* $\mathcal{C} \times \mathcal{D}$ by
$$\mathrm{ob}(\mathcal{C} \times \mathcal{D}) := \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$$
and
$$\mathcal{C}((x,y),(x',y')) := \mathcal{C}(x,x') \times \mathcal{C}(y,y').$$

1.2 Functors and natural transformations

Definition 1.2.1. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of

- a map $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$
- for every pair (x,y) of objects of $\mathcal{C}$, a map
$$F : \mathcal{C}(x,y) \to \mathcal{D}(F(x),F(y))$$

subject to the conditions

(1) For all $x \in \mathcal{C}$[3], $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$

(2) For every pair $x \xrightarrow{g} y \xrightarrow{f} z$ of composable morphisms,
$$F(f \circ g) = F(f) \circ F(g).$$

Examples 1.2.2. (1) A functor $F : \mathcal{P}(\{0,1\}) \to \mathcal{C}$ consists of a commutative square
$$\begin{tikzcd} F(\emptyset) \arrow[r] \arrow[d] \arrow[dr] & F(\{0\}) \arrow[d] \\ F(\{1\}) \arrow[r] & F(\{0,1\}). \end{tikzcd}$$
in $\mathcal{C}$.

(2) Let G be a group, and $\mathcal{C}$ a category. A functor
$$F : BG \to \mathcal{C}$$
is an object $F(*) \in \mathcal{C}$ together with a G-action on $F(*)$. In the special case where $\mathcal{C}$ is a category of vector spaces, F is a representation of G.

[3] Note that here, and in much of the rest of the text, we will abuse notation slightly by writing $x \in \mathcal{C}$ instead of $x \in \mathrm{ob}(\mathcal{C})$.

(3) The association

$$\mathbf{Top} \to \mathbf{Cat}, \qquad X \mapsto \pi_{\leq 1}(X)$$

can be extended to a functor, where **Cat** is the category whose objects are small categories, and whose morphisms are functors.

(4) Let K be a small field, and let $\mathbf{Vect}_K$ be the category of small K-vector spaces. Then the association

$$\mathbf{Vect}_K \to \mathbf{Vect}_K^{op}; \qquad V \mapsto V^* := \mathrm{Hom}_{\mathbf{Vect}_K}(V, K)$$

which sends each vector space to its dual vector space defines a functor.[4]

Definition 1.2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ from F to G consists of, for every $x \in \mathcal{C}$, a morphism $\eta_x : F(x) \to G(x)$ in $\mathcal{D}$ such that, for every morphism $f : x \to y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y). \end{tikzcd}$$

commutes.

Natural transformations are often represented visually as

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & B \end{tikzcd}$$

Examples 1.2.4. (1) For functors $F, G : \mathcal{P}(\{0\}) \to \mathcal{C}$, a natural transformation $\eta : F \Rightarrow G$ is a commutative square

$$\begin{tikzcd} F(\emptyset) \arrow[r] \arrow[d] \arrow[dr] & F(\{0\}) \arrow[d] \\ G(\emptyset) \arrow[r] & G(\{0\}) \end{tikzcd}$$

in $\mathcal{C}$.

(2) For G a group and functors $X, Y : BG \to \mathbf{Set}$, a natural transformation $\eta : X \to Y$ is a G-equivariant map. More precisely, it is a map $f = \eta_* : X(*) \to Y(*)$ such that, for ever $g \in G$, $f(g \cdot x) = g \cdot f(x)$.

(3) Let K be a small field. Consider the double dual functor

$$(-)^{**} : \mathbf{Vect}_K \to \mathbf{Vect}_K; \qquad V \mapsto (V^*)^*$$

and the identity functor

$$\mathrm{id} : \mathbf{Vect}_K \to \mathbf{Vect}_K; \qquad V \mapsto V.$$

For every V in $\mathbf{Vect}_K$, define the morphism $\eta_V : V \to (V^*)^*$ by

$$v \mapsto (f \mapsto f(v)).$$

Then the datum $\eta : \{\eta_V\}_{V \in \mathbf{Vect}_K}$ defines a natural transformation $\eta : \mathrm{id} \to (-)^{**}$. Furthermore, if we restrict η to the subcategory of finite dimensional vector spaces, then it becomes a *natural isomorphism*.

[4] Note that some authors would call this a *contravariant functor* $\mathbf{Vect}_K \to \mathbf{Vect}_K$. While this terminology is still sometimes used, we will not make use of it. In general a *covariant* functor is simply a functor as defined above, and a *contravariant* functor $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

(4) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then the collection of functors from $\mathcal{C}$ to $\mathcal{D}$ forms the set of objects of a category

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

with morphisms are given by natural transformations. The operation of composition is written as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=L, above}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "\eta"] \arrow[Rightarrow, from=M, to=L, "\gamma"] \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, bend right=60, ""{name=L, above}] & \mathcal{D} \arrow[Rightarrow, from=U, to=L, "\gamma * \eta"] \end{tikzcd}$$

where $\gamma * \eta := \{\gamma_x \circ \eta_x\}_{x \in \mathrm{ob}(\mathcal{C})}$.

We have further operations:

- Composing natural transformations with functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{E} \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G \circ F"] \arrow[r, bend right, "H \circ F"'] \arrow[r, phantom, "\Downarrow\eta \circ F"] & \mathcal{D} \end{tikzcd}$$

  where $\eta \circ F := \{\eta_{F(x)}\}_{x \in \mathcal{C}}$.
- Composing functors with natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H \circ F"] \arrow[r, bend right, "H \circ G"'] \arrow[r, phantom, "\Downarrow H \circ \eta"] & \mathcal{D} \end{tikzcd}$$

  where $H \circ \eta := \{H(\eta_x)\}_{x \in \mathcal{C}}$
- More generally, we can compose two natural transformations *horizontally*:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{E} \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H \circ F"] \arrow[r, bend right, "I \circ G"'] \arrow[r, phantom, "\Downarrow\gamma \circ \eta"] & \mathcal{D} \end{tikzcd}$$

  From the naturality of γ, we can see that the diagram

$$\begin{tikzcd} H(F(x)) \arrow[r, "\gamma_{F(x)}"] \arrow[d, "H(\eta_x)"'] & I(F(x)) \arrow[d, "I(\eta_x)"] \\ H(G(x)) \arrow[r, "\gamma_{G(x)}"'] & I(G(y)). \end{tikzcd}$$

  commutes, and thus,

$$\gamma \circ \eta := \left\{I(\eta_x) \circ \gamma_{F(x)}\right\}_{x \in \mathcal{C}} = \left\{\gamma_{G(x)} \circ H(\eta_x)\right\}_{x \in \mathcal{C}}$$

As will be shown on Problem Set 2, these operations make **Cat** a 2-*category*.

1.3 Equivalences and adjunctions

Within a category $\mathcal{C}$ we can define a notion of an isomorphism.

Definition 1.3.1. A morphism $f : x \to y$ in a category $\mathcal{C}$ is called an *isomorphism* if there exists a morphism $g : y \to x$ such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$.

However, applying this definition to **Cat** we come up with a very strict notion of when categories are 'the same'. Indeed, isomorphism of categories is far too strict a notion for most uses. A more flexible notion of identification is the following:

Definition 1.3.2. A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories is called an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $F \circ G \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$ and $G \circ F \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$.

We can, in fact, weaken this definition even further, by discarding the invertibility of the natural transformations.

Definition 1.3.3. An *adjunction* (F, G, η, ϵ) between categories $\mathcal{C}$ and $\mathcal{D}$ consists of

- a functor $F : \mathcal{C} \to \mathcal{D}$ called the *left adjoint*
- a functor $G : \mathcal{D} \to \mathcal{C}$ called the *right adjoint*
- a natural transformation
$$\eta : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$$
called the *counit.*
- a natural transformation
$$\epsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$$
called the *unit.*

subject to the conditions

(1) the composite
$$F \overset{F\circ\epsilon}{\Rightarrow} F \circ G \circ F \overset{\eta\circ F}{\Rightarrow} F$$
is the identity. That is, $(\eta \circ F) * (F \circ \epsilon) = \mathrm{id}_F$

(2) the composite
$$G \overset{\epsilon\circ G}{\Rightarrow} G \circ F \circ G \overset{G\circ\eta}{\Rightarrow} G$$
is the identity. That is, $(G \circ \eta) * (\eta \circ G) = \mathrm{id}_G$.

We will typically denote an adjunction $(F, G, \eta, \varepsilon)$ between categories $\mathcal{C}$ and $\mathcal{D}$ by
$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$$
leaving the choice of η and ε implicit.

To obtain a useful criterion for a functor to be an equivalence, which we will in particular use to understand the precise relation between Definitions 1.3.2 and 1.3.3, we introduce some further terminology. A functor $F : \mathcal{C} \to \mathcal{D}$ is called

(1) *full* (resp. *faithful*) if, for any pair (x, y) in $\mathrm{ob}(\mathcal{C})$, the map
$$F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$$
is surjective (resp. injective)

(2) *fully faithful* if it is both full and faithful

(3) *essentially surjective* if for all $y \in \mathcal{D}$ there exists $x \in \mathcal{C}$ and and isomorphism $F(x) \xrightarrow{\cong} y$ in $\mathcal{D}$.

Theorem 1.3.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

(1) F is an equivalence.

(2) F is fully faithful and essentially surjective.

(3) F is part of an adjunction (F, G, η, ϵ) with η and ϵ natural isomorphisms.

Proof. The implication (3)⇒(1) is immediate from the definitions.

(1)⇒(2): Assume F is an equivalence.

- Essential surjectivity: We are given $F \circ G \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$. Evaluation at any $y \in \mathcal{D}$ gives an isomorphism $F(G(y)) \cong y$.
- F is faithful: For a given morphism $f : x \to x'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} x \arrow[r, "\epsilon_x"] \arrow[d, "f"'] & G(F(x)) \arrow[d, "G(F(f))"] \\ x' \arrow[r, "\epsilon_{x'}"'] & G(F(x')). \end{tikzcd}$$

  commutes, which implies $f = \epsilon_{x'}^{-1} \circ G(F(f)) \circ \epsilon_x$. Hence, if $F(f) = F(g)$, we have that $f = g$. The same argument applied to η implies that G is faithful.
- F is full: Given a morphism $h : F(x) \to F(x')$, we define $f := \epsilon_{x'}^{-1} \circ G(h) \circ \epsilon_x$. In particular, the diagram

$$\begin{tikzcd} x \arrow[r, "\epsilon_x"] \arrow[d, "f"'] & G(F(x)) \arrow[d, "G(h)"] \\ x' \arrow[r, "\epsilon_{x'}"'] & G(F(x')). \end{tikzcd}$$

  commutes. Consequently, $G(h) = G(F(f))$. However, since G is faithful, this implies that $h = F(f)$.

(2)⇒(3): Suppose F is fully faithful and essentially surjective. Then, for every $y \in \mathcal{D}$, there exists an object $x_y \in \mathcal{C}$ equipped with an isomorphism $\eta_y : F(x_y) \to y$. Via the axiom of choice, we obtain a map

$$G : \mathrm{ob}\, \mathcal{D} \longrightarrow \mathrm{ob}\, \mathcal{C},\ y \mapsto x_y$$

defining the functor G on objects. Given a morphism $f : y \to y'$ in $\mathcal{D}$, we obtain a commutative square

$$\begin{tikzcd} F(x_y) \arrow[r, "\eta_y"] \arrow[d, dashed] & y \arrow[d, "f"] \\ F(x_{y'}) \arrow[r, "\eta_{y'}"] & y' \end{tikzcd}$$

by setting the dashed arrow to be $\eta_{y'}^{-1} g \eta_y$. We then define

$$G(g) = \psi^{-1}(\eta_{y'}^{-1} g \eta_y)$$

where ψ denotes the map $\mathcal{C}(x_y, x_{y'}) \to \mathcal{D}(F(x_y), F(x_{y'}))$ induced by F, which is a bijection since F is fully faithful.

By construction, the various isomorphisms $\{\eta_y\}_{y \in \mathcal{D}}$ organize into a natural isomorphism

$$\eta : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$$

which we declare to be the counit of the adjunction to be constructed. To obtain the unit

$$\varepsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$$

we note that, due to the fully faithfulness of F, it suffices to construct $F \circ \varepsilon : F \Rightarrow FGF$. To satisfy the compatibility constraints, we are forced to set

$$F \circ \varepsilon = (\eta \circ F)^{-1}.$$

It remains to verify the constraint

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\varepsilon \circ G"] \arrow[dr, Rightarrow, "\mathrm{id}"'] & GFG \arrow[d, Rightarrow, "G \circ \eta"] \\ & G. \end{tikzcd}$$

Again, using that F is faithful, it suffices to verify that the composite $(F \circ G \circ \eta) \circ (F \circ \varepsilon \circ G)$ equals the identity tranformation on FG. This follows from the formula

$$F \circ G \circ \eta = \eta \circ F \circ G$$

which is a consequence of the commutativity of the diagram

$$\begin{tikzcd} FGFG(y) \arrow[r, "FG(\eta_y)"] \arrow[d, "\eta_{FG(y)}"] & FG(y) \arrow[d, "\eta_y"] \\ FG(y) \arrow[r, "\eta_y"] & y \end{tikzcd}$$

for every $y \in \mathcal{D}$, noting that η_y is an isomorphism. □

Given an adjunction $(F, G, \eta, \varepsilon)$ between categories $\mathcal{C}$ and $\mathcal{D}$, we obtain, for every pair (x, y) with $x \in \mathcal{C}$ and $y \in \mathcal{D}$, a map

$$\varphi_{x,y} : \mathcal{D}(Fx, y) \longrightarrow C(x, Gy)$$

as the composite of

$$\mathcal{D}(Fx, y) \xrightarrow{G} \mathcal{C}(GFx, Gy) \xrightarrow{-\circ\varepsilon_y} \mathcal{C}(x, Gy).$$

Proposition 1.3.5. *For every pair (x, y) with $x \in \mathcal{C}$, $y \in \mathcal{D}$, the map*

$$\varphi_{x,y} : \mathcal{D}(Fx, y) \longrightarrow C(x, Gy)$$

is a bijection, natural in x and y. Setting $\varphi = \{\varphi_{x,y}\}_{x \in \mathcal{C}, y \in \mathcal{D}}$, the datum (F, G, φ) completely determines the adjunction $(F, G, \eta, \varepsilon)$ and, vice versa, any such datum (F, G, φ) comes from an adjunction. In other words, any adjunction between categories $\mathcal{C}$ and $\mathcal{D}$ is described equivalently by the datum

$$(F, G, \eta, \varepsilon)$$

or the datum

$$(F, G, \varphi).$$

Proof. To show that $\varphi_{x,y}$ is a bijection, we claim that the map

$$\psi_{x,y} : \mathcal{C}(x, Gy) \longrightarrow \mathcal{D}(Fx, y)$$

given as the composite of

$$\mathcal{C}(x, Gy) \xrightarrow{F} \mathcal{D}(Fx, FGy) \xrightarrow{\eta_y \circ -} \mathcal{D}(Fx, y)$$

is an inverse to $\varphi_{x,y}$. To this end, consider the commutative diagram

$$\begin{tikzcd} \mathcal{D}(Fx, y) \arrow[r, "G"] \arrow[dr] & \mathcal{C}(GFx, Gy) \arrow[r, "-\circ\varepsilon_x"] \arrow[d, "F"] & \mathcal{C}(x, Gy) \arrow[d, "F"] \\ & \mathcal{D}(FGFx, FGy) \arrow[r, "-\circ F(\varepsilon_x)"] \arrow[dr] & \mathcal{D}(Fx, FGy) \arrow[d, "\eta_y \circ -"] \\ & & \mathcal{D}(Fx, y) \end{tikzcd}$$

The statement that $\psi_{x,y}\varphi_{x,y} = \mathrm{id}$ amounts to the composite of the two diagonal arrows being the identity. By the commutativity of the diagram the image of $f \in \mathcal{D}(Fx, y)$ under this composite is the morphism

$$\eta_y \circ FG(f) \circ F(\varepsilon_x)$$

which, by the naturality of η is equal to

$$f \circ \eta_{Fx} \circ F(\varepsilon_x)$$

which, by the compatibility of η and ε, is equal to f. A similar argument shows that $\varphi_{x,y}\psi_{x,y} = \mathrm{id}$ so that $\varphi_{x,y}$ is a bijection.

To show that the datum (F, G, φ) determines the adjunction, we note that, for every $x \in \mathcal{C}$, we recover the value of the unit ε_x as the image of id_{Fx} under $\varphi_{x,Fx}$ and, similarly, we recover the value of the counit η_y at $y \in \mathcal{D}$, as the image of id_{Gy} under $\psi_{Gy,y}$.

The verification that every datum (F, G, φ), with φ a natural bijection, gives rise to an adjunction is left to the reader. □

Corollary 1.3.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $G, G' : \mathcal{D} \to \mathcal{C}$ be right adjoints of F, i.e., there exist adjunctions (F, G, φ) and (F, G, φ'). Then G and G' are naturally isomorphic.*

Proof. For $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we denote the composite of the bijections

$$\mathcal{C}(x, Gy) \xrightarrow{\varphi_{x,y}} \mathcal{D}(Fx, y) \xrightarrow{\varphi'^{-1}_{x,y}} \mathcal{C}(x, G'y)$$

by γ. Setting $x = Gy$ and let $\alpha_y : Gy \to G'y$ be the image of $\mathrm{id} : Gy \to Gy$ under γ. Similarly, setting $x = G'y$, we let $\beta_y : G'y \to Gy$ denote the inverse image of $\mathrm{id} : G'y \to G'y$. It is then an immediate consequence of the naturality of γ in x and y, that the morphisms $\alpha = \{\alpha_y\}_{y\in\mathcal{D}}$ and $\beta = \{\beta_y\}_{y\in\mathcal{D}}$ assemble to inverse natural transformations between G and G'. □

Examples 1.3.7. (1) For a small field K, there is an adjunction

$$F : \mathbf{Set} \longleftrightarrow \mathbf{Vect}_K : G$$

where G is the forgetful functor, associating to a vector space its underlying set, and F associates to a set X the vector space with basis X, i.e., $F(X) = \bigoplus_X K$.

(2) There is an adjunction

$$F : \mathbf{Set} \longleftrightarrow \mathbf{Grp} : G$$

where G is the forgetful functor and F associates to a set X the free group on X.

(3) There is an adjunction

$$F : \mathbf{Grp} \longleftrightarrow \mathbf{Ab} : G$$

where G is the forgetful functor (forgetting the commutativity of the group law) and F associates to a group H its abelianization $H/[H, H]$.

(4) Let R be a small commutative ring and let M a small R-module. Then there is an adjunction

$$- \otimes_R M : R - \mathbf{Mod} \longleftrightarrow R - \mathbf{Mod} : \underline{\mathrm{Hom}}_R(M, -)$$

with left adjoint given by the tensor product with M and right adjoint given by the internal Hom.

1.4 Limits and Kan extensions

For categories I and $\mathcal{C}$ we define the *category of I diagrams in* $\mathcal{C}$ to be the functor category

$$\mathcal{C}^I := \mathrm{Fun}(I, \mathcal{C}).$$

We then also have the *diagonal functor*

$$\Delta : \mathcal{C} \to \mathcal{C}^I, \quad x \mapsto \Delta(x)$$

which sends each object x to the constant diagram on x. Given an arbitrary I diagram $F \in \mathcal{C}^I$, we then say that a *cone over F* consists of

- an object $x \in \mathcal{C}$,
- a natural transformation $\eta : \Delta(x) \Rightarrow F$.

A *morphism of cones* between (x, η) and (x', η') is then a morphism $f : x \to x'$ such that the diagram

$$\begin{tikzcd} \Delta(x) \arrow[rr, Rightarrow, "\Delta(f)"] \arrow[dr, Rightarrow, "\eta"'] & & \Delta(x') \arrow[dl, Rightarrow, "\eta'"] \\ & F & \end{tikzcd}$$

commutes. Explicitly, such a morphism is $f : x \to x'$ such that, for every $i \in I$, the diagram

$$\begin{tikzcd} \Delta(x) \arrow[rr, "f"] \arrow[dr, "\eta_i"'] & & \Delta(x') \arrow[dl, "\eta'_i"] \\ & F & \end{tikzcd}$$

commutes.

Definition 1.4.1. Let $F \in \mathcal{C}^I$. A cone (x, η) over F is called a *limit cone* if it has the following universal property:

- For every cone (x', η') over F, there exists a unique morphism of cones (x', η').

If (x, η) is a limit cone, then we say the x is *a limit of F*.[5]

We can introduce the dual notion of a *colimit* by passing to opposite categories. More precisely, for $F : I \to \mathcal{C}$, we form the *opposite functor* $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. Then

- A *cone under F* is a cone over F^{op}.
- A *colimit cone of F* is a limit cone of F^{op}.
- A *colimit of F* is a limit of F^{op}.

Examples 1.4.2. (1) Given the *empty category* $\emptyset$, there is a unique empty diagram $F : \emptyset \to \mathcal{C}$. A limit (resp. colimit) of this diagram is called a final (resp. initial) object. For example

	Set	**Ab**	$\cdots$
initial	$\emptyset$	0	$\cdots$
finite	$* = \{\emptyset\}$	0	$\cdots$

[5] In a mild abuse of terminology, x is also sometimes said to be *the* limit of F. This has some justification, since the universal property implies that any two limits of F are canonically isomorphic.

(2) Let $\{0,1\}$ denote the discrete category with two objects, i.e. the category with two objects and no non-identity morphisms. A diagram $F : \{0,1\} \to \mathcal{C}$ is a pair (X, Y) of objects in $\mathcal{C}$. A limit (resp. colimit) of F is called a *product* (resp. *coproduct*) of X and Y, and is denoted by $X \times Y$ (resp. $X \coprod Y$). For example

	Set	**Ab**	**Grp**	$\cdots$
product	$X \times Y$	$A \oplus B$	$G \times H$	$\cdots$
coproduct	$X \coprod Y$	$A \oplus B$	$G \star H$	$\cdots$

Note that, even in categories (like **Ab** and **Grp**) where objects have underlying sets, the product and coproduct need *not* have as their underlying sets the Cartesian product or coproduct of the underlying sets of X and Y.

(3) Given a group G, consider a functor $F : BG \to \mathbf{Set}$. As we saw in example 1.2.2 (2), such a functor consists of a set X together with a G-action on X. A limit of F is then in particular a G-equivariant map

$$Z \to X$$

where Z carries a trivial G-action. This implies that every $z \in Z$ must be sent to a fixed point of X under the G action. Since this is the case, we might guess that the limit of F is the set X^G of *G-fixed points of* X. It is easy to check that the universal property is satisfied by the inclusion $X^G \to X$. Similarly one can check that the colimit is the set X/G of G-orbits in X.

Proposition 1.4.3. *Let I be a samll category, and let* **Set** *be the category of small sets. Then every I-diagram in* **Set** *has a limit. (We say* **Set** has small limits.*)*

Proof. Let $F : I \to \mathbf{Set}$ be any diagram. We have to produce a limit cone (X, ξ). Define X to be the set of cones over F of the form $(*, \eta)$, where $*$ represents a chosen singleton set. Note that this definition yields

$$X \subseteq \prod_{i \in I} F(i),$$

and hence, X is small. It only remains to construct $\eta : \Delta(X) \Rightarrow F$. For each $i \in I$, we specify the map

$$\eta_i : X \to F(i); \quad (*, \xi) \mapsto \xi_i(*).$$

One can then directly check the universal property. $\square$

Problem 1.4.4. Construct limits and colimits in **Set**, **Ab**, **Grp**, and **Cat**.

Proposition 1.4.5. *Let I and $\mathcal{C}$ be categories. Assume that every $F \in \mathcal{C}^I$ has a limit. Then there is an adjunction*

$$\Delta : \mathcal{C} \leftrightarrow \mathcal{C}^I : \lim$$

such that the functor lim *associates to each diagram $F \in \mathcal{C}^I$ a limit* $\lim F$ *of F.*

Proof. We first construct the functor lim. For every diagram $F \in \mathcal{C}^I$, we choose a limit cone $(\lim F, \eta_F : \Delta(\lim F) \Rightarrow F)$.[6] This gives us a definition of lim on $\mathrm{ob}(\mathcal{C}^I)$.

Given a morphism $\gamma : F \Rightarrow G$, we can then form the diagram

$$\begin{tikzcd} \lim F \arrow[r, Rightarrow, "\eta_F"] \arrow[dr, Rightarrow, "\gamma * \eta_F"] & F \arrow[d, Rightarrow, "\gamma"] \\ \lim G \arrow[r, Rightarrow, "\eta_G"'] & G \end{tikzcd}$$

This makes $\lim F$ a cone over G, and therefore, by universal property, there exists a unique morphism, which we will call $\lim(\gamma)$ which makes the diagram

$$\begin{tikzcd} \lim F \arrow[r, Rightarrow, "\eta_F"] \arrow[d, Rightarrow, "\lim(\gamma)"'] \arrow[dr, Rightarrow, "\gamma * \eta_F"] & F \arrow[d, Rightarrow, "\gamma"] \\ \lim G \arrow[r, Rightarrow, "\eta_G"'] & G \end{tikzcd}$$

commute. Functoriality follows from the uniqueness of this morphism.

By construction the functor comes equipped with a natural transformation

$$\eta : \Delta \circ \lim \Rightarrow \mathrm{id}_{\mathcal{C}^I}$$

which we take to be the counit. To define the unit $X \to \lim \circ \Delta(X)$, we use the constant cone $\mathrm{id}_{\Delta(X)}$. Checking the compatibilities is left as an exercise for the reader. $\square$

In light of proposition 1.4.5, we consider the following more general context. Let $\varphi : I \to J$ be a functor, and let $\mathcal{C}$ be a category. We then obtain a functor

$$\varphi^* : \mathcal{C}^J \to \mathcal{C}^I; \quad F \mapsto F \circ \varphi$$

that specializes to Δ when $J = *$ (the category with one object and no non-identity morphisms). We might reasonably then ask whether a right adjoint to φ^* exists, and, if so, how we might explicitly describe it. The answer to this question comes from considering *extensions* of functors.

[6] Note that this construction therefore requires the axiom of choice.

Definition 1.4.6. Let $F \in \mathcal{C}^I$. A *right extension of F along φ* consists of

(1) a diagram $G \in \mathcal{C}^J$

(2) a natural transformation $\eta : \varphi^* G \to F$.[7]

Such a right extension (G, η) is called a *right Kan extension* if it has the following universal property:

- For every right extension (G', η') of F along φ, there exists a unique natural transformation $\gamma : G' \Rightarrow G$ such that the diagram

$$\begin{tikzcd} \varphi^* G' \arrow[rr, Rightarrow, "\varphi^*(\gamma)"] \arrow[dr, Rightarrow, "\eta'"'] & & \varphi^* G \arrow[dl, Rightarrow, "\eta"] \\ & F & \end{tikzcd}$$

  commutes.

Note that this definition restricts to the definition of a limit cone in the case where $J = *$. Indeed, the definition is so analogous that we can carry proposition 1.4.5 over to Kan extensions almost verbatim:

Proposition 1.4.7. *Let I, J and $\mathcal{C}$ be categories, and let $\varphi : I \to J$ be a functor. Assume that every $F \in \mathcal{C}^I$ has a right Kan extension along φ. Then there is an adjunction*

$$\varphi^* : \mathcal{C}^J \leftrightarrow \mathcal{C}^I : \varphi_*$$

such that the functor φ_* associates to each diagram $F \in \mathcal{C}^I$ a right Kan extension G of F along φ.

Proof. The proof is a mutatis mutandis version of the proof of 1.4.5 and is left to the reader. □

We will now establish a formula to compute Kan extension, in particular, providing a criterion for their existence. To this end, we introduce some terminology.

Definition 1.4.8. Given a functor $\varphi : I \to J$ and an object $j \in J$, we introduce the *slice category j/I* as follows: the objects are pairs (i, f) with $i \in I$ and $f : j \to \varphi(i)$. A morphism $(i, f) \to (i', f')$ consists of a morphism $g : i \to i'$ such that the diagram

$$\begin{tikzcd} & \varphi(i) \arrow[dd, "\varphi(g)"] \\ j \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & \varphi(i') \end{tikzcd}$$

commutes.

[7] The term *extension* here is, in fact, quite an apt description of the functor so defined, though this may not be immediately apparent from the definition. Diagrammatically, an extension looks like

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[dr, "F"'] & J \arrow[d, "G"] \\ & \mathcal{C} \end{tikzcd}$$

In the case where the diagram commutes strictly, and the I is a subcategory of J, this is precisely a functor *extending* F to all of J.

Let $\varphi : I \to J$ be a functor and let $F \in \mathcal{C}^I$ be an I-diagram. For a given object $j \in J$, we denote the restriction of F along the forgetful functor

$$j/I \to I, (i, f) \mapsto i$$

by $F|(j/I)$. Now suppose that (G, η) is a right extension of F along φ. Then, for every $j \in J$, we may construct a cone

$$\xi : \Delta(G(j)) \Rightarrow F|(j/I) \tag{1.4.9}$$

as follows: To specify ξ, we need to provide, for every $(i, f) \in j/I$ a morhpism

$$\xi_{(i,f)} : G(j) \to F(i).$$

To this end, consider:

(1) The right extension G comes equipped with a natural transformation

$$\eta : \varphi^* G \Rightarrow F$$

which we may evaluate at $i \in I$ to obtain a morphism

$$g : G(\varphi(i)) \to F(i).$$

(2) The morphism f defines a morphism

$$G(f) : G(j) \to G(\varphi(i)).$$

We set $\xi_{(i,f)} = g \circ G(f)$. It is now straightforward to verify that these morphisms combine to define the desired cone ξ from (1.4.9).

Theorem 1.4.10. *Let $\varphi : I \to J$ be a functor, $F \in C^I$.*

(1) Let (G, η) be a right extension of F along φ and suppose that, for every $j \in J$, the associated cone (1.4.9) is a limit cone. Then (G, η) is a right Kan extension.

(2) Vice versa, assume that, for every $j \in J$, the diagram $F|(j/I)$ has a limit in $\mathcal{C}$. Then there exists a right Kan extension G of F along φ and it admits the pointwise formula

$$G(j) \cong \lim F|(j/I).$$

Proof. To verify (1) suppose that we are given a right extension (G, η) such that all cones (1.4.9) are limit cones. Let (G', η') be an extension of F. Then, for every $j \in J$, we obtain a unique morphism $\gamma_j : G'(j) \to G(j)$ such that the diagram

$$\begin{tikzcd} \Delta(G'(j)) \arrow[dd, Rightarrow, "\Delta(\gamma_j)"'] \arrow[dr, Rightarrow, "\xi'"] & \\ & F|(j/I) \\ \Delta(G(j)) \arrow[ur, Rightarrow, "\xi"] & \end{tikzcd} \tag{1.4.11}$$

commutes. To verify that the morphisms $\gamma = \{\gamma_j\}$ organize to define a natural transformation, we need to show that, for every morphism $g : j \to j'$, the diagram

$$\begin{tikzcd} G'(j) \arrow[r, "G'(g)"] \arrow[d, "\gamma_j"] & G'(j') \arrow[d, "\gamma_{j'}"] \\ G(j) \arrow[r, "G(g)"] & G(j') \end{tikzcd}$$

commutes. To this end, we apply Δ to this diagram and complete it to

$$\begin{tikzcd}
\Delta(G'(j)) \arrow[r, Rightarrow, "G'(g)"] \arrow[d, Rightarrow, "\gamma_j"] & \Delta(G'(j')) \arrow[d, Rightarrow, "\gamma_{j'}"] \arrow[dr, Rightarrow, "\xi'"] & \\
\Delta(G(j)) \arrow[r, Rightarrow, "G(g)"] & \Delta(G(j')) \arrow[r, Rightarrow, "\xi"] & F|(j'/I)
\end{tikzcd}$$

where the right triangle commutes. Due to the universal property of the limit cone ξ, we see that, to verify the commutativity of the square, it suffices to show that the diagram

$$\begin{tikzcd}
\Delta(G'(j)) \arrow[d, Rightarrow, "\gamma_j"] \arrow[dr, Rightarrow, "\xi' \circ G'(g)"] & \\
\Delta(G(j)) \arrow[r, Rightarrow, "\xi \circ G(g)"] & F|(j'/I)
\end{tikzcd}$$

commutes. Unravelling the definitions, it follows that this diagram commutes since it is the pullback of the commutative diagram

$$\begin{tikzcd}
\Delta(G'(j)) \arrow[dd, Rightarrow, "\Delta(\gamma_j)"'] \arrow[dr, Rightarrow, "\xi'"] & \\
& F|(j/I) \\
\Delta(G(j)) \arrow[ur, Rightarrow, "\xi"] &
\end{tikzcd}$$

under the functor $j'/I \to j/I$ induced by $g : j \to j'$. Therefore, the datum $\gamma = \{\gamma_j\}_{j \in J}$ defines a natural transformation $G' \Rightarrow G$. We need to verify that it is a morphism of right extensions of F, i.e., that, for every $i \in I$, the diagram

$$\begin{tikzcd}
G'(\varphi(i)) \arrow[d, "\gamma_{\varphi(i)}"] \arrow[dr, "\eta'_i"] & \\
G(\varphi(i)) \arrow[r, "\eta_i"] & F(i)
\end{tikzcd}$$

commutes. But this follows from the commutativity of (1.4.11), since $\eta_i = \xi_{(\varphi(i),\mathrm{id})}$ and $\eta'_i = \xi'_{(\varphi(i),\mathrm{id})}$. To show that γ is in fact the unique morphism from (G', η') to (G, η), we note that any natural transformation $\gamma' : G \Rightarrow G'$ such that, for every $i \in I$, the diagram

$$\begin{tikzcd}
G'(\varphi(i)) \arrow[d, "\gamma'_{\varphi(i)}"] \arrow[dr, "\eta'_i"] & \\
G(\varphi(i)) \arrow[r, "\eta_i"] & F(i)
\end{tikzcd}$$

commutes, also makes due to the naturality of γ', for every morphism $f : j \to \varphi(i)$, the diagram

$$\begin{tikzcd}
G'(j) \arrow[r, "G'(f)"] \arrow[d, "\gamma'_j"] & G'(\varphi(i)) \arrow[d, "\gamma'_{\varphi(i)}"] \arrow[dr, "\eta'_i"] & \\
G(j) \arrow[r, "G(f)"] & G(\varphi(i)) \arrow[r, "\eta_i"] & F(i)
\end{tikzcd}$$

commute. But this means that the morphism $\gamma'_j : G'(j) \to G(j)$ extends to a morphism between the $(G'(j), \xi')$ and $(G(j), \xi)$ and thus has to agree with γ_j, since the latter cone is a limit cone. This shows (1).

To verify (2), we construct, under the given hypothesis, a right extension (G, η) of F such that the corresponding cones (1.4.9) are limit cones. It then follows by (1), that (G, η) is a right Kan extension. To this end, we choose, for every $j \in J$, a limit cone $(G(j), \alpha^{(j)})$ over $F|(j/I)$ which in particular defines the functor G on objects via $j \mapsto G(j)$. Let $g : j \to j'$ be a morhpism in J. This morphism induces a functor

$$\psi_g : j'/I \to j/I, (i, f) \mapsto (i, fg)$$

along which we pull back $\alpha^(j) : \Delta(G(j)) \to F|(j/I)$ to obtain a cone over $F|(j'/I)$ with vertex $G(j)$. Due to the universal property of the limit cone $(G(j'), \alpha^{(j')})$ this cone defines a unique morphism $G(j) \to G(j')$ which we set to be $G(f)$. The functoriality follows from the uniqueness of this association. We exhibit the resulting functor $G : J \to \mathcal{C}$ as a right extension of F: for every $i \in I$, define

$$\eta_i : G(\varphi(i)) \to F(i)$$

to be the morphism $\eta_i = \alpha^{(\varphi(i))}_{(\varphi(i),\mathrm{id})}$ which is part of the limit cone defining $G(\varphi(i))$. It is now straightforward to verify that the various morphisms η_i define a natural transformation $\eta : \varphi^* G \to F$. The corresponding cones (1.4.9) agree with the cones $\alpha^{(j)}$ and are hence limit cones by construction, concluding the argument. □

Corollary 1.4.12. *Let $\varphi : I \to J$ be a functor of small categories and suppose that $\mathcal{C}$ is a category that has small limits. Then there is an adjunction*

$$\varphi^* : \mathcal{C}^J \longleftrightarrow \mathcal{C}^I : \varphi_*$$

so that φ_* associates to a diagram $F \in \mathcal{C}^I$ a right Kan extension of F along φ.

Proof. Immediate from Theorem 1.4.10 and Proposition 1.4.7. □

Problem 1.4.13. Under the hypothesis of Corollary 1.4.12 show that, if φ is fully faithful, then φ_* is fully faithful.

Example 1.4.14. Let $J = [1] \times [2]$ which we depict as

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \arrow[r] & (1,2) \end{tikzcd}$$

and let $I \subset J$ denote the full subcategory spanned by the objects $(1,1)$,$(1,2)$, and $(0,2)$ which we depict by

$$\begin{tikzcd} & (0,2) \arrow[d] \\ (1,1) \arrow[r] & (1,2). \end{tikzcd}$$

Suppose that $\mathcal{C}$ is a category with small limits. Then by Theorem 1.4.10, a diagram $F \in \mathcal{C}^J$ is a right Kan extension of its restriction to I if and only if it satisfies the following conditions:

(1) The morphism $F(1,0) \to F(1,1)$ is an isomorphism.

(2) The diagram

$$\begin{tikzcd} F(0,1) \arrow[r] \arrow[d] & F(0,2) \arrow[d] \\ F(1,1) \arrow[r] & F(1,2) \end{tikzcd}$$

is a limit cone with vertex $F(0,1)$.

(3) The diagram

$$\begin{tikzcd} F(0,0) \arrow[r] \arrow[d] & F(0,2) \arrow[d] \\ F(1,0) \arrow[r] & F(1,2) \end{tikzcd}$$

is a limit cone with vertex $F(0,0)$.

Problem 1.4.13 implies that the functor φ_* establishes an equivalence of categories between the category $\mathcal{C}^I$ and the full subcategory of $\mathcal{C}^J$ consisting of those J-diagrams that satisfy the conditions (1),(2), and (3).

Example 1.4.15. Let G be a small group and $H \subset G$ a subgroup. Let K be a small field and $F : BH \to \mathbf{Vect}_K$ a representation of H. We denote $V = F(*)$. By Theorem 1.4.10, a right Kan extension R of F along $BH \subset BG$ exists and can be computed as follows: We choose a set $\{g_i\}_{i \in I}$ of representatives of the cosets of H in G, so that the association $i \mapsto g_i H$ defines a bijection $I \cong G/H$. We set

$$R(*) = \bigoplus_{i \in I} V.$$

The action of an element $g \in G$ on a vector $v_i \in V$ in the component corresponding to the representative g_i is as follows: suppose that $gg_i = g_j h$ for some representative g_j. Then we define $g.v$ to be $h.v$ considered as a vector in the component of $R(*)$ corresponding to g_j. We leave it to the reader to exhibit the representation $R : BG \to \mathbf{Vect}_K$ as a right Kan extension of F.

Chapter 2

Simplicial Homotopy Theory

We develop the rudiments of simplicial homotopy theory, closely following the reference [1] which we highly recommend for further details.

2.1 Simplicial sets

The *simplex category* Δ has objects given by the standard linearly ordered sets $\{[n]\}_{n\in\mathbb{N}}$ and the set of morphisms $\Delta([m],[n])$ between $[m]$ and $[n]$ is defined to be the set of weakly monotone maps $f : \{0,1,\dots,m\} \to \{0,1,\dots,n\}$, i.e., maps of underlying sets such that, for $0 \le i \le j \le m$, we have $f(i) \le f(j)$.

Definition 2.1.1. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$.

We denote the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ of simplicial sets by $\mathbf{Set}_\Delta$.

Example 2.1.2. Let $n \ge 0$. Then the functor

$$\Delta(-,[n]) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set},\ [m] \mapsto \Delta([m],[n])$$

defines a simplicial set denoted by Δ^n, called the *standard n-simplex.* Representing a map $f : [m] \to [n]$ by the list $f(0)f(1)\dots f(m)$ of its values, we describe the low-dimensional simplices in Δ^2:

- The 0-simplices, or vertices, are $\Delta^2([0]) = \{0,1,2\}$.
- The 1-simplices, or edges, are $\Delta^2([1]) = \{01, 02, 12, 00, 11, 22\}$. Note the presence of the *degenerate edges* 00, 11, 22, corresponding to the constant maps $[1] \to [2]$.
- The 2-simplices are $\Delta^2([2]) = \{012, 011, 001, 002, 022, 112, 122, 000, 111, 222\}$.
- How many k-simplices does Δ^2 have?

Example 2.1.3. Let $n \ge 0$, and let $\mathcal{J} \subset \mathcal{P}(\{0,1,\dots,n\})$ be a collection of subsets of $\{0,1,\dots,n\}$. Then we define a simplicial subset $\Delta^{\mathcal{J}} \subset \Delta^n$ by letting

$$\Delta^{\mathcal{J}}([m]) \subset \Delta^n([m])$$

be the set of those morphisms $f : [m] \to [n]$ for which there exists $J \in \mathcal{J}$ such that $\mathrm{im}(F) \subset J$. Specific cases of interest are

- $\mathcal{J} = \mathcal{P}(\{0,1,\dots,n\})$: Then $\Delta^{\mathcal{J}} = \Delta^n$.
- $\mathcal{J} = \mathcal{P}(\{0,1,\dots,n\}) \setminus \{\{0,1,\dots,n\}\}$: Then

  $$\Delta^{\mathcal{J}} =: \partial\Delta^n$$

  is called the *boundary of the n-simplex.*

- $\mathcal{J} = \mathcal{P}(\{0, 1, \dots, n\}) \setminus \{\{0, 1, \dots, n\}, \{0, \dots, i-1, i+1, \dots, n\}\}$: Then
$$\Delta^{\mathcal{J}} =: \Lambda^n_i$$
is called the *ith horn of the n-simplex.*

- A collection $\mathcal{K} \subset \mathcal{P}(\{0, 1, \dots, n\})$ of nonempty subsets is called an *(abstract) simplicial complex* if
$$(J \in \mathcal{K} \text{ and } \emptyset \neq I \subset J) \quad \Rightarrow \quad I \in \mathcal{K}.$$
We thus obtain a simplicial set $\Delta^{\mathcal{K}}$ associated to any simplicial complex $\mathcal{K}$. Note that simplicial sets are more general than simplicial complexes. For example, the *simplicial circle* $\Delta^1/\partial\Delta^1$ defined via the formula
$$(\Delta^1/\partial\Delta^1)([m]) = \Delta^1([m])/\partial\Delta^1([m])$$
cannot arise from a simplicial complex: every simplex in a simplicial complex has distinct vertices.

Example 2.1.4. Let $\mathcal{C}$ be a small category. Note that we may interpret the ordinal $[n]$ as a category and a weakly monotone map $[m] \to [n]$ between ordinals as a functor. This defines a functor
$$\chi : \Delta \longrightarrow \mathbf{Cat}.$$
Restricting the functor
$$\mathrm{Fun}(-, \mathcal{C}) : \mathbf{Cat}^{\mathrm{op}} \longrightarrow \mathbf{Set}, I \mapsto \mathrm{Fun}(I, \mathcal{C})$$
along χ yields a simplicial set
$$\mathrm{N}(\mathcal{C}) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}, [n] \mapsto \mathrm{Fun}([n], \mathcal{C})$$
called the *nerve of* $\mathcal{C}$. Explicitly, in low dimensions, we have

- The vertices of $\mathrm{N}(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The edges of $\mathrm{N}(\mathcal{C})$ are the morphisms of $\mathcal{C}$ where the vertex 0 is the source and the vertex 1 the target of the morphism, respectively.
- The 2-simplices are commutative diagrams
$$\begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_2 \end{tikzcd}$$
- The n-simplices can be identified with composable chains of n morphisms in $\mathcal{C}$.

Example 2.1.5. Let Y be a small topological space. For $n \geq 0$, we define the *geometric n-simplex* to be the topological space
$$|\Delta^n| := \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 0; t_i \geq 0\}.$$
Given a morphism $f : [m] \to [n]$, we define a continuous map
$$f_* : |\Delta^m| \longrightarrow |\Delta^n|,\ (t_0, t_1, \dots, t_m) \mapsto (s_0, s_1, \dots, s_n)$$
with
$$s_i = \begin{cases} 0 & \text{if } f^{-1}(i) = \emptyset, \\ \sum_{j \in f^{-1}(i)} t_j & \text{else.} \end{cases}$$
This construction defines a functor
$$\rho : \Delta \longrightarrow \mathbf{Top},\ [n] \mapsto |\Delta^n|.$$

Restriction of the functor

$$\mathbf{Top}(-,Y) : \mathbf{Top}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

along ρ yields a simplicial set

$$\mathrm{Sing}(Y) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set},\ [n] \mapsto \mathbf{Top}(|\Delta^n|, Y)$$

called the *singular set of* Y. Explicitly, in low dimensions, we have

- The set of vertices of $\mathrm{Sing}(Y)$ is the underlying set of Y.
- The edges of $\mathrm{Sing}(Y)$ are continuous paths $\alpha : I \to Y$.
- The 2-simplices in $\mathrm{Sing}(Y)$ are $|\Delta^2| \to Y$ that can, up to reparametrization, be regarded as homotopies between the composite of the paths $0 \to 1$ and $1 \to 2$ and the path $0 \to 2$.
- ...

Let X be a simplicial set. We abbreviate $X_n := X([n])$. For every $0 \le i \le n$, there is a map

$$d_i : X_n \longrightarrow X_{n-1},$$

called the *ith face map*, that corresponds to the morphism

$$\partial_i : [n-1] \longrightarrow [n],\ j \mapsto \begin{cases} j & \text{for } j < i, \\ j+1 & \text{for } j \ge i. \end{cases}$$

in Δ. For every $0 \le i < n$, there is a map

$$s_i : X_{n-1} \longrightarrow X_n,$$

called the *ith degeneracy map*, that corresponds to the morphism

$$\sigma_i : [n-1] \longrightarrow [n],\ j \mapsto \begin{cases} j & \text{for } j \le i, \\ j-1 & \text{for } j > i. \end{cases}$$

These maps satisfy the following *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j, \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j, \\ d_j s_j &= 1 && \\ d_{j+1} s_j &= 1 && \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1, \\ s_i s_j &= s_{j+1} s_i && \text{if } i \le j. \end{aligned} \tag{2.1.6}$$

Problem 2.1.7. The datum of a simplicial set $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is equivalent to the data given by

- the sets $\{X_n\}_{n \ge 0}$,
- the face and degeneracy maps, satisfying the simplicial identities (2.1.6).

2.2 Geometric realization

Let $K \in \mathbf{Set}_\Delta$ be a simplicial set. We define the *geometric realization of* K to be the topological space

$$|K| := (\coprod_{n\geq 0} K_n \times |\Delta^n|)/\sim \tag{2.2.1}$$

where $\sim$ denote the equivalence relation generated by

$$(f^*(\sigma), y) \sim (\sigma, f_*(y))$$

where (f, σ, y) runs over all morphisms $f : [m] \to [n]$ in Δ, $\sigma \in K_n$, and $y \in |\Delta^m|$. This construction is functorial in K so that it defines a functor

$$|-| : \mathbf{Set}_\Delta \longrightarrow \mathbf{Top}.$$

Lemma 2.2.2. *Let $K \in \mathbf{Set}_\Delta$.*

(1) The simplicial set K is a colimit of the diagram

$$F : \Delta/K \longrightarrow \mathbf{Set}_\Delta,\ (\Delta^n \to K) \mapsto \Delta^n.$$

(2) The topological space $|K|$ is a colimit of the diagram

$$|F| : \Delta/K \longrightarrow \mathbf{Top},\ (\Delta^n \to K) \mapsto |\Delta^n|.$$

Proof. To show (1), we note that, unravelling the definitions, to provide a cone under F with vertex $S \in \mathbf{Set}_\Delta$, is equivalent to providing a map of simplicial sets $K \to S$. It is then immediate that the cone corresponding to the identity map $K \to K$ is a colimit cone under F.

Statement (2) follows by an explicit computation: assuming the existence of all small colimits in a given category $\mathcal{C}$, the colimit of any diagram $G : I \to \mathcal{C}$ with I small, can be expressed as the coequalizer of

$$\coprod_{f:i\to j \text{in } I} G(i) \underset{G(f)}{\overset{\text{id}}{\rightrightarrows}} \coprod_{i\in I} G(i)$$

Applying this formula to the functor $|F|$ describes its colimit as the coequalizer of

$$\coprod_{\Delta^m\to\Delta^n\to K} |\Delta^m| \underset{f_*}{\overset{\text{id}}{\rightrightarrows}} \coprod_{\Delta^n\to K} |\Delta^n|$$

which we may identify with the coequalizer of

$$\coprod_{f:[m]\to[n]} K_n \times |\Delta^m| \underset{\text{id}\times f_*}{\overset{f^*\times\text{id}}{\rightrightarrows}} \coprod_{n\geq 0} K_n \times |\Delta^n|$$

which in turn recovers precisely $|K|$ via its defining formula (2.2.1). $\square$

Corollary 2.2.3. *The functor*

$$|-| : \mathbf{Set}_\Delta \longrightarrow \mathbf{Top}$$

is a left Kan extension of the functor

$$\rho : \Delta \longrightarrow \mathbf{Top},\ [n] \mapsto |\Delta^n|$$

along the Yoneda embedding $\Delta \to \mathbf{Set}_\Delta$. In particular, the geometric realization functor is uniquely determined by

(1) $|-|$ commutes with small colimits.

(2) The restriction of $|-|$ along $\Delta \to \mathbf{Set}_\Delta$ is isomorphic to ρ.

Proof. The first part follows, in virtue of Lemma 2.2.2(2), from the pointwise formula for Kan extensions (Theorem 1.4.10). The second part is a general characterization of Kan extensions along the Yoneda embedding which is the content of Problem 4 on Exercise Sheet 3. □

Corollary 2.2.4. *There is an adjunction*

$$|-| : \mathbf{Set}_\Delta \longleftrightarrow \mathbf{Top} : \mathrm{Sing}\,.$$

Proof. Let K be a simplicial set, and let X be a topological space. Then we have

$$\begin{aligned}
\mathbf{Top}(|K|, X) &\cong \mathbf{Top}(\operatorname*{colim}_{\Delta/K} |\Delta^\bullet|, X) && \text{by Lemma 2.2.2(2)}\\
&\cong \lim_{\Delta/K} \mathbf{Top}(|\Delta^\bullet|, X) && \text{colimit cone}\\
&\cong \lim_{\Delta/K} \mathbf{Set}_\Delta(\Delta^\bullet, \mathrm{Sing}(X)) && \text{Yoneda + definition of Sing}\\
&\cong \mathbf{Set}_\Delta(\operatorname*{colim}_{\Delta/K} \Delta^\bullet, \mathrm{Sing}(X)) && \text{colimit cone}\\
&\cong \mathbf{Set}_\Delta(K, \mathrm{Sing}(X)) && \text{by Lemma 2.2.2(1)}
\end{aligned}$$

where naturality in K and X is straightforward to verify. □

Example 2.2.5. Let $n \geq 0$ and let $\mathcal{K} \subset \mathcal{P}(\{0, 1, \ldots, n\})$ be an abstract simplicial complex. Then we have

$$\begin{aligned}
|\Delta^{\mathcal{K}}| &\cong |\operatorname*{colim}_{J \in \mathcal{K}} \Delta^{\{J\}}|\\
&\cong \operatorname*{colim}_{J \in \mathcal{K}} |\Delta^{\{J\}}|\\
&\cong \bigcup_{J \in \mathcal{K}} |\Delta^{\{J\}}| \subset |\Delta^n|
\end{aligned}$$

so that $|\Delta^{\mathcal{K}}|$ is simply the union of the geometric simplices in $|\Delta^n|$ that comprise $\mathcal{K}$. In particular, we have that, for $n \geq 0$, the space $|\partial\Delta^n|$ is the geometric boundary of $|\Delta^n|$, i.e., the unions of all faces of $|\Delta^n|$. Similarly, the space $|\Lambda^n_i|$ is the union of those faces of $|\Delta^n|$ that contain the vertex i.

The idea of simplicial homotopy theory is to formulate a combinatorial approach to homotopy theory in terms of the category $\mathbf{Set}_\Delta$ and relate it to ordinary homotopy theory in $\mathbf{Top}$ via the adjunction

$$|-| : \mathbf{Set}_\Delta \longleftrightarrow \mathbf{Top} : \mathrm{Sing}\,. \tag{2.2.6}$$

We will now begin to set up the combinatorial language needed for this formulation.

2.3 Kan fibrations

Definition 2.3.1. A simplicial set K is called a *Kan complex* if it satisfies the following *horn filling condition:*

For all $0 \leq i \leq n$, $n > 0$, every map

$$\Lambda^n_i \xrightarrow{f} K$$

can be extended to a commutative diagram

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & K. \\
\Delta^n \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}$$

Proposition 2.3.2. *Let X be a topological space. Then the singular set* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. For every $0 \leq i \leq n$, the inclusion

$$\begin{tikzcd} {|\Lambda^n_i|} \arrow[r, "j"'] & {|\Delta^n|} \arrow[l, dashed, bend right, "p"'] \end{tikzcd}$$

admits a retraction $p : |\Delta^n| \to |\Lambda^n_i|$, i.e., $p \circ j = \mathrm{id}$. The map p is given by projection parallel to the vector from the barycenter of the ith face of $|\Delta^n|$ to the ith vertex of $|\Delta^n|$. Using the adjunction (2.2.6), the extension problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

is equivalent to the adjoint extension problem

$$\begin{tikzcd} {|\Lambda^n_i|} \arrow[r, "f'"] \arrow[d] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

which can be solved by setting the dashed arrow to be $f' \circ p$. □

Proposition 2.3.3. *Let $\mathcal{C}$ be a small category.*

(1) For all $0 < i < n$, every horn

$$\Lambda^n_i \xrightarrow{f} \mathrm{N}(\mathcal{C})$$

can be uniquely *extended to a commutative diagram*

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d] & \mathrm{N}(\mathcal{C}). \\ \Delta^n \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

(2) The simplicial set $\mathrm{N}(\mathcal{C})$ *is a Kan complex if and only if $\mathcal{C}$ is a groupoid. In this case, every horn in dimension $n \geq 2$ has a* unique *filler.*

Proof. (1) The fact that every inner horns $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ can be uniquely filled follows from the fact that a composable pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ has a unique composite $g \circ f$. The unique horn filling for horns $\Lambda^3_1 \to \mathrm{N}(\mathcal{C})$ and $\Lambda^3_2 \to \mathrm{N}(\mathcal{C})$ follows from the associativity of the composition law in $\mathcal{C}$.

For $n > 3$, we argue as follows. Let $j : \Delta_{\leq 2} \subset \Delta$ be the inclusion of the full subcategory spanned by the objects [0],[1], and [2] of Δ. We denote the restriction functor along j by

$$\mathrm{sk}_2 : \mathbf{Set}_\Delta \longrightarrow \mathbf{Set}_{\Delta_{\leq 2}}.$$

Then it is immediate to verify that, for every simplicial set K, the functor sk_2 induces a bijection

$$\mathbf{Set}_\Delta(K, \mathrm{N}(\mathcal{C})) \cong \mathbf{Set}_{\Delta_{\leq 2}}(\mathrm{sk}_2(K), \mathrm{sk}_2(\mathrm{N}(\mathcal{C}))).$$

The statement then follows immediately from the observation that, for $n > 3$ and every $0 \leq i \leq n$, the inclusion $\Lambda^n_i \to \Delta^n$ induces an isomorphism $\mathrm{sk}_2(\Lambda^n_i) \to \mathrm{sk}_2(\Delta^n)$.

The proof of part (2) is left to the reader. □

We now introduce a relative variant of the notion of a Kan complex:

Definition 2.3.4. A morphism $p : K \to S$ of simplicial sets is called a *Kan fibration* if, for every $0 \le i \le n$, $n > 0$, and every solid commutative diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S, \end{tikzcd}$$

there exists a dashed arrow making the diagram commute.

In the situation of Definition 2.3.4, the solid commutative square is often called a lifting problem while the dashed arrow is called its solution. We will now develop a rather abstract but effective approach to analyze such lifting problems.

2.4 The small object argument

Throughout this section, let $\mathcal{C}$ be a category with small colimits.

Definition 2.4.1. A set $\mathcal{S}$ of morphisms in $\mathcal{C}$ is called *saturated* if the following conditions are satisfied:

(1) $\mathcal{S}$ contains all isomorphisms.

(2) $\mathcal{S}$ is closed under pushouts: Given a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \end{tikzcd}$$

with $i \in \mathcal{S}$, then we also have $i' \in \mathcal{S}$.

(3) $\mathcal{S}$ is closed under retracts: Given a commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \end{tikzcd}$$

with $i \in \mathcal{S}$, then we also have $i' \in \mathcal{S}$.

(4) $\mathcal{S}$ is closed under countable composition: Given a chain of morphisms

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \dots$$

in $\mathcal{S}$, indexed by $\mathbb{N}$, then the canonical morphism

$$A_0 \longrightarrow \operatorname*{colim}_{i \in \mathbb{N}} A_i$$

is also contained in $\mathcal{S}$.

(5) $\mathcal{S}$ is closed under small coproducts: Let $\{i_j : A_j \to B_j\}_{j \in J}$ be a small subset of $\mathcal{S}$. Then the coproduct

$$\textstyle\coprod_{j \in J} A_j \xrightarrow{\coprod i_j} \coprod_{j \in J} B_j$$

is contained in $\mathcal{S}$.

The intersection of any set of saturated sets of morphisms in $\mathcal{C}$ is saturated. For a given set $\mathcal{M}$ of morphisms in $\mathcal{C}$, we then define the *saturated hull of* $\mathcal{M}$ to be the set

$$\overline{\mathcal{M}} := \bigcap_{\substack{\mathcal{M}\subset\mathcal{S} \\ \mathcal{S}\text{ saturated}}} \mathcal{S}.$$

Definition 2.4.2. Let $i : A \to B$ and $p : K \to S$ be morphisms in $\mathcal{C}$. We say that

- i has the left lifting property with respect to p, and
- p has the right lifting property with respect to i,

if, for every solid commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & S, \end{tikzcd}$$

there exists a dashed arrow making the diagram commute. In this case, we also say that the solid square forms a lifting problem and the dashed arrow provides a solution to the lifting problem.

Given a set $\mathcal{M}$ of morphisms in $\mathcal{C}$, we define

- $\mathcal{M}_\perp$ to be the set of morphisms in $\mathcal{C}$ that have the right lifting property with respect to every morphism in $\mathcal{M}$, and
- ${}_\perp\mathcal{M}$ to be the set of morphisms in $\mathcal{C}$ that have the left lifting property with respect to every morphism in $\mathcal{M}$.

Example 2.4.3. Let $\mathcal{C} = \mathbf{Set}_\Delta$ and let

$$\mathcal{M} = \{\Lambda^n_i \hookrightarrow \Delta^n \mid 0 \le i \le n, n > 0\}$$

denote the set of all horn inclusions. Then the set $\mathcal{M}_\perp$ is precisely the set of Kan fibrations.

Proposition 2.4.4. *Let $\mathcal{M}$ be a set of morphisms in $\mathcal{C}$. Then the set ${}_\perp\mathcal{M}$ is saturated.*

Proof. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & K \arrow[d] \\ B \arrow[r] & S \end{tikzcd}$$

with i isomorphism, then the morphism $f \circ i^{-1}$ solves the lifting problem, verifying that isomorphisms are contained in ${}_\perp\mathcal{M}$.

Suppose that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \end{tikzcd} \tag{2.4.5}$$

is a pushout square with $i \in {}_\perp\mathcal{M}$. Given a lifting problem

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"] & K \arrow[d, "p"] \\ B' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{2.4.6}$$

with $p \in \mathcal{M}$, we concatenate the squares (2.4.6) and (2.4.5) to obtain

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d, "p"] \\ B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] & S \end{tikzcd}$$

where the exterior rectangle provides a lifting problem that we can solve, since $i \in {}_\perp\mathcal{M}$. Using the universal property of B' as a colimit, we then immediately obtain a solution to the original lifting problem (2.4.6) thus showing that ${}_\perp\mathcal{M}$ is closed under pushouts.

Let $i : A \to B$ in ${}_\perp\mathcal{M}$ and let

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \end{tikzcd} \tag{2.4.7}$$

be a diagram that exhibits i' as a retract of i. Then, given a lifting problem

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"] & K \arrow[d, "p"] \\ B' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{2.4.8}$$

we concatenate the right square of (2.4.7) and (2.4.8), solve the resulting lifting problem, and then concatenate the solution with the left square in (2.4.7) to obtain a solution of the original lifting problem (2.4.8). This shows that ${}_\perp\mathcal{M}$ is closed under retracts.

Let

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \dots \tag{2.4.9}$$

be a chain of morphisms in ${}_\perp\mathcal{M}$ and let A_∞ denote its colimit. Given a lifting problem

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ A_\infty \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{2.4.10}$$

we build a cone under the diagram (2.4.9) by iteratively solving lifting problems of the form

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ A_{i+1} \arrow[r] \arrow[ur, dashed] & S. \end{tikzcd}$$

Due to the universal property of A_∞ as a colimit, this cone induces a unique map $A_\infty \to K$ which is easily seen to provide a solution of the lifting property (2.4.10).

The verification of the closure under small coproducts follows by a similar argument and is left to the reader. □

Corollary 2.4.11. *Let $\mathcal{M}$ be a set of morphisms in $\mathcal{C}$. Then we have*

$$\overline{\mathcal{M}} \subset {}_\perp(\mathcal{M}_\perp).$$

Proof. This is immediate from Proposition 2.4.4. □

Our next goal will be to show that under suitable hypotheses, the inclusion in Corollary 2.4.11 can be improved to an equality

$$\overline{\mathcal{M}} = {}_\perp(\mathcal{M}_\perp).$$

To this end, we need one more bit of further terminology. Recall that a partially ordered set I is called *filtered* if every finite subset of I has an upper bound.

Definition 2.4.12. An object X of $\mathcal{C}$ is called *compact* if, for every diagram

$$I \to \mathcal{C},\ i \mapsto Y_i$$

indexed by a small filtered partially ordered set I, with colimit Y_∞ and colimit cone $\{\eta_i : Y_i \to Y_\infty\}$, and for every morphism $f : X \to Y_\infty$, the following hold:

(1) There exists $i \in I$ and $f_i : X \to Y_i$ such that

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

commutes.

(2) Given $i, j \in I$, $f_i : X \to Y_i$, and $f_j : X \to Y_j$ such that the diagrams

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f"'] & Y_j \arrow[d, "\eta_j"] \\ & Y_\infty \end{tikzcd}$$

commute, then there exists $i, j \leq k$ such that

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "f_j"'] & Y_i \arrow[d] \\ Y_j \arrow[r] & Y_k \end{tikzcd}$$

commutes.

Remark 2.4.13. Suppose that $\mathcal{C}$ is locally small. Then an object X is compact if an only if the functor

$$\mathcal{C}(X, -) : \mathcal{C} \longrightarrow \mathbf{Set}$$

commutes with small filtered colimits.

Example 2.4.14. We claim that a set $X \in \mathbf{Set}$ is compact if and only if X is finite. Suppose first that X is compact. Consider the diagram

$$\mathcal{P}^{\mathrm{fin}}(X) \longrightarrow \mathbf{Set},\ Y \subset X \mapsto Y$$

where $\mathcal{P}^{\mathrm{fin}}(X)$ denotes the poset of finite subsets of X. This poset is filtered and X is a colimit of the diagram. Therefore, since X is compact, there exists a finite subset $Y \subset X$, and a factorization

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\mathrm{id}"'] & Y \arrow[d, hook] \\ & X. \end{tikzcd}$$

But this implies that $X \cong Y$ so that X is finite. Vice versa suppose that X is finite. Given a diagram

$$I \longrightarrow \mathbf{Set},\ i \mapsto Y_i$$

with I small filtered poset, we denote by

$$Y_\infty := (\coprod_{i\in I} Y_i)/\sim \tag{2.4.15}$$

its colimit. In virtue of the formula (2.4.15), for every element $x \in X$, there exists $i_x \in I$, such that $f(x)$ lies in the image of $Y_i \to Y_\infty$. Let i be an upper bound of the finite (!) set $\{i_x\}_{x\in X} \subset I$. Then it is clear that we have a factorization

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty, \end{tikzcd}$$

establishing property (1). Now suppose that we have two factorizations

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f"'] & Y_j \arrow[d, "\eta_j"] \\ & Y_\infty. \end{tikzcd}$$

Then, again in virtue of the explicit formula (2.4.15), for every $x \in X$, there exists $i, j \leq k_x$ such that the image of $f_i(x)$ and $f_j(x)$ in Y_{k_x} coincides. Let k denote an upper bound of the finite set $\{k_x\}_{x\in X} \subset I$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "f_j"'] & Y_i \arrow[d] \\ Y_j \arrow[r] & Y_k \end{tikzcd}$$

commutes as required.

Lemma 2.4.16 (Small object argument)**.** *Let $\mathcal{C}$ be a locally small category with small colimits. Suppose $\mathcal{M}$ is a small set of morphisms such that, for every $i : A \to B$ in $\mathcal{M}$, the object A is compact. Then every morphism $f : X \to Y$ in $\mathcal{C}$ admits a factorization*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"] & & Y \\ & Z \arrow[ur, "g"] & \end{tikzcd}$$

with $h \in \overline{\mathcal{M}}$ and $g \in \mathcal{M}_\perp$.

Before we give the proof, we show the following:

Corollary 2.4.17. *Under the hypothesis of Lemma 2.4.16, we have*

$$\overline{\mathcal{M}} = {}_\perp(\mathcal{M}_\perp).$$

Proof. Let $f \in {}_\perp(\mathcal{M}_\perp)$ and consider the factorization

$$f = g \circ h$$

from Lemma 2.4.16. This factorization defines the lifting problem

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

which has a solution since $g \in \mathcal{M}_\perp$ and $f \in {}_\perp(\mathcal{M}_\perp)$. We denote the solution by $r : Y \to Z$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & X \arrow[r, "\mathrm{id}"] \arrow[d, "h"] & X \arrow[d, "f"] \\ Y \arrow[r, "r"] & Z \arrow[r, "g"] & Y \end{tikzcd}$$

exhibits f as a retract of h so that $f \in \overline{\mathcal{M}}$. □

Proof of Lemma 2.4.16. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Consider the small (!) set

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"] & X \arrow[d, "f"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}_{j \in J}$$

of all commutative diagrams of the given form with $i_j \in \mathcal{M}$. We form the commutative square

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\coprod i_j"] & X \arrow[d, "f"] \\ \coprod_{j \in J} B_j \arrow[r] & Y \end{tikzcd}$$

and further

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\coprod i_j"] & X \arrow[d, "h_1"'] \arrow[ddr, "f"] & \\ \coprod_{j \in J} B_j \arrow[r] \arrow[drr] & X_1 \arrow[dr, "g_1"] & \\ & & Y \end{tikzcd}$$

where X_1 is a pushout so that we obtain a factorization $f = g_1 \circ h_1$ with $h_1 \in \overline{\mathcal{M}}$. We repeat this construction with f replaced by g_1: Consider the small set

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"] & X_1 \arrow[d, "g_1"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}_{j \in J_1}$$

and form the pushout

$$\begin{tikzcd} \coprod_{j \in J_1} A_j \arrow[r] \arrow[d, "\coprod i_j"] & X_1 \arrow[d, "h_2"'] \arrow[ddr, "g_1"] & \\ \coprod_{j \in J_1} B_j \arrow[r] \arrow[drr] & X_2 \arrow[dr, "g_2"] & \\ & & Y \end{tikzcd}$$

to obtain a factorization $f = g_2 \circ h_2 \circ h_1$ with $h_2 \in \overline{\mathcal{M}}$. Iterating this construction, we produce a sequence

$$\begin{tikzcd} X \arrow[r, "h_1"] \arrow[drr, "f"'] & X_1 \arrow[r, "h_2"] \arrow[dr, "g_1"] & X_2 \arrow[r] \arrow[d, "g_2"] & \dots \arrow[r] & X_\infty = \operatorname{colim} X_i \arrow[dll, "g"] \\ & & Y & & \end{tikzcd}$$

and henceforth a factorization $f = g \circ h$ with $h \in \overline{\mathcal{M}}$ as the countable composition of morphisms in $\overline{\mathcal{M}}$. We claim that this is the desired factorization, i.e., that $g \in \mathcal{M}_\perp$. To this end, consider a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_\infty \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2.4.18}$$

with $i \in \mathcal{M}$. Due to the compactness of A, we obtain a refinement of the square to

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "j"'] & & X_\infty \arrow[dd, "g"] \\ & X_k \arrow[ur] \arrow[dr] & \\ B \arrow[rr] & & Y \end{tikzcd}$$

for some $k \in \mathbb{N}$. But then the bottom left square is one of the commutative squares comprising the set J_k of the kth step of the iterative construction. In particular, we may further refine the square to the commutative diagram

$$\begin{tikzcd} A \arrow[rrr] \arrow[dr] \arrow[ddd, "j"'] & & & X_\infty \arrow[ddd, "g"] \\ & X_k \arrow[urr] \arrow[dr] & & \\ & & X_{k+1} \arrow[uur] \arrow[dr] & \\ B \arrow[urr] \arrow[rrr] & & & Y \end{tikzcd}$$

But then the composite of the morphisms $B \to X_{k+1} \to X_\infty$ provides a solution to the lifting problem (2.4.18). □

2.5 Anodyne morphisms

A morphism of simplicial sets is called *anodyne* if it has the left lifting property with respect to all Kan fibrations.

Corollary 2.5.1. *The set of anodyne morphisms is the saturated hull of the set*

$$\{\Lambda^n_i \hookrightarrow \Delta^n \mid n > 0, 0 \le i \le n\}$$

of horn inclusions.

Proof. Immediate from Corollary 2.4.17. □

Let $f : A \to A'$ and $g : B \to B'$ be morphisms of simplicial sets. The commutative square

$$\begin{array}{ccc} A \times B & \xrightarrow{f\times \mathrm{id}} & A' \times B \\ \big\downarrow{\scriptstyle \mathrm{id}\times g} & & \big\downarrow{\scriptstyle \mathrm{id}\times g} \\ A \times B' & \xrightarrow{f\times \mathrm{id}} & A' \times B' \end{array}$$

induces a map

$$f \wedge g : A' \times B \coprod_{A\times B} A \times B' \longrightarrow A' \times B', \tag{2.5.2}$$

called the *smash product of f and g.*

Example 2.5.3. Let $f : * \to X$ and $g : * \to Y$ be pointed simplicial sets. Then we have

$$f \wedge g : X \times * \coprod_{*\times *} * \times Y \hookrightarrow X \times Y.$$

Proposition 2.5.4. *Let f, g, h be morphisms of simplicial sets.*

(1) Suppose f and g are monic, then $f \wedge g$ is monic.

(2) We have $(f \wedge g) \wedge h \cong f \wedge (g \wedge h)$.

Proof. These are straightforward consequences of the definition of the smash product. □

Lemma 2.5.5. *Consider the following sets of morhpisms in* $\mathbf{Set}_\Delta$*:*

$$\mathcal{M}_1 = \{\Lambda^n_i \hookrightarrow \Delta^n \mid n > 0, 0 \le i \le n\}$$
$$\mathcal{M}_2 = \{\{e\} \times \Delta^n \coprod_{\{e\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \longrightarrow \Delta^1 \times \Delta^n \mid n \ge 0, e = 0, 1\}$$
$$\mathcal{M}_3 = \{\{e\} \times S \coprod_{\{e\}\times K} \Delta^1 \times K \longrightarrow \Delta^1 \times S \mid K \hookrightarrow S \textit{ monic}, e = 0, 1\}$$

Then we have

$$\overline{\mathcal{M}_1} = \overline{\mathcal{M}_2} = \overline{\mathcal{M}_3} = \{\textit{anodyne morphisms}\}.$$

Before giving the proof of the lemma, we deduce an important corollary:

Corollary 2.5.6. *Let f be an anodyne morphism and g a mononorphism of simplicial sets. Then $f \wedge g$ is an anodyne morphism.*

Proof. By Problem 5.3, for a fixed morphism g, the set of morphisms

$$\{f' \mid f' \wedge g \text{ is anodyne}\}$$

is saturated. It therefore suffices to verify that, for every $f' \in \mathcal{M}_3$, the morhpism $f' \wedge g$ is anodyne. But this follows, since f' is of the form $i \wedge j$ where $i : \{e\} \to \Delta^1$ and $j : K \hookrightarrow S$ monic:

$$f' \wedge g = (i \wedge j) \wedge g \cong i \wedge (j \wedge g) \in \mathcal{M}_3$$

where we apply both parts of Proposition 2.5.4. □

Proof of Lemma 2.5.5. We show $\mathcal{M}_2 \subset \overline{\mathcal{M}_1}$ and $\mathcal{M}_1 \subset \overline{\mathcal{M}_3}$. Combined with Problem 5.3, which shows $\overline{\mathcal{M}_2} = \overline{\mathcal{M}_3}$, this will complete the proof.

$\mathcal{M}_2 \subset \overline{\mathcal{M}_1}$: Consider the morphism

$$\{0\} \times \Delta^n \coprod_{\{0\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \longrightarrow \Delta^1 \times \Delta^n \tag{2.5.7}$$

in $\mathcal{M}_2$. We note that the cylinder $\Delta^1 \times \Delta^n$ consists of $n+1$ nondegenerate $(n+1)$-simplices which correspond to the maps

$$h_j : [n+1] \to [1] \times [n]$$

given by the chains

$$(0,0) \to (0,1) \to \cdots \to (0,j) \to (1,j) \to (1,j+1) \to \cdots \to (1,n).$$

We then observe that the morphism (2.5.7) can be described as a composite of maps

$$A^0 \to A^1 \to \cdots \to A^n$$

of simplicial sets given by attaching the above $(n+1)$-simplices in the order $h_n, h_{n-1}, \dots, h_0$ along horns inclusions. More precisely, we have pushout squares

$$\begin{tikzcd} A^i \arrow[r] & A^{i+1} \\ \Lambda^{n+1}_{n-i} \arrow[u] \arrow[r] & \Delta^{n+1} \arrow[u] \end{tikzcd}$$

so that each $A^i \to A^{i+1}$ is in $\overline{\mathcal{M}_1}$ and hence the composite is in $\overline{\mathcal{M}_1}$ as well.

$\mathcal{M}_1 \subset \overline{\mathcal{M}_3}$: For $k < n$, consider the maps

$$[n] \xrightarrow{i} [1] \times [n] \xrightarrow{p} [n]$$

where $i(j) = (1,j)$ and p is determined by the diagram

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & \dots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \dots \arrow[r] & n \\ 0 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \dots \arrow[r] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & \dots \arrow[r] & k. \arrow[u] \end{tikzcd}$$

Then we have $r \circ i = \mathrm{id}_{[n]}$ so that, passing to nerves, these maps exhibit Δ^n as a retract of $\Delta^1 \times \Delta^n$. It is then straightforward to verify that we have a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \{0\} \times \Delta^n \coprod_{\{0\} \times \Lambda^n_k} \Delta^1 \times \Lambda^n_k \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\ \Delta^n \arrow[r] & \Delta^1 \times \Delta^n \arrow[r] & \Delta^n \end{tikzcd} \tag{2.5.8}$$

where all rows compose to the identity. This exhibits the horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ as a retract of a morphism in $\mathcal{M}_3$. To show that the remaining horn inclusion $\Lambda^n_n \hookrightarrow \Delta^n$ is in $\overline{\mathcal{M}_3}$, we simply pass to opposite simplicial sets in (2.5.8). □

2.6 Mapping spaces

Let K, S be simplicial sets. We introduce a simplicial set $\mathrm{Map}(K,S)$ of maps from K to S via the formula

$$\mathrm{Map}(K,S)_n = \mathbf{Set}_\Delta(K \times \Delta^n, S).$$

The functoriality in $[n] \in \Delta$ is given by the Yoneda embedding

$$\Delta \longrightarrow \mathbf{Set}_\Delta, [n] \mapsto \Delta^n,$$

making $\mathrm{Map}(K,S)$ a simplicial set. The simplicial set $\mathrm{Map}(K,S)$ comes equipped with an evaluation map

$$\mathrm{ev} : K \times \mathrm{Map}(K,S) \longrightarrow S$$

given by associating to a pair (σ, f) with $\sigma \in K_n$ and $f \in \mathrm{Map}(K,S)_n$, the n-simplex $f(\sigma, \mathrm{id}_{[n]})$ in S.

Proposition 2.6.1. *Let K be a simplicial set. Then there is an adjunction*

$$K \times - : \ \mathbf{Set}_\Delta \longleftrightarrow \mathbf{Set}_\Delta \ : \mathrm{Map}(K, -).$$

Proof. We need to show that there are bijections

$$\mathbf{Set}_\Delta(K \times X, S) \cong \mathbf{Set}_\Delta(X, \mathrm{Map}(K, S)),$$

natural in X and S. To this we construct two inverse maps: The map

$$\psi : \mathbf{Set}_\Delta(X, \mathrm{Map}(K, S)) \to \mathbf{Set}_\Delta(K \times X, S)$$

is obtained by associating to a morphism $f : X \to \mathrm{Map}(K, S)$, the map $\mathrm{ev} \circ (K \times f) : K \times X \to S$. Its inverse

$$\varphi : \mathbf{Set}_\Delta(K \times X, S) \to \mathbf{Set}_\Delta(X, \mathrm{Map}(K, S))$$

associates to a map $g : K \times X \to S$ the map $\varphi(g) : X \to \mathrm{Map}(K, S)$ that maps an n-simplex $\sigma : \Delta^n \to X$ to the n-simplex of $\mathrm{Map}(K, S)$ determined by $g \circ (K \times \sigma)$. It is straightforward to verify that φ and ψ are inverse to oneanother. □

Proposition 2.6.2. *Let $i : A \to B$ be a monomorphism and let $p : K \to S$ be a Kan fibration of simplicial sets. Then the map*

$$\mathrm{Map}(B, K) \longrightarrow \mathrm{Map}(A, K) \times_{\mathrm{Map}(A,S)} \mathrm{Map}(B, S)$$

induced by the commutative square

$$\begin{tikzcd}
\mathrm{Map}(B, K) \arrow[r] \arrow[d] & \mathrm{Map}(A, K) \arrow[d] \\
\mathrm{Map}(B, S) \arrow[r] & \mathrm{Map}(A, S)
\end{tikzcd}$$

is a Kan fibration.

Proof. We verify that, for every anodyne morphism $j : A' \to B'$, every lifting problem of the form

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & \mathrm{Map}(B, K) \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(A, K) \times_{\mathrm{Map}(A,S)} \mathrm{Map}(B, S)
\end{tikzcd}$$

has a solution. But, in virtue of Proposition 2.6.1, this lifting problem is equivalent to the "adjoint" lifting problem

$$\begin{tikzcd}
A' \times B \coprod_{A' \times A} B' \times A \arrow[r] \arrow[d] & K \arrow[d, "p"] \\
B' \times B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

which has a solution since p is a Kan fibration and the left hand vertical map is the smash product of the monomorphism $i : A \to B$ and the anodyne morphism $j : A' \to B'$, hence anodyne by Corollary 2.5.6. □

Example 2.6.3. Let K be a Kan complex.

(1) Let B be a simplicial set. Then the simplicial set $\mathrm{Map}(B, K)$ is a Kan complex. This follows from Proposition 2.6.2 by setting $A = \emptyset$ and $S = *$.

(2) Let $A \to B$ be a monomorphism of simplicial sets. Then the restriction map

$$\mathrm{Map}(B, K) \to \mathrm{Map}(A, K)$$

is a Kan fibration. This follows from Proposition 2.6.2 by setting $S = *$

2.7 Simplicial homotopy

Let $f, g : B \to K$ be morphisms of simplicial sets. A *homotopy from f to g* is a morphism

$$H : \Delta^1 \times B \longrightarrow K$$

such that $H|\{0\} \times B = f$ and $H|\{1\} \times B = g$. We say that *f is homotopic to g* if there exists a homotopy from f to g. In this case we write

$$f \overset{H}{\simeq} g$$

Suppose in addition that we are given a monomorphism $i : A \hookrightarrow B$ and that $f|A = g|A =: u$. Then we say that H is a *homotopy relative to A* if the diagram

$$\begin{tikzcd} \Delta^1 \times B \arrow[r, "H"] & K \\ \Delta^1 \times A \arrow[u, "\mathrm{id}\times i"] \arrow[r, "p_A"] & A \arrow[u, "u"'] \end{tikzcd}$$

commutes. We say that *f is homotopic to g relative to A* if there exists a homotopy from f to g relative to A. In this case, we write

$$f \overset{H}{\simeq} g \quad (\text{rel. } A)$$

Proposition 2.7.1. *Let $i : A \to B$ be a monomorphism of simplicial sets, K a simplicial set, and $u : A \to K$ a morphism. Then the relation of homotopy relative to A defines an equivalence relation on the set of those morphisms $f : B \to K$ that satisfy $f|A = u$.*

Proof. We first note that Kan fibrations are stable under pullback, i.e., given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y' \arrow[r] & y \end{tikzcd}$$

where p is a Kan fibration then p' is a Kan fibration. This is an immediate consequence of the defining right lifting property of Kan fibrations. We apply this to the pullback square

$$\begin{tikzcd} \mathrm{Map}(B,K)^u \arrow[r] \arrow[d] & \mathrm{Map}(B,K) \arrow[d, "p"] \\ \{u\} \arrow[r] & \mathrm{Map}(A,K) \end{tikzcd}$$

which defining $\mathrm{Map}(B,K)^u$ as the fiber of the map p over the vertex $\{u\}$. Since, by Proposition 2.6.2, the map p is a Kan fibration, the simplicial set $\mathrm{Map}(B,K)^u$ is a Kan complex. Now the set of maps $f : B \to K$ such that $f|A = u$ is precisely the set of vertices of $\mathrm{Map}(B,K)^u$, i.e., maps $\Delta^0 \to \mathrm{Map}(B,K)^u$. Further, two such maps f and g are homotopic relative to A if and only if the correspoding maps $\Delta^0 \to \mathrm{Map}(B,K)^u$ are homotopic in the absolute sense. In other words, we have reduced the prove to the statement that the relation of homotopy on the set of vertices of a Kan complex is an equivalence relation, i.e., we may assume $A = \emptyset$ and $B = \Delta^0$. This, we verify explicitly as follows:

(1) *Reflexive.* We need to show that every vertex x of K is homotopic to itself. Indeed, the corresponding homotopy is given by the degenerate edge $s_0(x)$.

(2) *Symmetric.* Suppose $x, y \in K_0$ with $x \simeq y$. This means that there exists an edge $H : \Delta^1 \to K$ with source x and target y. We extend this edge to a horn $\Lambda^2_0 \to K$ by mapping the edge $\{0,1\}$ of Λ^2_0 to H and the edge $\{0,2\}$ to the degenerate edge $s_0(x)$. Since K is a Kan complex, we may fill the horn with a 2-simplex σ in K. The face $d_0\sigma$ defines a homotopy from y and x.

(3) *Transitive.* Similarly, an edge $H : \Delta^1 \to K$ between vertices x and y, and an edge $H' : \Delta^1 \to K$ between y and z, define an inner horn $\Lambda^2_1 \to K$ which can be filled to a 2-simplex σ in K. The face $d_1\sigma$ then defines the needed homotopy from x to z.

□

Definition 2.7.2. Let K be a Kan complex.

(1) We define $\pi_0(K)$ to be the set of homotopy classes of vertices of K.

(2) For $n \geq 1$ and $v \in K_0$, we define $\pi_n(K, v)$ to be the set of homotopy classes relative to $\partial\Delta^n$ of those maps $\alpha : \Delta^n \to K$ that satisfy $\alpha|\partial\Delta^n = v$. Here v denotes the constant map given by the composite

$$\partial\Delta^n \longrightarrow \Delta^0 \xrightarrow{\{v\}} K.$$

We now construct a composition law on the sets $\pi_n(K, v)$: To this end, let $\alpha, \beta : \Delta^n \longrightarrow K$ such that $\alpha|\partial\Delta^n = \beta|\partial\Delta^n = v$. For an $(n+1)$-simplex σ in K let us denote the set of n-simplices in its boundary by the notation

$$\partial\sigma = (d_0\sigma, d_1\sigma, \ldots, d_{n+1}\sigma).$$

The n-tupel of n-simplices

$$(v, v, \ldots, v, \alpha, -, \beta)$$

assembles to define a horn $\Lambda^{n+1}_n \to K$ which can be filled to a full $(n+1)$-simplex σ in K such that

$$\partial\sigma = (v, v, \ldots, v, \alpha, d_n\sigma, \beta). \tag{2.7.3}$$

Proposition 2.7.4. *The association*

$$(\alpha, \beta) \mapsto d_n\sigma$$

from (2.7.3) *yields a well-defined map*

$$\pi_n(K, v) \times \pi_n(K, v) \to \pi_n(K, v),\ ([\alpha], [\beta]) \mapsto [d_n\sigma].$$

Proof. Suppose we are given $\alpha \overset{H}{\simeq} \alpha'$, $\beta \overset{H'}{\simeq} \beta'$, and $(n+1)$-simplices σ and σ', such that

$$\begin{aligned} \partial\sigma &= (v, v, \ldots, v, \alpha, d_n\sigma, \beta) \\ \partial\sigma' &= (v, v, \ldots, v, \alpha', d_n\sigma', \beta'). \end{aligned}$$

We have to show that there exists a homotopy H'' between $d_n\sigma$ and $d_n\sigma'$. To this end, we note that the above given data assemble to define a map

$$\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Delta^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \longrightarrow K$$

which we can extend along the anodyne map

$$\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Delta^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \longrightarrow \Delta^1 \times \Delta^{n+1}$$

to obtain a map

$$\Delta^1 \times \Delta^{n+1} \longrightarrow K.$$

restricting this map along $\mathrm{id} \times d_n : \Delta^1 \times \Delta^n \to \Delta^1 \times \Delta^{n+1}$ defines the desired homotopy

$$H'' : \Delta^1 \times \Delta^n \longrightarrow K$$

between $d_n\sigma$ and $d_n\sigma'$. □

Proposition 2.7.5. *Let K be a Kan complex, $v \in K_0$ a vertex, and let $n \geq 1$. Then the operation*

$$\pi_n(K,v) \times \pi_n(K,v) \to \pi_n(K,v)$$

from Proposition 2.7.4 makes $\pi_n(K,v)$ a group.

Proof. To show associativity, suppose that α,β, and γ represent elements of $\pi_n(K,v)$. We choose $(n+1)$-simplices σ_{n-1}, σ_{n+1}, and σ_{n+2} such that

$$\begin{aligned}\partial\sigma_{n-1} &= (v,\dots,v,\alpha,\mu,\beta)\\ \partial\sigma_{n+1} &= (v,\dots,v,\mu,\xi,\gamma)\\ \partial\sigma_{n+2} &= (v,\dots,v,\beta,\psi,\gamma)\end{aligned}$$

Then the list of $(n+1)$-simplices $(v,\dots,v,\sigma_{n-1},-,\sigma_{n+1},\sigma_{n+2})$ defines a horn

$$\Lambda^{n+2}_n \longrightarrow K$$

which can thus be extended to a full $(n+2)$-simplex $\sigma : \Delta^{n+1} \to K$. We observe that

$$\partial(d_n\sigma) = (v,\dots,v,\alpha,\xi,\psi)$$

from which we deduce

$$([\alpha][\beta])[\gamma] = [\xi] = [\alpha]([\beta][\gamma]).$$

Further, we observe that, for $\alpha : \Delta^n \to K$ representing an element in $\pi_n(K,v)$, the degenerate simplices $s_{n-1}(\alpha)$ and $s_{n-2}(\alpha)$ satisfy

$$\partial s_{n-1}(\alpha) = (v,\dots,v,\alpha,\alpha)$$

and

$$\partial s_{n-2}(\alpha) = (v,\dots,\alpha,\alpha,v)$$

so that we have $[\alpha][v] = [\alpha]$ and $[v][\alpha] = [\alpha]$ showing unitality. To construct an left inverse of α, we fill the horn determined by the n-simplices $(v,\dots,-,v,\alpha)$ to a full $(n+1)$-simplex σ so that $[d_{n-1}\sigma]$ provides the inverse. A similar argument yields the existence of a right inverse. □

We state two further results which can be proven using similar explicit combinatorial techniques. Detailed proofs can be found in [1].

Theorem 2.7.6. *Let $p : K \to S$ be a Kan fibration, let $v \in K_0$ be a vertex, set $w = p(v)$, and consider the corresponding pullback square*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \{w\} \arrow[r] & S \end{tikzcd}$$

Then there is an associated long exact sequence

$$\dots \longrightarrow \pi_{n+1}(K,v) \longrightarrow \pi_n(F,v) \longrightarrow \pi_n(K,v) \longrightarrow \pi_n(S,w) \longrightarrow \pi_{n-1}(F,v) \longrightarrow \dots$$

of homotopy groups (and pointed sets for $n = 0$).

Theorem 2.7.7. *Let K be a Kan complex and $v \in K_0$. Then $\pi_n(K,v)$ is abelian for $n \geq 2$.*

Chapter 3

Model categories

3.1 Localization of categories

In the previous section, we have developed the rudiments of a homotopy theory of Kan complexs. The question arises how to relate this combinatorial version of homotopy theory to the usual one based on topological spaces. In this section, we will introduce a categorical framework in which a precise comparison can be achieved.

Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a set of morphisms in $\mathcal{C}$. A *localization* of $\mathcal{C}$ along $\mathcal{W}$ is a category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a functor $\pi : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ satisfying the following universal property:

- For every category $\mathcal{D}$, the functor

  $$\pi^* : \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

  is fully faithful with essential image given by those functors $F : \mathcal{C} \to \mathcal{D}$ that send all morphisms in $\mathcal{W}$ to isomorphisms in $\mathcal{D}$.

Example 3.1.1. Denote by $B\mathbb{N}$ the category with a single object and endomorphisms given by the monoid $(\mathbb{N}, +)$. Then the category $B\mathbb{Z}$ is a localization of $B\mathbb{N}$ along $\mathcal{W} = \{1\}$.

Proposition 3.1.2. *Let $\mathcal{C}$ be a category and $\mathcal{W}$ a set of morphisms in $\mathcal{C}$. Then a localization of $\mathcal{C}$ along $\mathcal{W}$ exists.*

Proof. The existence of $\mathcal{C}[\mathcal{W}^{-1}]$ is shown by constructing it explicitly. We define a graph to be a pair $\Gamma = (A, V)$ of sets equipped with a pair of maps $s, t : A \to V$. To any category $\mathcal{D}$, we may associate a graph where A is the set $\mathrm{mor}(\mathcal{C})$ of all morphisms is $\mathcal{C}$, V is the set of objects, and the maps s and t map a morphism to its source and target, respectively. This construction has a left adjoint which associates to a graph Γ, a category $\mathcal{D}(\Gamma)$ with objects V and morphisms between objects x and y given by tupels

$$(f_1, \dots, f_n) \quad n \geq 0$$

of arrows $f_i \in A$ with $s(f_1) = x$, $t(f_n) = y$, and, for $i < n$, $t(f_i) = s(f_{i+1})$. The identity morphisms are given by empty strings and the composition is given by concatenation.

In the given context, we now define a graph $\Gamma = (\mathrm{mor}(\mathcal{C}) \amalg \mathcal{W}, \mathrm{ob}(\mathcal{C}))$ where the restriction of s and t to $\mathrm{mor}(\mathcal{C})$ are the usual source and target. For $w \in \mathcal{W}$, on the other hand, we set $s(w)$ to be the target of w and $t(w)$ to be the source of w. We then construct $\mathcal{C}[\mathcal{W}^{-1}]$ from $\mathcal{D}(\Gamma)$ by imposing the following relations:

(1) $\dots x \xrightarrow{f} y \xrightarrow{g} z \dots = \dots x \xrightarrow{gf} z \dots \quad$ for $f, g \in \mathrm{mor}(\mathcal{C})$,

(2) $\dots x \xleftarrow{v} y \xleftarrow{w} z \dots = \dots x \xleftarrow{vw} z \dots \quad$ for $v, w \in \mathcal{W}$,

(3) $\dots x \xrightarrow{\mathrm{id}} x \dots \dots = \dots x \dots \quad$ for $x \in \mathrm{ob}(\mathcal{C})$,

(4) $\dots x \xrightarrow{w} y \xleftarrow{w} x \dots = \dots x \dots$ for $w \in \mathcal{W}$,

(5) $\dots x \xleftarrow{w} y \xrightarrow{w} x \dots = \dots x \dots$ for $w \in \mathcal{W}$,

(6) $\dots x \xleftarrow{w} y \dots = \dots x \xrightarrow{w^{-1}} y \dots$ for $w \in \mathcal{W}$ such that w is an isomorphism.

We further have a functor $\pi : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ which is the identity on objects and associates to a morphism f the sequence (f). It is then immediate to verify that the pullback $\pi^* : \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ induces an isomorphism (in particular an equivalence) between the category $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D})$ and the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ that send morphisms in $\mathcal{W}$ to isomorphisms in $\mathcal{D}$. □

Using the notion of localization, we can formulate the comparison between the two versions of homotopy theory alluded to above: A continuous map $f : X \to Y$ of topological spaces is called a *weak homotopy equivalence* if, for every $n \geq 0$ and every $x \in X$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism. We denote the set of weak homotopy equivalences by $\mathcal{W}$. A map $f : K \to S$ of simplicial sets is called a *weak homotopy equivalence* if the induced map $|f| : |K| \to |S|$ on geometric realizations is a weak homotopy equivalence. By a slight abuse of notation, we also refer to the set of weak homotopy equivalences of simplicial sets by $\mathcal{W}$.

Theorem 3.1.3 (Quillen)**.** *The adjunction* $(|-|, \mathrm{Sing})$ *induces an equivalence of localizations*

$$|-| : \mathbf{Set}_\Delta[\mathcal{W}^{-1}] \overset{\simeq}{\longleftrightarrow} \mathbf{Top}[\mathcal{W}^{-1}] : \mathrm{Sing}\,.$$

The explicit description of the localization $\mathcal{C}[\mathcal{W}^{-1}]$ given in the proof of Proposition 3.1.2 may give the misleading impression that localizations of categories can be computed easily. However, the description of the sets of morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ is rather unwieldy and difficult to handle in practice.

The problem of providing effective means to compute localizations of categories can be seen as one motivation to introduce model categories, the central notion of this chapter. This is the framework in which Quillen provided a proof of Theorem 3.1.3, as will be discussed in more detail below.

3.2 The model category axioms

Definition 3.2.1. A *model category* is a category $\mathcal{C}$ equipped with three distinguished sets of morphisms:

- $\mathcal{W}$, called *weak equivalences*,
- $\mathcal{C}\mathrm{of}$, called *cofibrations*,
- $\mathcal{F}\mathrm{ib}$, called *fibrations*,

satisfying the following list of axioms:

(M1) The category $\mathcal{C}$ has small limits and colimits.

(M2) *Two out of three* (2/3): Given a pair $f : X \to Y$, $g : Y \to Z$ of composable morphisms in $\mathcal{C}$, if two of the morphisms $f, g, g \circ f$ are weak equivalences then so is the third.

(M3) $\mathcal{F}\mathrm{ib}$, $\mathcal{C}\mathrm{of}$, $\mathcal{W}$ are stable under retracts.

(M4) The lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution provided that

- either $i \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$ and $p \in \mathcal{F}\mathrm{ib}$,

- or $i \in \mathcal{C}\mathrm{of}$ and $p \in \mathcal{F}\mathrm{ib} \cap \mathcal{W}$.

(M5) Any map $X \to Z$ in $\mathcal{C}$ admits factorizations

- $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$ and $g \in \mathcal{F}\mathrm{ib}$,
- $X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ with $f \in \mathcal{C}\mathrm{of}$ and $g \in \mathcal{F}\mathrm{ib} \cap \mathcal{W}$.

We say that the sets $\mathcal{W}$, $\mathcal{C}\mathrm{of}$ and $\mathcal{F}\mathrm{ib}$ define a *model structure* on the category $\mathcal{C}$.

Example 3.2.2. Any category $\mathcal{C}$ admits a model structure with $\mathcal{W}$ given by the set of isomorphisms, $\mathcal{C}\mathrm{of}$ and $\mathcal{F}\mathrm{ib}$ given by all morphisms in $\mathcal{C}$.

Example 3.2.3. If $\mathcal{C}$ is a model category, then the opposite category $\mathcal{C}^{\mathrm{op}}$ carries a model structure with weak equivalences $\mathcal{W}^{\mathrm{op}}$, cofibrations $\mathcal{F}\mathrm{ib}^{\mathrm{op}}$ and fibrations $\mathcal{C}\mathrm{of}^{\mathrm{op}}$.

Lemma 3.2.4. *Let $\mathcal{C}$ be a model category. Then we have*

(1) $\mathcal{C}\mathrm{of}_{\perp} = \mathcal{F}\mathrm{ib} \cap \mathcal{W}$,

(2) $\mathcal{C}\mathrm{of} = {}_{\perp}(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$,

(3) $(\mathcal{C}\mathrm{of} \cap \mathcal{W})_{\perp} = \mathcal{F}\mathrm{ib}$,

(4) $\mathcal{C}\mathrm{of} \cap \mathcal{W} = {}_{\perp}\mathcal{F}\mathrm{ib}$.

In particular, the sets $\mathcal{C}\mathrm{of}$ *and* $\mathcal{F}\mathrm{ib}$ *of a model structure are determined by one another.*

Proof. The proof of these statements is a mutatis mutandis version of the proof of Corollary 2.4.17 formulated in terms of the model category axioms: we demonstrate this by showing $\mathcal{C}\mathrm{of} \cap \mathcal{W} = {}_{\perp}\mathcal{F}\mathrm{ib}$. The other statements are proved analogously. The inclusion $\mathcal{C}\mathrm{of} \cap \mathcal{W} \subset {}_{\perp}\mathcal{F}\mathrm{ib}$ follows from (M4). Vice versa, let $f \in {}_{\perp}\mathcal{F}\mathrm{ib}$ and consider the factorization

$$f = g \circ h$$

with $h \in \mathcal{C}\mathrm{of} \cap W$ and $g \in {}_{\perp}\mathcal{F}\mathrm{ib}$ which exists by (M5). This factorization defines the lifting problem

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

which has a solution since $g \in {}_{\perp}\mathcal{F}\mathrm{ib}$ and $f \in \mathcal{F}\mathrm{ib}$. We denote the solution by $r : Y \to Z$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & X \arrow[r, "\mathrm{id}"] \arrow[d, "h"] & X \arrow[d, "f"] \\ Y \arrow[r, "r"] & Z \arrow[r, "g"] & Y \end{tikzcd}$$

exhibits f as a retract of h so that $f \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$. □

Corollary 3.2.5. *In any model category, the sets* $\mathcal{C}\mathrm{of}$ *and* $\mathcal{C}\mathrm{of} \cap \mathcal{W}$ *are saturated.*

In virtue of Lemma 3.2.4, to specify a model structure on a category $\mathcal{C}$, it suffices to specify $\mathcal{W}$ and either $\mathcal{F}\mathrm{ib}$ or $\mathcal{C}\mathrm{of}$. Note however, that it is not automatic that a given choice say of sets $(\mathcal{W}, \mathcal{C}\mathrm{of}, (\mathcal{C}\mathrm{of} \cap \mathcal{W})_{\perp})$ defines a model structure. We provide a list of examples of model structures without giving proofs that the model category axioms hold in each case. The verification of the axioms requires substantial effort, proofs can be found in [2, 1, 3]. The theory we have developed in §2 enters the verification of the axioms in the first two examples. The arguments needed to complete the verification are of a similar nature and can be found in [1].

Example 3.2.6. (1) The cateory **Top** has a model structure, called the *Quillen model structure*, with

- $\mathcal{W}$ = weak homotopy equivalences,
- $\mathcal{F}\mathrm{ib}$ = Serre fibrations,
- $\mathcal{C}\mathrm{of} = {}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$.

(2) The category $\mathbf{Set}_\Delta$ has a model structure, called the *Kan model structure*, with

- $\mathcal{W}$ = weak homotopy equivalences,
- $\mathcal{F}\mathrm{ib}$ = Kan fibrations,
- $\mathcal{C}\mathrm{of}$ = monomorphisms.

Further, the set $\mathcal{C}\mathrm{of} \cap \mathcal{W}$ is given by the anodyne morphisms.

(3) Let R be a small ring and let $\mathbf{Ch}(R)$ denote the category of chain complexes of small left R-modules. Then $\mathbf{Ch}(R)$ has a model structure, called the *projective model structure*, with

- $\mathcal{W}$ = quasi-isomorphisms, i.e., morphisms of complexes that induce isomorphisms on all homology modules,
- $\mathcal{F}\mathrm{ib}$ = morphisms f of complexes, such that, for every $n \in \mathbb{Z}$, the map f_n is a surjection,
- $\mathcal{C}\mathrm{of} = {}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$.

3.3 The homotopy category of a model category

Let $\mathcal{C}$ be a model category. A *cylinder object* for an object $A \in \mathcal{C}$ is an object C equipped with a factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(i_0,i_1)"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

such that i is a cofibration and σ is a weak equivalence. Note that, due to (M5), every object in a model category has a cylinder object.

Example 3.3.1. Consider the category $\mathbf{Set}_\Delta$ equipped with the Kan model structure. Let K be a simplicial set. Then $\Delta^1 \times K$ is a cylinder object with $i_0 = \{0\} \times \mathrm{id}$, $i_1 = \{1\} \times \mathrm{id}$, and σ given by projection onto the factor K.

A *left homotopy* between morphisms $f, g : A \to B$ in a model category $\mathcal{C}$ consists of a cyclinder object C for A and a map $H : C \to B$ such that the diagram

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

commutes. We say that f and g are *left homotopic* if there exists a left homotopy between f and g. In this case, we write

$$f \overset{l}{\sim} g.$$

Proposition 3.3.2. *(1) Let A be a cofibrant object and let C be a cylinder object for A. Then the maps i_0 and i_1 are trivial cofibrations.*

(2) Let A be a cofibrant object. Then left homotopy of maps $A \to B$ is an equivalence relation.

Proof. To show (1), note that we have a pushout square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "j_1"] \\ A \arrow[r, "j_0"'] & A \amalg A \end{tikzcd}$$

so that j_0 and j_1 are cofibrations, since $\emptyset \to A$ is a cofibration and cofibrations are stable under pushout by Corollary 3.2.5. But then $i_0 = i \circ j_0$ and $i_1 = i \circ j_1$ are cofibrations, since, by the same corollary, cofibrations are stable under composition. Finally, we have $\sigma \circ i_k = \mathrm{id}$ so that i_0 and i_1 are weak equivalences by ((M2)).

To show (2), we verify that $\overset{l}{\sim}$ is reflexive, symmetric, and transitive. Given a morphism $f : A \to B$ and a cylinder C for A, set $H : C \to B$ to be $H = f \circ \sigma$. Then H is a left homotopy from f to f verifying reflexivity. To show symmetry, suppose that $f \overset{l}{\sim} g$ via a homotopy $H : C \to B$ with cylinder object C. Let $\tau : A \amalg A \cong A \amalg A$ be the automorphism that flips the components. Then the object C equipped with the factorization

$$A \amalg A \xrightarrow{i\circ\tau} C \xrightarrow{\sigma} A$$

and the same homotopy H, interpreted with respect to this flipped cylinder object, defined a left homotopy from g to f. Finally, to show transitivity, suppose that

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} A \amalg A \arrow[r, "(g,h)"] \arrow[d, "i'"'] & B \\ C' \arrow[ur, "H'"'] & \end{tikzcd}$$

are left homotopies. Then consider the pushout

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i'_0"'] & C \arrow[d, "j"] \\ C' \arrow[r] & C \amalg_A C'. \end{tikzcd}$$

The maps (H, H') define, by the universal property of the pushout, a map $H'' : C \amalg_A C' \to B$ making the diagram

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,h)"] \arrow[d, "(i_0,i'_1)"'] & B \\ C \amalg_A C' \arrow[ur, "H''"'] & \end{tikzcd}$$

commute. It remains to verify that the object $C \amalg_A C'$ is a cylinder object via the maps $(i_0, i'_1) : A \amalg A \to C \amalg_A C'$ and $(\sigma, \sigma') : C \amalg_A C' \to A$. First note that the map $j : C \to C \amalg_A C'$ is a trivial cofibration, since by (1), the map i'_0 is a trivial cofibration, and the set of trivial cofibrations is saturated. It follows from ((M2)) that (σ, σ') is a weak equivalence as required. To show that (i_0, i'_1) is a cofibration, consider the pushout

$$\begin{tikzcd} (A \amalg A) \amalg (A \amalg A) \arrow[r, "i \amalg i'"] \arrow[d] & C \amalg C' \arrow[d] \\ A \amalg A \amalg A \arrow[r, "k"] & C \amalg_A C' \end{tikzcd}$$

showing that k is a cofibration. Finally, the pushout

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ A \amalg A \arrow[r, "l"] & A \amalg A \amalg A \end{tikzcd}$$

shows that l is a cofibration so that $(i_0, i_1') = k \circ l$ is a cofibration as well. □

Dually, we define a *path object* for an object B of a model category, to be an object P equipped with a factorization

$$\begin{tikzcd} & P \arrow[d, "p=(p_0,p_1)"] \\ B \arrow[ur, "s"] \arrow[r, "(\mathrm{id},\mathrm{id})"'] & B \times B \end{tikzcd}$$

where p is a fibration and s is a weak equivalence. A *right homotopy* between morphisms $f, g : A \to B$ consists of a path object P for B together with a map $H : A \to P$ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur, "H"] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd}$$

commutes. We then have the following dual version of Proposition 3.3.2. If there exists a right homotopy between f and g, we write $f \overset{r}{\sim} g$.

Proposition 3.3.3. *(1) Let B be a fibrant object and let P be a cylinder object for B. Then the maps p_0 and p_1 are trivial fibrations.*

(2) Let B be a fibrant object. Then right homotopy of maps $A \to B$ is an equivalence relation.

Proof. Apply Proposition 3.3.2 to $\mathcal{C}^{\mathrm{op}}$. □

Lemma 3.3.4. *Let $\mathcal{C}$ be a model category and A a cofibrant object. Suppose that $f, g : A \to B$ are left homotopic. Then, given any path object P for B, there exists a right homotopy between f and g with underlying path object P.*

Proof. Let

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

be a left homotopy between f and g. The map i_0 is a trivial cofibration by Proposition 3.3.2. Therefore, the lifting problem

$$\begin{tikzcd} A \arrow[r, "sf"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[ur, dashed] \arrow[r, "(f\sigma,H)"'] & B \times B \end{tikzcd}$$

has a solution $h : C \to P$. Then the map $H' = h \circ i_1$ defines a right homotopy between f and g. □

Corollary 3.3.5. *Suppose $f, g : A \to B$ are morphisms in a model category with A cofibrant and B fibrant. Then the following are equivalent:*

(1) f and g are left homotopic.

(2) For every path object P for B, there exists a right homotopy between f and g with underlying path object P.

(3) f *and* g *are right homotopic.*

(4) For every cylinder object C *for* A*, there exists a left homotopy between* f *and* g *with underlying cylinder object* C*.*

Proof. The implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are immediate. The implication (1) $\Rightarrow$ (2) is the content of Lemma 3.3.4 while (3) $\Rightarrow$ (4) is its dual. $\square$

In light of Corollary 3.3.5, for morphisms $f, g : A \to B$ in a model category with A cofibrant and B fibrant, we do not distinguish notationally between left and right homotopy and write $f \sim g$.

Theorem 3.3.6 (Whitehead)**.** *Let* $f : X \to Y$ *be a weak equivalence in a model category* $\mathcal{C}$ *and suppose that* X *and* Y *are both fibrant and cofibrant. Then* f *is a homotopy equivalence, i.e., there exists a morphism* $g : Y \to X$ *such that* $fg \sim \mathrm{id}_Y$ *and* $gf \sim \mathrm{id}_X$*.*

Proof. First assume that f is a trivial fibration. Let $g : Y \to X$ a solution to the lifting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

so that we have $fg = \mathrm{id}_Y$. Further, let C be a cylinder object. Then a solution $H : C \to X$ to the lifting problem

$$\begin{tikzcd} X \amalg X \arrow[r, "(gf,\mathrm{id})"] \arrow[d] & X \arrow[d, "f"] \\ C \arrow[r, "f\sigma"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

yields a left homotopy between gf and id_X. Dually, if f is a trivial cofibration, then f is a homotopy equivalence. Now let $f : X \to Y$ be a general weak equivalence. Then we factor $f = gh$ with h cofibration and g trivial fibration. By (M2), h is a weak equivalence and hence a trivial cofibration. By the above, g and h are homotopy equivalences and it is immediate to check that composites of homotopy equivalences are homotopy equivalences, concluding the argument. $\square$

Definition 3.3.7. Let $\mathcal{C}$ be a model category. We define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ as follows:

- The objects of $\mathrm{Ho}(\mathcal{C})$ are the fibrant-cofibrant objects of $\mathcal{C}$.
- Given two objects X, Y, the set of morphisms $\mathrm{Ho}(\mathcal{C})(X, Y)$ is defined as the set of homotopy classes of morphisms from X to Y.
- Using Corollary 3.3.5 it is immediate to verify that the composition of homotopy classes of morphisms between fibrant-cofibrant objects via the formula $[f][g] = [fg]$ is well-defined.

Our next goal is to show that the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category provides an explicit model for the localization $\mathcal{C}[\mathcal{W}^{-1}]$. To show this, we need some preparatory work.

Let $\mathcal{C}$ be a model category. For every object $X \in \mathcal{C}$, we choose a factorization

$$\emptyset \to QX \overset{p_X}{\to} X$$

with QX cofibrant and p_X a trivial fibration. Here we assume that $p_X = \mathrm{id}_X$ if X is already cofibrant. QX is called a cofibrant replacement of X. Further we choose, for every object $X \in \mathcal{C}$, a factorization

$$QX \overset{i_X}{\to} RQX \to *$$

with RQX fibrant and i_X a trivial cofibration. Again we assume that, if QX is already fibrant, then $i_X = \mathrm{id}_{QX}$. Note that RQX is fibrant-cofibrant. Given a morphism $f : X \to Y$, we choose a lift $Qf : QX \to QY$ by means of solving the lifting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & QY \arrow[d, "p_Y"] \\ QX \arrow[r, "fp_X"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

and further, and extension $RQf : RQX \to RQY$ by means of solving the lifting problem

$$\begin{tikzcd} QX \arrow[r, "i_Y f"] \arrow[d, "i_X"] & RQY \arrow[d] \\ RQX \arrow[r] \arrow[ur, dashed] & *. \end{tikzcd}$$

Theorem 3.3.8. *Let $\mathcal{C}$ be a model category.*

(1) The associations

$$\begin{aligned} \pi : &\mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C}), \\ &X \mapsto RQX \\ &f \mapsto [RQf] \end{aligned}$$

yield a well-defined functor.

(2) The functor π exhibits $\mathrm{Ho}(\mathcal{C})$ as a localization of $\mathcal{C}$ along $\mathcal{W}$.

Proof. To show (1), we show that the homotopy class $[RQf]$ only depends on f and is independent of the choices made in the construction of RQf. Suppose that $f_1, f_2 : QX \to QY$ are different choices of lifts of f. Let C be a cyclinder object for QX. A solution to the lifting problem

$$\begin{tikzcd} QX \amalg QX \arrow[r, "(f_1,f_2)"] \arrow[d] & QY \arrow[d] \\ C \arrow[r, "fp_x\sigma"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

shows that f_1 and f_2 are left homotopic. Therefore, $i_Y f_1$ and $i_Y f_2$ are left homotopic, hence also right homotopic by Lemma 3.3.4. Let P be a path object for RQY and let $H : QX \to P$ a right homotopy between $i_Y f_1$ and $i_Y f_2$. Suppose further that $g_1, g_2 : RQX \to RQY$ are extensions of f_1 and f_2, respectively. Then a solution to the lifting problem

$$\begin{tikzcd} QX \arrow[r, "H"] \arrow[d, "i_X"] & P \arrow[d, "p"] \\ RQX \arrow[r, "(g_1,g_2)"'] \arrow[ur, dashed] & RQY \times RQY \end{tikzcd}$$

yields a right homotopy between g_1 and g_2, showing the claim that $[RQf]$ only depends on f. The functoriality of the association $f \mapsto [RQf]$ follows immediately, since, given composable morphisms f and g, both $RQ(fg)$ and $RQ(f)RQ(g)$ are potential choices for $RQ(fg)$ and must therefore be homotopic.

To show (2), we introduce an auxiliary category $\overline{\mathcal{C}}$ with objects the objects of $\mathcal{C}$ and morphisms $\overline{\mathcal{C}}(X,Y) := \mathrm{Ho}(\mathcal{C})(RQX, RQY)$. The functor $\pi : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ factors as

$$\mathcal{C} \xrightarrow{\overline{\pi}} \overline{\mathcal{C}} \xrightarrow{\xi} \mathrm{Ho}(\mathcal{C})$$

where $\overline{\pi}$ is the identity on objects and given by $f \mapsto [RQf]$ on morphisms, while ξ is given by $X \mapsto RQX$ on objects and the identity on morphisms. The functor ξ is surjective on objects and fully faithful, in particular, it is an equivalence. It thus suffices to show that $\overline{\pi}$ exhibits $\overline{\mathcal{C}}$ as a localization of $\mathcal{C}$ along $\mathcal{W}$. To this end, suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends morphisms in $\mathcal{W}$ to isomorphisms in $\mathcal{D}$. We will show that there exists a unique functor $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ such that $\overline{F}\pi = F$. First note that, on objects, we are forced to set $\overline{F}(X) = F(X)$. Further, note that F identifies homotopic morphisms: Given, e.g., a left homotopy

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "f"] \arrow[r, "H"] \arrow[dr, "g"'] & P \arrow[u, "p_0"] \arrow[d, "p_1"'] & Y \arrow[ul, "\mathrm{id}"'] \arrow[l, "s"] \arrow[dl, "\mathrm{id}"] \\ & Y & \end{tikzcd}$$

between f and g, we have that p_i are weak equivalences so that $F(p_i)$ are both inverses to $F(s)$ and must therefore be equal. But this in turn implies $F(f) = F(g)$. Therefore, given a morphism $[g] : RQX \to RQY$ in $\overline{\mathcal{C}}$ from X to Y, the formula

$$\overline{F}([g]) := F(p_Y)F(i_y)^{-1}F(g)F(i_X)F(p_X)^{-1} \tag{3.3.9}$$

is well-defined and yields a functor $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$. For a morphism $f : X \to Y$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ QX \arrow[u, "p_X"] \arrow[r, "Qf"] \arrow[d, "i_X"] & QY \arrow[u, "p_Y"'] \arrow[d, "i_Y"] \\ RQX \arrow[r, "RQf"] & RQY \end{tikzcd}$$

which implies that $\overline{F}\overline{\pi} = F$ as desired. But vice versa, every functor $\overline{F}$ with this property has to satisfy formula (3.3.9) showing the uniqueness of $\overline{F}$. □

3.4 Quillen adjunctions

Proposition 3.4.1. *Let $\mathcal{C}, \mathcal{D}$ be model categories and let*

$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$$

be an adjunction of underlying categories. Then the following are equivalent:

(1) F preserves cofibrations and trivial cofibrations.

(2) G preserves fibrations and trivial fibrations.

Proof. Immediate from the lifting properties. □

An adjunction of model categories satisfying the equivalent conditions of Proposition 3.4.1 is called a *Quillen adjunction.*

Lemma 3.4.2. *Let $F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$ be a Quillen adjunction. Then:*

(1) F preserves weak equivalences between cofibrant objects.

(2) F preserves weak equivalences between fibrant objects.

Proof. We show that any functor $F : \mathcal{C} \to \mathcal{D}$ between model categories that sends trivial cofibrations between cofibrant objects to weak equivalences, sends all weak equivalences between cofibrant objects to weak equivalences: Suppose $f : A \to B$ is a weak equivalence between cofibrant objects. We apply (M5) to obtain a factorization

$$\begin{tikzcd} A \amalg B \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, "q"] & B \\ C \arrow[ur, "p"'] & \end{tikzcd}$$

where q is a cofibration and p is a trivial fibration. The pushout

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "i_A"] \\ B \arrow[r, "i_B"] & A \amalg B \end{tikzcd}$$

shows that i_A and i_B are cofibrations. Further, by (M2), we have that qi_A and qi_B are weak equivalences, hence trivial cofibrations, so that $F(qi_A)$ and $F(qi_B)$ are weak equivalences. Again by (M2), we deduce that $F(p)$ is a weak equivalence. Finally, we have $F(f) = F(p)F(qi_A)$ is a weak equivalence. $\square$

Given a Quillen adjunction

$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$$

the inclusion $\mathcal{C}_c \subset \mathcal{C}$ of the full subcategory of cofibrant objects induces an equivalence

$$\mathcal{C}_c[\mathcal{W}^{-1}] \xrightarrow{\simeq} \mathcal{C}[\mathcal{W}^{-1}]$$

of localizations. This is a direct consequence of the description of $\mathcal{C}[\mathcal{W}^{-1}]$ via the homotopy category $\mathrm{Ho}(\mathcal{C})$. We choose an inverse of this equivalence and denote it by

$$Q : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_c[\mathcal{W}^{-1}].$$

We then define the *left derived functor* LF of F as the composite

$$\mathcal{C}[\mathcal{W}^{-1}] \xrightarrow{Q} \mathcal{C}_c[\mathcal{W}^{-1}] \xrightarrow{F} \mathcal{D}[\mathcal{W}^{-1}]$$

where the second functor F is well-defined by Lemma 3.4.2. Dually, we define the *right derived functor* RG of G as the composite

$$\mathcal{D}[\mathcal{W}^{-1}] \xrightarrow{R} \mathcal{D}_f[\mathcal{W}^{-1}] \xrightarrow{G} \mathcal{C}[\mathcal{W}^{-1}]$$

where R is an inverse to the equivalence $\mathcal{D}_f[\mathcal{W}^{-1}] \simeq \mathcal{D}[\mathcal{W}^{-1}]$.

Proposition 3.4.3. *The functors LG and RG form an adjunction*

$$LF : \mathcal{C}[\mathcal{W}^{-1}] \longleftrightarrow \mathcal{D}[\mathcal{W}^{-1}] : RG$$

of localized categories.

Proof. [2] $\square$

Definition 3.4.4. A Quillen adjunction

$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$$

of model categories is called a Quillen equivalence if LF (or equivalently RG) is an equivalence of localized categories.

We ave now introduced the necessary language to give precise meaning to the statement that the "homotopy theories" described by simplicial sets and topological spaces, respectively, are equivalent:

Theorem 3.4.5 (Quillen)**.** *The adjunction*

$$|-| : \mathbf{Set}_\Delta \longleftrightarrow \mathbf{Top} : \mathrm{Sing}$$

is a Quillen equivalence with respect to the model structures described in Example 3.2.6.

Proof. [1] $\square$

Chapter 4

∞-categories

In this chapter we closely follow [3]. We have seen two different classes of simplicial sets which satisfy interesting horn filling conditions:

(1) For the singular set $\mathrm{Sing}(X)$ of a topological space X, for every $n \geq 1$, $0 \leq i \leq n$, every horn

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a filling (which is not necessarily unique).

(2) For the nerve $\mathrm{N}(\mathcal{C})$ of a category $\mathcal{C}$, for every $0 < i < n$, every horn

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a *unique* filling.

The notion of an ∞-category incorporates both of these examples:

Definition 4.0.1. An ∞-category is a simplicial set $\mathcal{C}$ such that, for every $0 < i < n$, every horn $\Lambda^n_i \to \mathcal{C}$ has a filling.

On problem set 4, it was shown that to every ∞-category $\mathcal{C}$ we may associate a category $\mathrm{h}\mathcal{C}$ called the *homotopy category* of $\mathcal{C}$ with

(1) objects of $\mathrm{h}\mathcal{C}$ are the vertices of $\mathcal{C}$,

(2) the morphisms between vertices x and y are homotopy classes of edges from x to y where two edges f, f' are declared homotopic if there exists a 2-simplex σ in $\mathcal{C}$ with $d_0\sigma = s_0(y)$, $d_1\sigma = f'$ and $d_2\sigma = f$,

(3) the composition law of $\mathrm{h}\mathcal{C}$ is induced by the filling of horns $\Lambda^2_1 \to \mathcal{C}$.

Example 4.0.2. (1) For every small category $\mathcal{C}$, we have $\mathrm{hN}(\mathcal{C}) \cong \mathcal{C}$.

(2) For every topological space X, the category $\mathrm{hSing}(X)$ is isomorphic to the fundamental groupoid $\pi_{\leq 1}(X)$ of X.

(3) For every Kan complex, the category $\mathrm{h}\mathcal{C}$ is a groupoid.

4.1 The coherent nerve

We therefore see that both topological spaces and categories can be interpreted as ∞-categories that satisfy strengthened horn filling conditions. We will now introduce a hybrid of the notion of a topological space and a category: a topological category. Again, we may produce an ∞-category from any topological category via a coherent version of the nerve. It will then turn out that, in a suitable sense, this construction will be exhaustive: up to a certain notion of weak equivalence, every ∞-category is the coherent nerve of a topological category.

Definition 4.1.1. Let $(\mathcal{M}, \otimes)$ be a monoidal category. An $\mathcal{M}$-*enriched category* $\mathcal{C}$ consists of

(1) a set $\operatorname{ob}\mathcal{C}$ of objects,

(2) for every pair of objects (x, y) of $\mathcal{C}$, a morphism object $\mathcal{C}(x, y) \in \mathcal{M}$,

(3) for every object x of $\mathcal{C}$, a morphism
$$\mathbb{1}_{\mathcal{M}} \longrightarrow \mathcal{C}(x, x)$$
in $\mathcal{M}$, called the *unit*,

(4) for every triple of objects (x, y, z) of $\mathcal{C}$, a morphism
$$\mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, z)$$
in $\mathcal{M}$, called the *composition*,

such that these data yield a unital, associative composition law in the apparent sense.

The following examples will be of interest to us:

Example 4.1.2. Consider the monoidal category **Top** of topological spaces equipped with the Cartesian product. Then a **Top**-enriched category is called a *topological category*.

Example 4.1.3. Consider the monoidal category $\mathbf{Set}_\Delta$ of simplicial sets equipped with the Cartesian product. Then a $\mathbf{Set}_\Delta$-enriched category is called a *simplicial category*.

From any topological category $\mathcal{T}$, we may produce a simplicial category $\operatorname{Sing}(\mathcal{T})$ obtained by taking singular sets of all mapping objects in $\mathcal{T}$. Here, to carry over the composition law from $\mathcal{T}$ to $\operatorname{Sing}(\mathcal{T})$, we use the morphisms
$$\operatorname{Sing}(X) \times \operatorname{Sing}(Y) \longrightarrow \operatorname{Sing}(X \times Y)$$
obtained by pulling back along the diagonal embeddings $|\Delta^n| \to |\Delta^n| \times |\Delta^n|$. We will now define the simplicial nerve $\mathrm{N}_\Delta(\mathcal{C})$ of a $\mathbf{Set}_\Delta$-enriched category $\mathcal{C}$, therefore also obtaining the nerve of a topological category as the composite $\mathrm{N}_{\mathrm{top}} = \mathrm{N}_\Delta \circ \operatorname{Sing}$.

Remark 4.1.4. Small simplicial categories (i.e. the set of objects and all sets of simplices in the mapping objects are small) form a category which we denote by $\mathbf{Cat}_\Delta$. There is a slight abuse of notation here: simplicial categories are not the same as simplicial objects in **Cat**. But, in fact, simplicial categories can be identified with those simplicial objects in **Cat** such that the simplicial set of underlying objects is constant. By an analogous argument to case of ordinary categories, the category $\mathbf{Cat}_\Delta$ has small limits and colimits.

To construct the simplicial nerve N_Δ, we proceed as usual when constructing a nerve: we begin by defining a cosimplicial object
$$\mathfrak{C} : \Delta \longrightarrow \mathbf{Cat}_\Delta, [n] \mapsto \mathfrak{C}[\Delta^n]$$
. We define $\mathfrak{C}[\Delta^n]$ to be the simplicial category described as follows:

- The set of objects of $\mathfrak{C}[\Delta^n]$ is the set $\{0, 1, \ldots, n\}$.
- To define the simplicial set of morphisms between i and j, consider the poset $P_{i,j}$ of subsets $I \subset [n]$ satisfying:

(1) $i, j \in I$,
(2) for every $k \in I$, we have $i \leq k \leq j$.

We then set $\mathfrak{C}[\Delta^n](i,j) = \mathrm{N}(P_{i,j})$.

- The composition law is induced by

$$P_{i,j} \times P_{j,k} \longrightarrow P_{i,k},\ (I, J) \mapsto I \cup J.$$

It is evident how the assignment $[n] \mapsto \mathfrak{C}[\Delta^n]$ organizes into a cosimplicial object.

Definition 4.1.5. Let $\mathcal{C}$ be a simplicial category. We define its simplicial nerve to be

$$\mathrm{N}_\Delta(\mathcal{C}) = \mathbf{Cat}_\Delta(\mathfrak{C}[\Delta^\bullet], \mathcal{C}).$$

Proposition 4.1.6. *Let $\mathcal{C}$ be a simplicial category such that, for every pair (x, y) of objects, the simplicial set $\mathcal{C}(x, y)$ is a Kan complex. Then the simplicial nerve $\mathrm{N}_\Delta(\mathcal{C})$ is an ∞-category.*

Proof. Via left Kan extension, we lift the functor $\mathfrak{C}$ along the Yoneda embedding to an adjunction

$$\mathfrak{C} : \mathbf{Set}_\Delta \longleftrightarrow \mathbf{Cat}_\Delta.$$

We need to show that, for every inner horn $\Lambda^n_i \subset \Delta^n$, any corresponding extension problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{N}_\Delta(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. But via the above adjunction, this lifting problem is equivalent to the lifting problem

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i] \arrow[r] \arrow[d] & \mathcal{C} \\ \mathfrak{C}[\Delta^n]. \arrow[ur, dashed] & \end{tikzcd} \tag{4.1.7}$$

It is immediate to solve this lifting problem for $n = 2$. For $n \geq 3$, we prodeed by an explicit computation of the functor

$$F : \mathfrak{C}[\Lambda^n_i] \to \mathfrak{C}[\Delta^n].$$

This is a combinatorially tedious but straightforward task, solved by expressing Λ^n_i as a colimit of its nondegenerate simplices, and then verifying that the following description has the universal property of the colimit: The functor F is the identity on the sets of objects. For objects $(j, k) \neq (0, n)$, we have

$$F : \mathfrak{C}[\Lambda^n_i](j,k) \xrightarrow{\mathrm{id}} \mathfrak{C}[\Delta^n](j,k).$$

The most interesting case is when $(j, k = (0, n)$. In this case, we have $\mathfrak{C}[\Delta^n](0, n) = (\Delta^1)^{n-1}$, and the map

$$F : \mathfrak{C}[\Lambda^n_i](0,n) \xrightarrow{\mathrm{id}} \mathfrak{C}[\Delta^n](0,n).$$

is the inclusion of a cube with the interior and one face missing: the face whose vertices are given by those subsets $I \subset [n]$ such that $i \notin I$. Thus, this inclusion is the smash product of the anodyne morphism $\{1\} \to \Delta^1$ with the monomorphism $\partial((\Delta^1)^{n-2}) \subset (\Delta^1)^{n-2}$, hence anodyne (cf. Corollary 2.5.6). From this, it follows immediately that the lifting problem (4.1.7) can be solved, since, by assumption all mapping simplicial sets in $\mathcal{C}$ are Kan complexes. $\square$

Example 4.1.8. Let $\mathbf{Kan} \subset \mathbf{Set}_\Delta$ denote the full subcategory of Kan complexes. We may interpret $\mathbf{Kan}$ as a simplicial category via the simplicial mapping space $\mathrm{Map}(K, L)$ which we defined via

$$\mathrm{Map}(K, L)_n = \mathbf{Set}_\Delta(\Delta^n \times K, L).$$

By Example 2.6.3, the simplicial set $\mathrm{Map}(K, L)$ is a Kan complex as long as L is a Kan complex. In particular, the simplicial category $\mathbf{Kan}$ satisfied the hypothesis of Proposition 4.1.6. We define

$$\mathcal{S} = \mathrm{N}_\Delta(\mathbf{Kan})$$

to be the *∞-category of spaces.*

4.2 ∞-groupoids

Let $\mathcal{C}$ be an ∞-category. The vertices of $\mathcal{C}$ are called *objects* and the edges of $\mathcal{C}$ are called morphisms of $\mathcal{C}$. A morphism in $\mathcal{C}$ is called an *equivalence* if it becomes an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$. An ∞-category $\mathcal{C}$ is called an ∞-groupoid if all morphisms in $\mathcal{C}$ are equivalences, i.e., if $\mathrm{h}\mathcal{C}$ is a groupoid.

Example 4.2.1. Every Kan complex is an ∞-groupoid.

Proposition 4.2.2. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is an ∞-groupoid if and only if $\mathcal{C}$ is a Kan complex.*

Proof. Later. □

Let $\mathcal{C}$ be an ∞-category. We denote by

$$\mathcal{C}^{\simeq} \subset \mathcal{C}$$

the simplicial subset consisting of those simplices all of whose edges are equivalences. Note that $\mathcal{C}^{\simeq}$ is an ∞-groupoid which we call the *maximal ∞-groupoid in* $\mathcal{C}$.

4.3 Functors and diagrams

A functor between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is defined to be a map $f : \mathcal{C} \to \mathcal{D}$ of simplicial sets. More generally, given a simplicial set K and an ∞-category $\mathcal{C}$, we define a K-diagram in $\mathcal{C}$ to be a map $K \to \mathcal{C}$ of simplicial sets.

Proposition 4.3.1. *Let K be a simplicial set and let $\mathcal{C}$ be an ∞-category. Then the simplicial set*

$$\mathrm{Fun}(K, \mathcal{C}) := \mathrm{Map}(K, \mathcal{C})$$

is an ∞-category.

Proof. Later. □

Example 4.3.2. Let $\mathcal{T}$ be a topological category, I an ordinary category. We set $\mathcal{C} := \mathrm{N}_{\mathrm{top}}(\mathcal{T})$ and define a *homotopy coherent I-diagram in* $\mathcal{T}$ to be a functor

$$f : \mathrm{N}(I) \longrightarrow \mathcal{C}$$

of ∞-categories. Let $\mathrm{h}\mathcal{T} := \mathrm{h}\mathcal{C}$ be the homotopy category of $\mathcal{T}$. Then any homotopy coherent I-diagram in $\mathcal{T}$ induces an ordinary I-diagram in $\mathrm{h}\mathcal{T}$ by means of passing to

$$\mathrm{h}f : I \longrightarrow \mathrm{h}\mathcal{C}.$$

We illustrate in a specific example that the original homotopy coherent I-diagram may contain substantially more information than its shadow in the homotopy category: Let $\mathcal{T} = \mathbf{Top}$ and $I = [1] \times [1]$. The datum of a homotopy coherent I-diagram in $\mathbf{Top}$ consists of morphisms

$$\begin{tikzcd} X_{00} \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, "h"] & X_{10} \arrow[d, "g'"] \\ X_{01} \arrow[r, "f'"] & X_{11} \end{tikzcd}$$

as depicted, together with homotopies $H : h \simeq g' \circ f$ and $H' : h \simeq g \circ f'$. Its image in the homotopy category consists of the maps $[f]$, $[f']$,$[g]$,$[g']$ such that $[f][g'] = [f'][g]$ without specific choice of homotopy that realizes this latter equality. For a specific example of the relevance of the fact that we include choices of the homotopies H and H', let X be a topological space, $* \in X$ and ΩX be the space of loops in X based at $*$. Then, there are two homotopy coherent squares of the form

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr] & * \arrow[d, "g'"] \\ * \arrow[r, "f'"] & X \end{tikzcd}$$

which are obtained by setting all morphisms to be constant but the homotopies H and H' to be

(1) H and H' are constant homotopies,

(2) $H : I \times \Omega X \to X, (t, \alpha) \mapsto \alpha(t)$ and H' is constant.

Both these diagrams have the same shadow in the homotopy category of topological spaces. But the homotopy coherent diagrams themselves are not equivalent as objects if $\mathrm{Fun}(I, \mathcal{C})$. Later, we will see that the diagram (2) is a pullback diagram in the ∞-categorical sense (and (1) is not).

Example 4.3.3. Let $\mathbf{Cat}_\infty$ denote the set of all small ∞-categories. We interpret $\mathbf{Cat}_\infty$ as a simplicial category where the simplicial set of maps between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is defined as the maximal ∞-groupoid $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ which is, in particular, a Kan complex. Therefore, we obtain an ∞-category

$$\mathcal{C}\mathrm{at}_\infty := \mathrm{N}_\Delta(\mathbf{Cat}_\infty)$$

called the *∞-category of small ∞-categories.*

4.4 Overcategories and undercategories

Let $\mathcal{C}$ and $\mathcal{C}'$ be ordinary categories. We introduce the *join* $\mathcal{C} * \mathcal{C}'$ to be the category with

- objects $\mathrm{ob}(\mathcal{C} * \mathcal{C}') = \mathrm{ob}(\mathcal{C}) \amalg \mathrm{ob}(\mathcal{C}')$
- morphisms
$$(\mathcal{C} * \mathcal{C}')(x, y) = \begin{cases} \mathcal{C}(x, y) & \text{if } x, y \in \mathcal{C}, \\ \mathcal{C}'(x, y) & \text{if } x, y \in \mathcal{C}', \\ * & \text{if } x \in \mathcal{C} \text{ and } y \in \mathcal{C}', \\ \emptyset & \text{if } x \in \mathcal{C}' \text{ and } y \in \mathcal{C}. \end{cases}$$

Example 4.4.1. We have $[m] * [n] \cong [m + n + 1]$.

Our goal is to introduce a version of the join construction for ∞-categories which extends the above join construction in the sense that $\mathrm{N}(\mathcal{C}) * \mathrm{N}(\mathcal{C}') = \mathrm{N}(\mathcal{C} * \mathcal{C}')$.

Remark 4.4.2. In what follows, it is useful to use slightly more flexible coordinates for simplicial objects: let $\widetilde{\Delta}$ denote the category of small finite linearly ordered nonempty sets with weakly monotone maps as morphisms. Then we have an inclusion $\Delta \subset \widetilde{\Delta}$ which is an equivalence of categories. So that we may identify simplicial objects in a category $\mathcal{C}$ with functors $\widetilde{\Delta}^{\mathrm{op}} \to \mathcal{C}$. In particular, we may construct simplicial objects as functors defined on $\widetilde{\Delta}$ which can sometimes be useful to make the functoriality more transparent. We adopt this point of view in the following definition.

Definition 4.4.3. Let K, K' be simplicial sets. We define a simplicial set $K * K'$ via

$$(K * K')(I) = \coprod_{I = J \cup J'} K(J) \times K'(J') \tag{4.4.4}$$

where $I \in \widetilde{\Delta}$ and the coproduct runs over all partitions of I into unions of J and J' where every element of J is smaller than every element of J'. Here, we also allow $J = \emptyset$ and $J' = \emptyset$ where we agree that $K(\emptyset) = K'(\emptyset) = *$. This construction has an apparent functoriality in $\widetilde{\Delta}$ so that we obtain a simplicial set via Remark 4.4.2. We call $K * K'$ the *join* of K and K'.

Proposition 4.4.5. *Let $\mathcal{C}$, $\mathcal{C}'$ be small categories. Then we have* $\mathrm{N}(\mathcal{C}) * \mathrm{N}(\mathcal{C}') \cong \mathrm{N}(\mathcal{C} * \mathcal{C}')$.

Proof. Let $\sigma : \Delta^n \to \mathrm{N}(\mathcal{C}) * \mathrm{N}(\mathcal{C}')$ be an n-simplex. Then, by (4.4.4), there are three different cases to be considered:

(1) $J = [n]$, $J' = \emptyset$. In this case, σ corresponds to an n-simplex in $\mathrm{N}(\mathcal{C})$ and hence to a functor $[n] \to \mathcal{C}$,

(2) $J = \emptyset$, $J' = [n]$. In this case, σ corresponds to an n-simplex in $\mathrm{N}(\mathcal{C}')$ and hence to a functor $[n] \to \mathcal{C}'$,

(3) $J = [m]$,$J' = [m']$. In this case, the simplex σ corresponds to a pair given by a simplex $[m] \to \mathcal{C}$ in $\mathrm{N}(\mathcal{C})$ and a simplex $[m'] \to \mathcal{C}'$ in $\mathrm{N}(\mathcal{C}')$.

We now conclude by noting that in each of these cases, the given data determines a unique map $[n] \to \mathcal{C} * \mathcal{C}'$ and, in fact, every such map arises in one of these forms. Therefore, we have constructed a bijection $(\mathrm{N}(\mathcal{C}) * \mathrm{N}(\mathcal{C}'))_n \cong \mathrm{N}(\mathcal{C} * \mathcal{C}')_n$. It is easy to check that this bijection is further functorial in $[n]$. □

Example 4.4.6. We have $\Delta^m * \Delta^n = \mathrm{N}([m]) * \mathrm{N}([n]) = \mathrm{N}([m] * [n]) = \mathrm{N}([m+n+1]) = \Delta^{m+n+1}$.

Remark 4.4.7. The join construction is functorial in both coordinates. In fact, via the natural inclusion $K \subset K * K'$, we may interpret it as a functor

$$\mathbf{Set}_\Delta \longrightarrow (\mathbf{Set}_\Delta)_{K/},\ K' \mapsto K * K'.$$

As follows directly from the defining formula of the join, this functor commutes with colimits. This statement, together with the analogous statement in the first variable and the formula for the join of simplices from Example 4.4.6 determines the functor $*$ uniquely: every simplicial set can be expressed as a colimit of its simplices.

Proposition 4.4.8. *Let $\mathcal{C}$, $\mathcal{C}'$ be ∞-categories. Then $\mathcal{C} * \mathcal{C}'$ is an ∞-category.*

Proof. Homework. □

We introduce some notation. Given a simplicial set K, we introduce

$$K^{\triangleleft} = \Delta^0 * K$$

called the *left cone of* K and

$$K^{\triangleright} = K * \Delta^0$$

called the *right cone of* K. We will now use the join construction to introduce an ∞-categorical version of over- and undercategories.

Definition 4.4.9. Let K, S be simplicial sets and let $p : K \to S$ be a morphism. We introduce a simplicial set $S_{p/}$ via

$$(S_{p/})_n = \{\text{maps } f : K * \Delta^n \to \mathcal{C} \text{ satisfying } f|K = p\}$$

where the functoriality in $[n]$ is given by the functoriality of the join construction. Dually, we introduce a simplicial set $S_{/p}$ via

$$(S_{/p})_n = \{\text{maps } f : \Delta^n * K \to \mathcal{C} \text{ satisfying } f|K = p\}.$$

Proposition 4.4.10. *Let K be a simplicial set, $\mathcal{C}$ an ∞-category, and $p : K \to \mathcal{C}$ a diagram in $\mathcal{C}$. Then the simplicial sets $\mathcal{C}_{p/}$ and $\mathcal{C}_{/p}$ are ∞-categories.*

Proof. Later. □

Under the hypothesis of the proposition, we call $\mathcal{C}_{p/}$ the *∞-category of cones under p* and $\mathcal{C}_{/p}$ the *∞-category of cones over p.*

4.5 Limits and colimits

Having already provided the ∞-categorical counterparts of categories of cones under and over a given diagram, to provide a definition of colimits and limits, it suffices to introduce initial and final objects in the ∞-categorical context.

Definition 4.5.1. Let $\mathcal{C}$ be an ∞-category and x an object of $\mathcal{C}$. We say that x is an *initial object* if the map

$$\mathcal{C}_{x/} \longrightarrow \mathcal{C}$$

obtained by restricting an n-simplex $f : \Delta^0 * \Delta^n \to \mathcal{C}$ to Δ^n is a trivial Kan fibration. Dually, we say that x is a *final object* if the map

$$\mathcal{C}_{/x} \longrightarrow \mathcal{C}$$

is a trivial Kan fibration.

Example 4.5.2. Let $\mathcal{C}$ be an ∞-category and let x be an initial object. Suppose y is another object of y. By solving the lifting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

we obtain an edge $x \to y$ in $\mathcal{C}$. Suppose that $f, f' : x \to y$ are edges from x to y in $\mathcal{C}$. We may organize this data into a lifting problem

$$\begin{tikzcd} \Delta^0 \amalg \Delta^0 \arrow[r, "(f,f')"] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^1 \arrow[r, "\mathrm{id}_y"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

whose solution provides a homotopy between f and f' so that the images of f and f' in the homotopy category $\mathrm{h}\mathcal{C}$ coincide. In particular, we have shown that any initial object in $\mathcal{C}$ defines an initial object in the homotopy category. However, the condition that x be an initial object in the ∞-category $\mathcal{C}$ is much stronger: The next question we may ask is whether two different choices of homotopies between f and f' can be identified by a homotopy of homotopies - the answer is positive and the desired homotopy can be obtained by means of another lifting problem for the map $\partial\Delta^2 \subset \Delta^2$. A similar question then appears in the next dimension, etc.

Proposition 4.5.3. *Let $\mathcal{C}$ be an ∞-category and suppose that $\mathcal{C}$ has an initial object. Let $\mathcal{C}' \subset \mathcal{C}$ be the simplicial subset consisting of those simplices all of whose vertices are initial objects ($\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ on the initial objects). Then $\mathcal{C}'$ is a contractible Kan complex.*

Proof. We have to show that every lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}' \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. For $n = 0$, this simply means that $\mathcal{C}'$ is nonempty which follows by assumption. For $n > 0$, the above lifting problem is equivalent to a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

where x is the vertex of $\mathcal{C}'$ given by restricting $\partial\Delta^n \to \mathcal{C}$ to the vertex $\{0\}$. Since this vertex is an initial object, we may solve this latter lifting problem and hence also the original one. □

Remark 4.5.4. In ordinary category theory, we have that initial objects are unique up to unique isomorphism. The conclusion of Proposition 4.5.3 is the ∞-categorical version of this statement: initial objects are unique up to contractible choice.

Definition 4.5.5. Let $\mathcal{C}$ be an ∞-category, K a simplicial set, and $p : K \to \mathcal{C}$ a diagram in $\mathcal{C}$.

(1) an initial object in $\mathcal{C}_{p/}$ is called a *colimit cone for* p,

(2) a final object in $\mathcal{C}_{/p}$ is called a *limit cone for* p.

Remark 4.5.6. From Proposition 4.5.3, we therefore immediately deduce that colimits and limits of a diagram in an ∞-category $\mathcal{C}$ are, if they exist, unique up to contractible choice.

4.6 Left fibrations

We follow more or less verbatim the content of §2.1 in [3] including the proofs so that we do not include lecture notes on this part of the course.

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