Introduction to to higher category theory — Winter Semester 18/19
Prof. Dr. Tobias Dyckerhoff — Due: 4. December

Exercise sheet 7

Problem 1. Let $K \in \mathbf{Kan}$ and $v \in K_0$. Recall from exercise 4.2.5 that there is a functor $\pi_{\leq 1} : \mathbf{Kan} \to \mathbf{Grpd}$ left adjoint to the nerve (denoted by γ in exercise 4.2.5).

1. Show that there is an isomorphism $\pi_0(K) \simeq \pi_0(|K|)$.
2. Show that there is an isomorphism
$$\mathrm{Hom}_{\pi_{\leq 1}(K)}(v, v) \cong \pi_1(K, v).$$
Conclude that if K is connected (i.e., $\pi_0(K) = *$), there is an equivalence of categories
$$B\pi_1(K, v) \simeq \pi_{\leq 1}(K).$$

Problem 2. A *simplicial group* is an object $G \in \mathbf{Grp}_\Delta := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Grp})$. The forgetful functor $\iota : \mathbf{Grp} \to \mathbf{Set}$ induces a functor
$$\mathbf{Grp}_\Delta \to \mathbf{Set}_\Delta, \quad G \mapsto \hat{G} := \iota \circ G.$$

1. Let G be a simplicial group. Given a horn $(g_1, g_2, \dots, g_{k-1}, -, g_{k+1}, \cdots g_n)$ with $g_i \in G_{n-1}$, define an element inductively as follows:
   - Set $h_0 := s_0(g_0)$
   - Define h_i for $i < k$ by $h_i := h_{i-1} \cdot (s_i d_i(h_{i-1})) \cdot s_i(g_i)$
   - Set $h_n := h_{k-1} \cdot (s_{n-1} d_n(h_{k-1}))^{-1} \cdot s_{n-1}(g_n)$.
   - Define h_i for $i > k$ by $h_i := h_{i+1} \cdot (s_{i-1} d_i(h_{i+1}))^{-1} \cdot s_{i-1}(g_i)$.

   Show that:
   - if $k < n$, then $d_i(h_{k+1}) = g_i$;
   - if $k = n$, then $d_i(h_{n-1}) = g_i$.

   Conclude that $\hat{G}$ is a Kan complex.
2. For a simplicial set $X \in \mathbf{Set}_\Delta$, an *action* of G on X consists of an action of G_n on X_n for every $n \geq 0$ such that, for every morphism $f : [m] \to [n]$ in Δ, the induced map $f^* : X_n \to X_m$ is G-equivariant in the sense that for $(g, x) \in G_n \times X_n$, we have
$$f^*(g \cdot x) = f^*(g) \cdot f^*(x).$$
Given an action of G on X, show that there is a simplicial set X/G with
$$(X/G)_n = X_n/G_n$$
Such that there is a well defined quotient map
$$p_G : X \to X/G$$
3. Suppose given an action of G on X such that X_n is a free G_n set for all $n \geq 0$. Let $\sigma : \Delta^n \to X/G$ be an n-simplex and define the *fiber* F_σ of X over σ to be the pullback.
$$\begin{array}{ccc} F_\sigma & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p_G} \\ \Delta^n & \xrightarrow[\sigma]{} & X/G \end{array}$$
Show that there is an isomorphism $F_\sigma \xrightarrow{\cong} G \times \Delta^n$ such that the diagram
$$\begin{array}{ccc} F_\sigma & \longrightarrow & G \times \Delta^n \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[\mathrm{id}]{} & \Delta^n \end{array}$$
commutes. Conclude that $p_G : X \to X/G$ is a Kan fibration.

Problem 3. Let $G \in \mathbf{Grp}_\Delta$ be a simplicial group. Define a simplicial set EG by

$$EG_n := G_n \times G_{n-1} \times \cdots G_0$$

with face maps

$$d_i(g_n, \ldots, g_0) = \begin{cases} (d_i(g_n), d_{i-1}(g_{n-1}), \ldots, d_0(g_{n-i})g_{n-i-1}, g_{n-i-2}, \ldots, g_0) & i < n \\ (d_n(g_n), \ldots, d_1(g_1)) & i = n \end{cases}$$

and degeneracy maps

$$s_i(g_n, \ldots, g_0) = (s_i(g_n), s_{i-1}(g_{n-1}), \ldots, s_0(g_{n-i}), e, g_{n-i-1}, \ldots, g_0)$$

where e is the unit of G_{n-i}.

1. Show that the maps
$$g \cdot (g_n, \ldots, g_0) \mapsto (gg_n, \ldots, g_0)$$
equip EG with a free G-action, yielding a simplicial $BG := EG/G$ and a Kan fibration
$$\pi : EG \to BG$$

2. Let $e \in BG_0$ be the vertex represented by the identity in G_0. Show that the fiber of EG over e is isomorphic to G.

3. Consider the functor
$$\kappa_n : [n] \times [1] \to [n+1], \quad (i, j) \mapsto \begin{cases} 0 & j = 0 \\ i+1 & j = 1. \end{cases}$$
Show that given a morphism $f : [n] \to [m]$ in Δ, the diagram
$$\begin{array}{ccc} [n] \times [1] & \xrightarrow{\kappa_n} & [n+1] \\ {\scriptstyle f \times \mathrm{id}_{[1]}} \downarrow & & \downarrow {\scriptstyle \tilde{f}} \\ [m] \times [1] & \xrightarrow[\kappa_m]{} & [m+1] \end{array}$$
commutes, where $\tilde{f} : [n+1] \to [m+1]$ is defined by
$$\tilde{f}(i) = \begin{cases} f(i-1) + 1 & i \neq 0 \\ 0 & i = 0. \end{cases}$$

4. Show that the map $EG_n \to P_e(BG)_n$ given by
$$(g_n, \ldots, g_0) \mapsto \kappa_n^*(\pi(e, g_n, \ldots, g_0))$$
defines a map of simplicial sets such that the diagram
$$\begin{array}{ccc} EG & \longrightarrow & P_e(BG) \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$
commutes. Conclude that there is a commutative diagram
$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\ \Omega_e(BG) & \longrightarrow & P_e(BG) & \longrightarrow & BG \end{array}$$