Exercise sheet 6

Problem 1. Let I be the groupoid with two objects 0 and 1 and a unique isomorphism between them. Call a functor $F : \mathcal{C} \to \mathcal{D}$ an *isofibration* if it has the right lifting property with respect to the functor

$$\{0\} \hookrightarrow I.$$

1. Show that for any functor $F : \mathcal{C} \to \mathcal{D}$ between categories, the nerve $\mathrm{N}(F) : \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ is an inner fibration, i.e., it has the right lifting property with respect to all horn inclusions $\Lambda^n_i \subset \Delta^n$ with $0 < i < n$.
2. Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration between two groupoids. Show that $\mathrm{N}(F) : \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ is a Kan fibration.

Problem 2. Let G be a discrete group, let BG be the associated category.

1. Show that the nerve $\mathrm{N}(BG)$ of BG is a Kan complex.
2. Show that $\pi_1(\mathrm{N}(BG)) \cong G$.
3. Show that $\pi_n(\mathrm{N}(BG)) \cong 0$ for $n > 1$.

Problem 3. A *compactly generated Hausdorff space* is a Hausdorff space X such that a subset $U \subset X$ is closed if and only if, for every compact subset $C \subset X$, $U \cap C$ is closed. Let **CGHaus** denote the full subcategory of **Top** on the compactly generated Hausdorff spaces.

1. For a Hausdorff space X with topology τ, define a space $X_C = (X, \tau_C)$, whose underlying set is the same as that of X, and where τ_C has as its closed sets precisely those $U \subset X$ such that, if $C \subset X$ is compact in τ, then $U \cap C$ is closed in C (with the topology induced by τ). Show that $X_C \in$ **CGHaus**, and show that the assignment $X \mapsto X_C$ defines a functor

   $$F : \mathbf{Haus} \to \mathbf{CGHaus},$$

   where **Haus** is the full subcategory of **Top** on the Hausdorff spaces.

2. Show that F is right adjoint to the inclusion $i : \mathbf{CGHaus} \to \mathbf{Haus}$. Conclude that **CGHaus** has all limits and colimits, and that if a colimit of compactly generated Hausdorff spaces in **Top** is Hausdorff, then it is the colimit in **CGHaus**.
3. For $X \in \mathbf{CGHaus}$, let $\mathcal{C}(X)$ be the poset of compact subsets of X. Let $F_{X,Y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathbf{Top}$ be the functor that sends $C, D \mapsto C \times D$. Show that

   $$(X \times Y)_C \cong \operatorname*{colim}_{\mathcal{C}(X)\times\mathcal{C}(Y)} F_{X,Y} \cong \operatorname*{colim}_{\mathcal{C}(X)} \operatorname*{colim}_{\mathcal{C}(Y)} F_{X,Y}.$$

4. Let $\mathrm{Map}(X, Y)$ be the set of continuous maps $X \to Y$ equipped with the compact-open topology. Show that

   $$\mathbf{CGHaus}((X \times Y)_C, Z) \cong \mathbf{CGHaus}\,(X, (\mathrm{Map}(Y, Z))_C)$$

   Conclude that $X, Y \mapsto (X \times Y)_C$ preserves colimits in each variable.

5. Show that $|\Delta^n| \in \mathbf{CGHaus}$, and consequently that the adjuction $|-| \dashv \mathrm{Sing}$ descends to an adjunction

$$|-| : \mathbf{Set}_\Delta \leftrightarrow \mathbf{CGHaus} : \mathrm{Sing}\,.$$

Show that there is a natural isomorphism $(|\Delta^n| \times |\Delta^m|)_C \cong |\Delta^n \times \Delta^m|$ in **CGHaus**.

6. Expressing X as $\mathrm{colim}_{\Delta/X}\, \Delta^n$, show that $|-| : \mathbf{Set}_\Delta \to \mathbf{CGHaus}$ preserves products.

Problem 4. 1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between small categories and let $\eta : F \Rightarrow G$ be a natural transformation. Denote by $f, g : |\,\mathrm{N}(\mathcal{C})| \to |\,\mathrm{N}(\mathcal{D})|$ the maps induced by F and G respectively. Show that η induces a homotopy between f and g. (Hint: view η as a functor on $[1] \times \mathcal{C}$.)

2. Show that an adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ induces a homotopy equivalence $|\,\mathrm{N}(\mathcal{C})| \simeq |\,\mathrm{N}(\mathcal{D})|$.

3. Show that if $\mathcal{C}$ has an initial or a final object, then $|\,\mathrm{N}(\mathcal{C})|$ is contractible.