Exercise sheet 3

Problem 1. Recall that a groupoid is a category in which all morphisms are invertible. A $(2,1)$-*category* is a 2-category $\mathcal{C}$ such that, for all $x, y \in \mathcal{C}$, the category $\mathcal{C}(x, y)$ is a groupoid.

1. Given a groupoid $\mathcal{G}$, let
$$\pi_0(\mathcal{G}) := \mathrm{ob}(\mathcal{G})/_\sim$$
be the set of equivalence classes of objects under the relation $x \sim y$ if there exists $g : x \to y$ in $\mathcal{G}$. Show that the assignment
$$\pi_0 : \mathbf{Grpd} \to \mathbf{Set}; \quad \mathcal{G} \mapsto \pi_0(\mathcal{G})$$
on objects defines a functor.

2. Show that, given a (2,1)-category $\mathcal{C}$, there is a 1-category $\mathrm{h}\mathcal{C}$ such that
   - $\mathrm{ob}(\mathrm{h}\mathcal{C}) = \mathrm{ob}(\mathcal{C})$
   - for all $x, y \in \mathcal{C}$, $\mathrm{h}\mathcal{C}(x, y) = \pi_0(\mathcal{C}(x, y))$.

Problem 2. An *equalizer* in a category $\mathcal{C}$ is a limit over a diagram of the form
$$x \rightrightarrows y \tag{1}$$
A *product* is a limit over a diagram $F : I \to \mathcal{C}$, where I has only identity morphisms, written as $\prod_{i \in I} F(i)$. A category $\mathcal{C}$ is said to *have equalizers* (resp. have products) if any diagram of the form 1 (resp. any diagram $F : I \to \mathcal{C}$, where I has only identity morphisms) has a limit.

1. Let $\mathcal{C}$ be a category with equalizers and small products. Let I be a small category and $F \in \mathcal{C}^I$. Show that the equalizer of
$$\prod_{i \in \mathrm{ob}(I)} F(i) \rightrightarrows \prod_{i \in \mathrm{morph}(I)} t(f),$$
where $t(f)$ denotes the target of the morphism f, is a limit of F. Conclude that $\mathcal{C}$ has small limits.

2. Show that the categories **Set**, **Grp**, **Cat**, and $\mathbf{Vect}_K$ have small limits and colimits.

Problem 3. Let $\varphi : I \to J$ be a fully faithful functor between small categories, and $\mathcal{C}$ a category with all small limits. Show that the counit
$$\varphi^* \circ \varphi_* \Rightarrow \mathrm{id}_{\mathcal{C}^I}$$
is a natural isomorphism. Conclude that the right Kan extension functor φ_* is fully faithful.

Problem 4. Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be a category which has small colimits. Let $Y : \mathcal{C} \to \mathbf{Set}_{\mathcal{C}}, x \mapsto \mathcal{C}(-, x)$ denote the Yoneda embedding.

1. Show that $\mathbf{Set}_{\mathcal{C}}$ has small colimits, in particular, show that, given a diagram $F : I \to \mathbf{Set}_{\mathcal{C}}$ a cone under F is a colimit cone, if and only if, for every $x \in \mathcal{C}$, the induced cone obtained by evaluating at x is a colimit cone in $\mathbf{Set}$.

2. Show that a functor $F : \mathbf{Set}_{\mathcal{C}} \to \mathcal{D}$ is a left Kan extension along Y of its restriction Y^*F if and only if F *preserves colimits*, i.e., F maps colimit cones in $\mathbf{Set}_{\mathcal{C}}$ to colimit cones in $\mathcal{D}$.

3. Deduce that the functor

$$Y^* : \mathrm{Fun}(\mathbf{Set}_{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

restricts to an equivalence between the full subcategory of $\mathrm{Fun}(\mathbf{Set}_{\mathcal{C}}, \mathcal{D})$ given by the colimit preserving functors and the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Problem 5. Let $f : H \to G$ be an arbitrary homomorphism between small groups and $\phi : BH \to BG$ the corresponding functor of groupoids. For a small field K and a representation $F : BH \to \mathrm{Vect}_K$, describe the right Kan extension $\phi_* F$.