Exercise Sheet 12

Problem 1. Let $p : \mathcal{C} \to \mathcal{D}$ be a left fibration of ∞-categories.

(1) Let f and g be homotopic 1-simplices of f, show that $\chi(f)$ and $\chi(g)$ are homotopic 1-simplices of $\mathcal{S}$. Conclude that there is a functor $h\mathcal{D} \to h\mathcal{S}$ which sends $y \in \mathcal{D}$ to the fiber of p over y.

(2) Suppose p is a Kan fibration. Show that, for every $f : x \to y$ in $\mathcal{D}$, there is an induced map $\chi(y) \to \chi(x)$ which is homotopy inverse to $\chi(f)$.

Problem 2. For a simplicial set X, define $\mathcal{M}_X := \mathfrak{C}\left[X^{\triangleleft} \coprod_X \Delta^0\right]$.

(1) Let $x \in X^{\triangleleft}$ denote the cone point, and $y \in \Delta^0$ be the unique 0-simplex. Show that

$$\mathrm{St}_* : \mathbf{Set}_\Delta \to \mathbf{Set}_\Delta; \quad X \mapsto \mathrm{Map}_{\mathcal{M}_X}(x, y)$$

is a functor.

(2) Show that St_* preserves colimits.

(3) Conclude that there is a functor $Q_\bullet : \Delta \to \mathbf{Set}_\Delta$ such that St_* is the left Kan extension of $Q_\bullet$ along the Yoneda embedding and that St_* has a right adjoint. (Hint: See Sheet 3, problem 4.) Give a description of $Q_\bullet$.

Problem 3. Let $\mathcal{C}$ be a small category, and $F : \mathcal{C} \to \mathbf{Set}_\Delta$ a functor. We define a simplicial set $N_F(\mathcal{C})$ as follows. An n-simplex $\Delta^n \to N_F(\mathcal{C})$ consists of the data of:

- a functor $\sigma : [n] \to \mathcal{C}$
- for every nonempty $J \subset \{0, 1, \ldots, n\}$ with maximal element j, a morphism $\tau_J : \Delta^J \to F(\sigma(j))$ in $\mathbf{Set}_\Delta$.

Subject to the condition that, for every nonempty $J \subset K \subset \{0, 1, \ldots, n\}$, with maximal elements j and k respectively, the diagram

$$\begin{array}{ccc} \Delta^J & \longrightarrow & F(\sigma(j)) \\ \downarrow & & \downarrow \\ \Delta^k & \longrightarrow & F(\sigma(k)) \end{array}$$

commutes.

(1) Show that, for the constant functor $* : \mathcal{C} \to \mathbf{Set}_\Delta$ with value Δ^0, we have $N_*(\mathcal{C}) \cong N(\mathcal{C})$.

(2) Let $F, G : \mathcal{C} \to \mathbf{Set}_\Delta$ be two functors, and let $\eta : F \to G$ be a natural transformation. Show that η induces a map of simplicial sets $N_F(\mathcal{C}) \to N_G(\mathcal{C})$.

(3) Show that if for all $c \in \mathcal{C}$ the morphism $\eta_c : F(c) \to G(c)$ is a Kan fibration then the induced morphism $N_F(\mathcal{C}) \to N_G(\mathcal{C})$ is a left fibration.

(4) Deduce that if, for every $c \in \mathcal{C}$, the simplicial set $F(c)$ is a Kan complex, then the induced map $N_F(\mathcal{C}) \to N_*(\mathcal{C}) = N(\mathcal{C})$ is a left fibration. Show that the functor $\mathcal{C} \to h\mathcal{S}$ associated to this left fibration is isomorphic to the functor $\overline{F} : \mathcal{C} \to h\mathcal{S}$ obtained from F by postcomposing with the localization functor from the category of Kan complexes to its homotopy category.

Remark. In the case where $\mathcal{C}$ is a 1-category, the construction which sends $F : \mathcal{C} \to \mathbf{Set}_\Delta$ to $N_F(\mathcal{C}) \to N(\mathcal{C})$ is a model for the ∞-categorical Grothendieck construction.