Exercise Sheet 11

Problem 1. Let $\mathcal{D}$ be a small category.

1. Explicitly define a "contravariant" version of the Grothendieck construction which associates to a diagram
$$\chi : \mathcal{D}^{\text{op}} \to \mathbf{Grp}$$
a functor $\pi : \mathcal{C}^{\chi} \to \mathcal{D}$ that exhibits $\mathcal{C}^{\chi}$ as fibered in groupoids over $\mathcal{D}$.

2. Express the contravariant Grothendieck construction in terms of the covariant Grothendieck construction from class.

3. Let $X : \mathcal{D}^{\text{op}} \to \mathbf{Set}$ be a diagram of sets which we interpret as discrete groupoids. Show that there is an equivalence of categories between the contravariant Grothendieck construction of X and the slice category $\mathcal{D}/X$ with respect to the Yoneda embedding $\mathcal{D} \to \mathbf{Set}_{\mathcal{D}}$.

Problem 2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of small categories. A morphism $f : c \to c'$ in $\mathcal{C}$ is called *F-coCartesian* if, for every object c'' of $\mathcal{C}$, the natural map

$$\mathcal{C}(c', c'') \longrightarrow \mathcal{C}(c, c'') \times_{\mathcal{D}(F(c), F(c''))} \mathcal{D}(F(c'), F(c''))$$

is bijective. The functor F is called a *coCartesian fibration* if, for every $x \in \mathcal{C}$, and every morphism $\overline{f} : F(x) \to y$ in $\mathcal{D}$, there exists an F-coCartesian morphism $f : x \to \widetilde{y}$ such that $F(f) = \overline{f}$.

1. Show that F exhibits $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$ if and only if every morphism in $\mathcal{C}$ is F-coCartesian.

2. Modify the construction of the functor $\pi : \underline{\mathbf{Vect}} \to \mathbf{Top}$ from class so that

   (i) Its fiber over a space Y is isomorphic to the category of (say real) vector bundles over Y and a morphism is a morphism $E \to E'$ of bundles over Y.

   (ii) The functor π^{op} is a coCartesian fibration (we say π is a Cartesian fibration).

3. Let $\mathcal{D}$ be a small category and let
$$\chi : \mathcal{D} \to \mathbf{Cat}$$
be a diagram of categories. Introduce a category $\mathcal{C}_{\chi}$ with objects given by pairs (d, x) with $d \in \mathcal{D}$ and $x \in \chi(d)$ and a morphism from (d, x) to (d', x') given by a morphism $f : d \to d'$ together with a morphism $\alpha : \chi(f)(x) \to x'$ in $\chi(d')$. Show that the natural forgetful functor
$$F : \mathcal{C}_{\chi} \longrightarrow \mathcal{D}$$
is a coCartesian fibration.

4. Given a coCartesian fibration F, construct a pseudo-functor $\chi_F : \mathcal{D} \to \mathbf{Cat}$ which, in the case $F : \mathcal{C}_{\chi} \to \mathcal{D}$ recovers χ up to natural isomorphism.

5. Provide a definition of a coCartesian fibration of ∞-categories such that, when applied to the nerve of a functor of categories, it recovers the above notion.

Problem 3. Complete the arguments from class to show that, given a small category $\mathcal{D}$, there is an equivalence between suitably defined categories of functors $F : \mathcal{C} \to \mathcal{D}$ that exhibit $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$ and pseudo-functors $\chi : \mathcal{D} \to \mathbf{Grp}$.