Exercise sheet 10

Problem 1. Let $\mathcal{C} \in \mathbf{Set}_\Delta$ be an ∞-category.

1. Let $x \in \mathcal{C}$ be an initial object. Show that x is an initial object in $\tau_1(\mathcal{C})$.
2. Find a counterexample that shows that if x is an initial object in $\tau_1(\mathcal{C})$, x is not necessarily initial in $\mathcal{C}$.
3. For $y, z \in \mathcal{C}$, define $\mathrm{Map}^L_{\mathcal{C}}(y, z)$ to be the simplicial set whose n-simplices are $n+1$-simplices $\sigma : \Delta^{n+1} \to \mathcal{C}$ such that $\sigma(0) = y$ and $d_0\sigma = z$. Show that if $x \in \mathcal{C}$ is an initial object, then for any $y \in \mathcal{C}$, $\mathrm{Map}^L_{\mathcal{C}}(x, y)$ is a contractible Kan complex.
4. Suppose $\mathcal{C}$ is a Kan complex. Show that $\mathcal{C}$ has an initial object if and only if $\mathcal{C}$ is contractible.

Problem 2. Recall that there is a equivalence of categories $O : \Delta \to \Delta$, which sends each ordered set $\{0 < 1 < \cdots < n\}$ to the ordered set $\{n < n-1 < \cdots < 1 < 0\}$. Given a simplicial set $X : \Delta \to \mathbf{Set}$, define the *opposite simplicial set* X^{op} of X to be the simplicial set $X \circ A^{\mathrm{op}}$. Let $\mathcal{C}$ be an ∞-category.

1. Let $X, Y \in \mathbf{Set}_\Delta$. Show that $(X \star Y)^{\mathrm{op}} \cong X^{\mathrm{op}} \star Y^{\mathrm{op}}$.
2. Suppose $K \in \mathbf{Set}_\Delta$ and $p : K \to \mathcal{C}$. Show that $(\mathcal{C}_{/p})^{\mathrm{op}} \cong (\mathcal{C}^{\mathrm{op}})_{p/}$
3. Suppose $K \in \mathbf{Set}_\Delta$ and $p : K \to \mathcal{C}$. Show that a colimit of p^{op} in $\mathcal{C}^{\mathrm{op}}$ is a limit of p in $\mathcal{C}$.

Problem 3. Let K be a simplicial set.

1. Show that
$$K \star - : \mathbf{Set}_\Delta \to (\mathbf{Set}_\Delta)_{K/}, \quad X \mapsto (K \to K \star X)$$
defines a functor. Show that this functor preserves colimits.
2. Show that $- \star K : \mathbf{Set}_\Delta \to (\mathbf{Set}_\Delta)_{K/}$ likewise preserves colimits.
3. Conclude that the join $\star$, is uniquely determined by the properties (1) and (2), together with the fact that $\Delta^n \star \Delta^m \cong \Delta^{n+m+1}$. (Hint: See exercise sheet 3, problem 4.)

Problem 4. A *dg-category* $\mathcal{D}$ over a small field k is defined to be a category enriched in chain complexes over k, i.e.:

- A set $\mathrm{ob}(\mathcal{D})$ of objects.
- For every $x, y \in \mathrm{ob}(\mathcal{D})$, a chain complex $\mathcal{D}(x, y)_\bullet$ over k.

- For every triple of objects $x, y, z \in \mathrm{ob}(\mathcal{D})$, a composition morphism

$$\circ : \mathcal{D}(x, y)_\bullet \otimes_k \mathcal{D}(y, z)_\bullet \to \mathcal{D}(x, z).$$

  (i.e. a collection of bilinear maps $\circ : \mathcal{D}(x, y)_n \times \mathcal{D}(y, z)_m \to \mathcal{D}(x, z)_{n+m}$ such that $d(f \circ g) = df \circ g + (-1)^m f \circ dg$), such that $(f \circ g) \circ h = f \circ (g \circ h)$ for all composable morphisms.

- For each $x \in \mathrm{ob}(\mathcal{D})$ an element $\mathrm{id}_x \in \mathcal{D}(x, x)_0$ such that for all $f \in \mathcal{D}(y, x)_n$ and $g \in \mathcal{D}(x, z)_m$ $\mathrm{id}_x \circ f = f$ and $g \circ \mathrm{id}_x = g$.

Given a dg-category $\mathcal{D}$, define the *dg-nerve* of $\mathcal{D}$ to be the simplicial set $N_{dg}(\mathcal{D})$ whose n-simplices given by

- A set $\{x_i\}_{0 \le i \le n}$ of objects of $\mathcal{D}$.
- For each ordered subset $I = \{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subset [n]$ with $m \ge 0$, an element $f_I \in \mathcal{D}(x_{i_-}, x_{i_+})_m$, such that

$$df_I = \sum_{1 \le j \le m} (-1)^j \left(f_{I \setminus \{i_j\}} - f_{i_j < i_{j-1} < \cdots < i_+} \circ f_{i_- < i_m < \cdots < i_j} \right)$$

1. Let $\mathcal{D}$ be a dg-category. Given a map $\alpha : [m] \to [m]$ in Δ and $(\{x_i\}_{0 \le i \le n}, \{f_I\}) \in N_{dg}(\mathcal{D})_n$, define $\alpha^*(\{x_i\}_{0 \le i \le n}, \{f_I\})$ to be the object $(\{x_{\alpha(j)}\}, \{g_J\})$ where

$$g_J = \begin{cases} f_{\alpha(J)} & \alpha|_J \text{ injective} \\ \mathrm{id}_{x_i} & J = \{j, j'\} \text{ with } \alpha(j) = \alpha(j') = i \\ 0 & \text{otherwise.} \end{cases}$$

   Show that the maps α^* equip $N_{dg}(\mathcal{D})$ with the structure of a simplicial set.

2. Show that $N_{dg}(\mathcal{D})$ is an ∞-category.