Exercise sheet 1

Problem 1. By construction, categories that arise from posets have the following property: between any given pair of objects, there is at most one morphism. Does every category with this property arise from a poset? What additional properties are needed to characterize those categories that arise from posets?

Problem 2. Let $Y \subset \mathbb{R}^3$ be a simplicial complex, and define $\Delta(Y)$ to be the poset of subsimplices of Y ordered by inclusion.

1. For $n = 0, 1, 2, 3$, draw the category $\Delta(Y)$, where Y is the standard n-simplex in $\mathbb{R}^3$.
2. Draw the category $\Delta(Y)$ when Y is your favorite simplicial complex in $\mathbb{R}^3$.

Problem 3. Show that, as a consequence of the axiomatic properties from Definition 1.1.8 in the lecture notes, any universe $\mathcal{U}$ has the following further properties:

1. If $x \in \mathcal{U}$ then $x \subset \mathcal{U}$.
2. If $y \in \mathcal{U}$ and $x \subset y$ then $x \in \mathcal{U}$.
3. $\varnothing \in \mathcal{U}$.
4. If $x, y \in \mathcal{U}$ then $(x, y) := \{x, \{x, y\}\} \in \mathcal{U}$.
5. If $x, y \in \mathcal{U}$ then $x \cup y$ and $x \times y \in \mathcal{U}$.
6. If $x, y \in \mathcal{U}$ then $\mathrm{Map}(x, y) \in \mathcal{U}$.
7. If $I \in \mathcal{U}$ and $\{x_i\}_{i \in I}$ is a family of elements $x_i \in \mathcal{U}$ then $\prod_{i \in I} x_i$, $\coprod_{i \in I} x_i$, and $\bigcap_{i \in I} x_i$ are elements of $\mathcal{U}$.
8. If $x \in \mathcal{U}$ then $x \cup \{x\} \in \mathcal{U}$.
9. $\mathbb{N} \subset \mathcal{U}$.

Problem 4. We make the following recursive definition: A set X is called *hereditarily finite* if X is finite and all subsets of X are hereditarily finite. More explicitly, define $V_0 = \varnothing$, $V_1 = \mathcal{P}(V_0)$, $V_2 = \mathcal{P}(V_1)$, $\dots$ so that $V_0 \subset V_1 \subset V_2 \dots$ and set

$$V_\omega := \bigcup_{k \geq 0} V_k.$$

Then the elements of the set V_ω are precisely the hereditarily finite sets. Show that V_ω is a universe.

Problem 5. Let $\mathcal{U}$ be an infinite universe. Show that $\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$, and $\mathbb{C}$ are elements of $\mathcal{U}$.