Exercise Sheet 9

Problem 1. Properties of tensor products. Let K be a field, let V,W be K-vector spaces, and let $V \otimes_K W$ denote their tensor product, considered as a K-vector space.

1. Formulate the universal property of the map $u : V \times W \to V \otimes W, (v, w) \mapsto v \otimes w$.
2. Let $v_1, v_2 \in V \setminus \{0\}$ and $w_1, w_2 \in W \setminus \{0\}$. Show that $v_1 \otimes w_1 = v_2 \otimes w_2$ implies that there exists an element $a \in K$ such that $v_1 = av_2$ and $w_2 = aw_1$.
3. Show that, if $\{v_i\}_{i \in I}$ is a basis of V and $\{w_j\}_{j \in J}$ is a basis of W then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.
4. Let $V_1 \subset V_2$ inclusion of vector spaces, show that there is an isomorphism

$$(V_2 \otimes W)/(V_1 \otimes W) \cong (V_2/V_1) \otimes W$$

of vector spaces.

Problem 2. Show that every linear algebraic group G is isomorphic to the Galois group of a Picard-Vessiot extension E/F.

Problem 3. Let $A \in \mathrm{GL}(n, K)$ be a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$ and let $G = \langle A \rangle$ denote the subgroup generated by A.

1. Show that, in general, we have $G \neq \overline{G}$.
2. Show that G consists of those diagonal matrices with diagonal entries $d_1, \dots, d_n$ satisfying: if $(m_1, \dots, m_n) \in \mathbb{Z}^n$ such that $\lambda_1^{m_1} \dots \lambda_n^{m_n} = 1$ then $d_1^{m_1} \dots d_n^{m_n} = 1$.

Problem 4. A *character* of a linear algebraic group G is a homomorphism $\chi : G \longrightarrow \mathrm{GL}(1, K)$ of linear algebraic groups.

1. Show that the set of characters of G forms a group with multiplication $(\chi\chi')(g) = \chi(g)\chi'(g)$.
2. Show that a character of G is uniquely determined by the element $a_\chi = \chi^*(x \in \mathcal{O}(G))$ where $\mathcal{O}(\mathrm{GL}(1, K)) = K[x, x^{-1}]$.
3. Show that the conditions for an element $a \in \mathcal{O}(G)$ to be of the form a_χ for a character χ are that a is invertible and $m^*(a) = a \otimes a$ where $m : G \times G \to G$ is the multiplication morphism.
4. Let $T \subset \mathrm{GL}(n, K)$ be the group of diagonal matrices. Show that the group of characters of T is isomorphic to $\mathbb{Z}^n$.
5. What is the group of characters of $\mathrm{GL}(n, K)$?