Exercise Sheet 8

This problem set has fewer problems to leave sufficient time for questions + discussions.

Problem 1. Let R/F be a Picard-Vessiot ring and let $G = \mathrm{Gal}^{\partial}(R/F)$ be the ∂-Galois group, considered as a linear algebraic group, and let X denote the affine F-scheme corresponding to R.

1. Let L be an F-algebra and let $p \in X(L)$ be an L-valued point. Show that there is an isomorphism of L-algebras
$$X_L \cong G_L$$
such that the induced map of L-valued points maps p to the identity element $e \in G_L(L)$.

2. Let $\overline{F}$ denote an algebraic closure of F. Show that there exists $p \in X_{\overline{F}}(\overline{F})$. *Hint.* Use Hilbert's Nullstellensatz.

Problem 2. Let K be a field and let L/K be a finite Galois extension. Let X be the affine K-scheme corresponding to the K-algebra L.

1. Show that we may interpret the Galois group $G = \mathrm{Gal}(L/K)$ as the group of K-valued points of an affine K-group scheme with corresponding K-algebra
$$\prod_G K.$$

2. Show that X admits the structure of a G-torsor in the category of affine K-schemes.

3. Show that there is an isomorphism $X_L \cong G_L$ of affine L-schemes. Could it be that there is an isomorphism $X \cong G$ of affine K-schemes?