Exercise Sheet 6

Problem 1. Let K be an algebraically closed field and let V be an affine K-variety. For a commutative K-algebra R, denote by $V(R)$ the set of R-valued points of V.

1. Show that the association

$$V(-) : \mathbf{Calg}_K \longrightarrow \mathbf{Set}, R \mapsto V(R)$$

   extends to a functor.

2. Show that a morphism $f : V \to W$ of affine K-varieties defines a natural transformation f_* from $V(-)$ to $W(-)$.

3. Show that, for every natural transformation η from $V(-)$ to $W(-)$, there exists a unique morphism $f : V \to W$ such that $\eta = f_*$.

Problem 2. Let K be an algebraically closed field. For each of the following affine K-varieties compute the dimensions of all tangent spaces.

1. $\{p\} \subset K^n$ for $p \in K^n$.
2. $\{(t, t^2, t^3) | t \in K\} \subset K^3$.
3. $\{(t^2, t^3) | t \in K\} \subset K^2$.

Problem 3. Let K be an algebraically closed field and $n \geq 1$. Show that each of the following groups is a linear algebraic group, determine the corresponding Lie algebra and its dimension as a K-vector space.

1. A finite subgroup

$$G \subset \mathrm{GL}(n, K).$$

2. The orthogonal group

$$\mathrm{O}(n, K) = \{X \in \mathrm{GL}(n, K) \mid X^{\mathrm{tr}} X = I_n\} \subset \mathrm{GL}(n, K).$$

3. The symplectic group

$$\mathrm{Sp}(2n, K) = \{X \in \mathrm{GL}(2n, K) \mid X^{\mathrm{tr}} \Omega X = \Omega\} \subset \mathrm{GL}(2n, K)$$

   where

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Problem 4. Let K be an algebraically closed field.

1. Let $\varphi : V \to W$ be a morphism of affine K-varieties. Show that φ is continuous with respect to the Zariski topology.

2. Let $\varphi : G \to G'$ be a morphism of linear algebraic groups, i.e., a morphism of affine K-varieties which is also a group homomorphism. Show that $\ker(\varphi) \subset G$ is a Zariski-closed subgroup.

3. Let G be a linear algebraic group over K and let $H \subset G$ be a subgroup. Denote by $\overline{H}$ the closure of H in G with respect to the Zariski topology on G. Show that $\overline{H} \subset G$ is a subgroup (and hence a linear algebraic group).

4. Recall that a topological space is called connected if it cannot be expressed as the union of two disjoint non-empty open subsets. Every topological space admits a partition into its connected components which are the maximal connected subspaces. For a linear algebraic group G, let G^0 denote the connected component (with respect to the Zariski topology) which contains the identity element $e \in G$. Show that $G^0 \subset G$ is a Zariski closed normal subgroup.