Homogeneous Quaternionic Kähler Manifolds of Unimodular Group

D.V. Alekseevsky*
Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-53225 Bonn

V. Cortés†
Mathematisches Institut
der Universität Bonn
Beringstr. 6
D-53115 Bonn

To the memory of Franco Tricerri

1 Introduction

A **quaternionic structure** on a vector space V^{4n} is a 3-dimensional linear Lie algebra $\mathfrak{q} \subset End(V)$ with a basis J_1, J_2, J_3 satisfying the quaternionic relations

$$J_\alpha^2 = -1\,, \quad J_\alpha J_\beta = -J_\beta J_\alpha = J_\gamma\,.$$

Here (α, β, γ) is a cyclic permutation of $(1, 2, 3)$. The basis $(J_\alpha)_\alpha$ is called a **standard basis** of $\mathfrak{q}$. A quaternionic Kähler manifold is a Riemannian manifold (M^{4n}, g) together with a field of quaternionic structures $\mathfrak{q} : x \mapsto \mathfrak{q}_x \subset \mathfrak{so}(T_x M)$ such that:

1) $\mathfrak{q}$ is parallel with respect to the Levi-Civita connection.

2) The curvature tensor R_x, $x \in M$, of the metric g is invariant under the natural action of $\mathfrak{q}_x$.

It is known that 1) implies 2) if $n > 1$ and that any quaternionic Kähler manifold is Einstein.

The main result of the paper is the following theorem.

*e-mail: daleksee@mpim-bonn.mpg.de; partially supported by Max-Planck-Institut für Mathematik (Bonn).

†Fax: +49-228-737916; e-mail: V.Cortes@uni-bonn.de or vicente@rhein.iam.uni-bonn.de; partially supported by SFB 256 (Bonn University).

Theorem 1.1 *Let M be a quaternionic Kähler manifold admitting a transitive unimodular group G of isometries. Then either M is flat and hence is the Riemannian product of a torus and an Euclidean space or it is a quaternionic Kähler symmetric space G/H, where G is a simple Lie group and H is the normalizer of a regular 3-dimensional subgroup G_α associated with a long root α.*

The proof of the theorem reduces to the case of negative scalar curvature $s < 0$ and semisimple Lie group G . Indeed, if $s > 0$ the manifold M is compact and in this case the theorem was proved in [A]. In the case $s = 0$, the Ricci curvature $Ric = 0$ and the result follows from the fact that any Ricci-flat homogeneous Riemannian manifold is flat [A-K]. Hence, we may assume that $s < 0$ and hence $Ric < 0$.

The following result of I. Dotti Miatello shows that the group G is semisimple.

Theorem 1.2 *[Do] Let M be a Riemannian manifold admitting a transitive unimodular group G of isometries. If $Ric < 0$ then the group G is semisimple.*

To prove the main theorem we need some basic facts concerning homogeneous quaternionic Kähler manifolds.

2 Basic facts about homogeneous quaternionic Kähler manifolds

2.1. Let M be a quaternionic Kähler manifold which admits a transitive group G of isometries. Then we identify $M = G/H$, where H is the stabilizer of a point. We will say that $M = G/H$ is a homogeneous quaternionic Kähler manifold. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition, where $\mathfrak{g} = Lie\, G$, $\mathfrak{h} = Lie\, H$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We identify $\mathfrak{m} \cong T_H M$ and denote by $< \cdot, \cdot >$ the Ad_H-invariant scalar product on $\mathfrak{m}$ induced by the Riemannian metric on M. For any $a \in \mathfrak{g}$ we define a skew-symmetric endomorphism L_a (Nomizu operator) on $\mathfrak{m}$ by the formula

$$2 < L_a x, y > = < \pi[a, x], y > - < x, \pi[a, y] > - < \pi a, \pi[x, y] > ,$$

$x, y \in \mathfrak{m}$, where $\pi : \mathfrak{g} \to \mathfrak{m}$ is the natural projection.

Remark that for $a \in \mathfrak{h}$ the Nomizu operator $L_a = ad_a|\mathfrak{m}$ is exactly the isotropy operator. The following proposition is known.

Proposition 2.1 *[A] A homogeneous Riemannian manifold $M^{4n} = G/H$ ($n > 1$) is quaternionic Kähler iff the Nomizu operators belong to the normalizer $\mathfrak{n}(\mathfrak{q}) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ in $\mathfrak{so}(\mathfrak{m})$ of some quaternionic structure $\mathfrak{q} = span\{J_1, J_2, J_3\}$ on $\mathfrak{m}$.*

2.2. Structure equations. Let $M = G/H$ be a homogeneous quaternionic Kähler manifold. We will always assume that the group G is connected and semisimple. Then the Cartan-Killing form B of $\mathfrak{g}$ is non degenerate on $\mathfrak{g}$ and $\mathfrak{h}$ and we fix the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the B-orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. Let J_α, $\alpha = 1, 2, 3$, be a standard basis of the corresponding quaternionic structure on $\mathfrak{m}$. Then for any $a \in \mathfrak{g}$ we can write

$$L_a = \sum_{\alpha=1}^{3} \omega_\alpha(a) J_\alpha + \bar{L}_a \,,$$

where $\bar{L}_a$ belongs to the centralizer $\mathfrak{z}(\mathfrak{q}) \cong \mathfrak{sp}(n)$ of $\mathfrak{q}$ in $\mathfrak{so}(\mathfrak{m})$ and the 1-forms ω_α satisfy the following structure equations

$$\nu \pi^* \rho_\alpha = d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma \,. \tag{1}$$

Here $\rho_\alpha = < \cdot, J_\alpha \cdot >$ is the Hermitian form associated with the complex structure J_α; (α, β, γ) is a cyclic permutation of $(1, 2, 3)$ and $\nu = s/4n(n+2)$ is the reduced scalar curvature, see [A].

We denote by Ω the Kraines 4-form on $\mathfrak{m}$, given by

$$\Omega = \sum_{\alpha=1}^{3} \rho_\alpha \wedge \rho_\alpha \,.$$

It is $L_{\mathfrak{g}}$-invariant and defines a parallel 4-form on M (the Kraines form of M). The 4-form $\pi^*\Omega$ on $\mathfrak{g}$ is exact:

$$\pi^*\Omega = d\psi,$$

$$\psi = \sum_{\alpha=1}^{3} \omega_\alpha \wedge d\omega_\alpha + 4\omega_1 \wedge \omega_2 \wedge \omega_3 \,.$$

Denote by $\bar{\mathfrak{h}}$ the kernel of the homomorphism

$$\phi : \mathfrak{h} \to \mathfrak{q}, \quad h \mapsto L_h - \bar{L}_h = \sum_{\alpha=1}^{3} \omega_\alpha(h) J_\alpha$$

and by $\mathfrak{a}$ the orthogonal complement of $\bar{\mathfrak{h}}$ in $\mathfrak{h}$ with respect to the Cartan-Killing form B. Since $\phi : \mathfrak{a} \hookrightarrow \mathfrak{q} \cong \mathfrak{sp}(1)$ is an embedding, $d = \dim \mathfrak{a} = 0, 1$ or 3. We will say that the homogeneous quaternionic Kähler manifold $M = G/H$ is of type 1, 2 or 3, if $d = 0$, 1 or 3 respectively. Passing to the universal covering, if needed, we may assume that M is simply connected and hence that H is connected.

3 Proof of the theorem for manifolds of type 1 and 2

3.1. Type 1 We assume now that $\mathfrak{a} = 0$. Then $\omega_\alpha(\mathfrak{h}) = 0$, $\alpha = 1, 2, 3$, and the structure equations show that the 1-forms ω_α are invariant under the isotropy representation of the Lie algebra $\mathfrak{h}$ and hence of the Lie group H, since H is connected. This implies that ψ defines some invariant form on M whose differential is the Kraines form Ω on M. In particular, the volume form Ω^n is the differential of some invariant form. This contradicts the following result of Koszul [Ko], [Ha].

Theorem 3.1 *Let $M = G/H$ be an orientable Riemannian homogeneous space of a connected unimodular Lie group G. Then the Riemannian volume form is not cohomological to zero in the complex of invariant differential forms.*

3.2. Totally geodesic Kähler and quaternionic Kähler submanifolds

Definition 3.1 *Let $(M, g, \mathfrak{q})$ be a quaternionic Kähler manifold.*

1) *A submanifold N of M is called a* **Kähler submanifold** *if there exists a section J of the quaternionic structure $\mathfrak{q}$ along N such that $(N, g|N, J)$ is a Kähler manifold, i.e. J is a parallel complex structure on N.*

2) *A submanifold N of M is called a* **quaternionic Kähler submanifold** *if $\mathfrak{q}_x T_x N \subset T_x N$ for any $x \in N$.*

Recall that any quaternionic Kähler submanifold N of a quaternionic Kähler manifold $(M, g, \mathfrak{q})$ is totally geodesic with the same reduced scalar curvature, in particular, $(N, g|N, \mathfrak{q}|N)$ is a quaternionic Kähler manifold.

Let $M = G/H$ be a homogeneous quaternionic Kähler manifold and

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = \mathfrak{a} + \bar{\mathfrak{h}} + \mathfrak{m}$$

be the corresponding reductive decomposition as before. Denote by $Z_G^0(b)$ the connected component of the centralizer of an element $b \in \mathfrak{h}$ in G.

Proposition 3.2 *Let $M = G/H$ be a homogeneous quaternionic Kähler manifold of type k.*

1) *For any $b \in \bar{\mathfrak{h}} \subset \mathfrak{h} = \mathfrak{a} + \bar{\mathfrak{h}}$ the orbit $N = Z_G^0(b)o$ of the point $o = eH$ is a quaternionic Kähler submanifold of the same type k or a point.*

2) For any $a \in \mathfrak{a} - \{0\}$ the orbit $N = Z_G^0(a)o$ is a totally geodesic Kähler submanifold or a point.

3) Assume $k = 2$. Then for any $b \in \mathfrak{h} \setminus \bar{\mathfrak{h}}$ the orbit $N = Z_G^0(b)o$ is a totally geodesic Kähler submanifold or a point.

Proof. It is known (see e.g. [A], Assertion 4) that the orbit $N = Z_G^0(b)o$ of the centralizer of any element $b \in \mathfrak{h}$ in a homogeneous Riemannian manifold $M = G/H$ is totally geodesic. In the case 1), the reductive decomposition of the Lie algebra $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(b)$ corresponding to N can be written as

$$\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{z}_{\bar{\mathfrak{h}}}(b) + \mathfrak{n}\,, \qquad \mathfrak{n} = \mathfrak{z}_{\mathfrak{m}}(b).$$

Since $L_b \in \mathfrak{z}(\mathfrak{q}) \cong \mathfrak{sp}(n)$, the subspace $\mathfrak{n}$ is quaternionic, i.e. $\mathfrak{q}\mathfrak{n} \subset \mathfrak{n}$. Now it is immediate to check that N is a homogeneous quaternionic Kähler manifold of type k, using the trivial fact that the image of $b \in \mathfrak{h} \cap \mathfrak{g}_0$ under the isotropy representation on $\mathfrak{n} \cong T_oN$ equals $ad_b|\mathfrak{n} = L_b|n = \sum_{\alpha=1}^{3} \omega_\alpha(b) J_\alpha|\mathfrak{n} + \bar{L}_b|\mathfrak{n}$.

In the case 3), the reductive decomposition of $\mathfrak{g}_0$ reads:

$$\mathfrak{g}_0 = \mathbb{R}a + \mathfrak{z}_{\bar{\mathfrak{h}}}(\bar{b}) + \mathfrak{n}\,, \qquad \mathfrak{n} = \mathfrak{z}_{\mathfrak{m}}(b),$$

where $b = a \oplus \bar{b} \in \mathfrak{a} \oplus \bar{\mathfrak{h}}$. Without restriction of generality we can choose a standard base $(J_\alpha)_\alpha$ of $\mathfrak{q}$ such that $L_b = J_1 + \bar{L}_b$, $\bar{L}_b \in \mathfrak{z}(\mathfrak{q})$. Since $[L_b, J_1] = 0$, $\mathfrak{n}$ is a J_1-invariant subspace of $\mathfrak{m}$. The structure equations (1) show that $\omega_2|\mathfrak{n} = \omega_3|\mathfrak{n} = 0$, e.g.

$$0 = \omega_2([b, x]) = 0 + 2(0 - \omega_3(x) \cdot 1) = -2\omega_3(x), \quad x \in \mathfrak{n}.$$

This shows that $[L_x, J_1] = 0$ for all $x \in \mathfrak{g}_0$. Since the Lie algebra generated by the Nomizu operators contains the holonomy algebra, this implies that J_1 defines an invariant parallel complex structure on N and hence N is a Kähler submanifold.

In the case 2), $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(a)$ has the reductive decomposition

$$\mathfrak{g}_0 = \mathbb{R}a + \bar{\mathfrak{h}} + \mathfrak{n}\,, \qquad \mathfrak{n} = \mathfrak{z}_{\mathfrak{m}}(a)$$

and the proof is the same as for the case 3). □

Remark that in the cases 2) and 3) the N is a totally complex manifold in the sense of Tsukada [T].

3.3. Invariant symplectic structure on quaternionic Kähler manifolds of type 2 Now we consider the case when $\dim \mathfrak{a} = 1$. Choosing an appropriate standard basis $(J_\alpha)_\alpha$ we may assume $\mathfrak{a} = \mathbb{R}a$, $B(a, a) = -1$ and

$L_a = J_1 + \bar{L}_a$. The reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{g}$ induces a decomposition $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{m}^*$ of the dual space. For any k-form $\sigma \in \wedge^k \mathfrak{g}^*$ we denote by σ^{pq}, $(p+q=k)$ the natural projection onto

$$\wedge^{pq} := \wedge^p \mathfrak{h}^* \otimes \wedge^q \mathfrak{m}^* \,.$$

If σ is Ad_H-invariant, σ^{pq} is also Ad_H-invariant and, in particular, σ^{0q} is an Ad_H-invariant k-form on $\mathfrak{m}$ and hence defines an invariant form on M. The 1-forms ω_α associated to the basis $(J_\alpha)_\alpha$ have the following properties:

$$\omega_1 = \omega_1^{10} + \omega_1^{01} \quad \text{is } Ad_H\text{-invariant and} \quad \omega_1^{10} = -B(a, \cdot) \neq 0 \,,$$

$$\omega_2 = \omega_2^{01} \quad \text{and} \quad \omega_3 = \omega_3^{01} \,.$$

Lemma 3.3 *1) The 2-form $d\omega_1^{10}(x,y) = B(a,[x,y])$ belongs to $\wedge^{02}$, is Ad_H-invariant and hence defines an invariant 2-form σ on M.*

2) The forms $\omega_2 \wedge \omega_3$, $\omega_2 \wedge d\omega_2 + \omega_3 \wedge d\omega_3$ and ψ are Ad_H-invariant.

3) The Kraines form Ω on M is cohomological to $\sigma \wedge \sigma$.

Proof. The form $d\omega_1^{10}$ is Ad_H-invariant, since ω_1 is Ad_H-invariant. Let $h \in \mathfrak{h}$, $x \in \mathfrak{m}$, then $d\omega_1^{10}(h,x) = -\omega_1^{10}([h,x]) = 0$, since $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$. Hence $(d\omega_1^{10})^{11} = 0$. The component $(d\omega_1^{10})^{20} = 0$, because $[\mathfrak{h},\mathfrak{h}] \subset \bar{\mathfrak{h}} = ker\, \omega_1$. This proves 1).

2) The structure equations (1) imply

$$\begin{aligned} ad_h \omega_2 &= 2\omega_1(h)\omega_3 \,, \\ ad_h \omega_3 &= -2\omega_1(h)\omega_2 \end{aligned}$$

for $h \in \mathfrak{h}$. From this 2) immmediately follows.

3) From the structure equations we obtain the following equalities:

$$\begin{aligned} d\omega_1 &= d\omega_1^{02} = \pi^*\rho_1 - 2\omega_2 \wedge \omega_3 \,, \\ d\omega_2 &= d\omega_2^{02} + d\omega_2^{11} \,, \\ d\omega_3 &= d\omega_3^{02} + d\omega_3^{11} \,, \\ d\omega_2^{02} &= \pi^*\rho_2 - 2\omega_3 \wedge \omega_1^{01} \,, \\ d\omega_3^{02} &= \pi^*\rho_3 - 2\omega_1^{01} \wedge \omega_2 \,, \\ d\omega_2^{11} &= -2\omega_3 \wedge \omega_1^{10} \,, \\ d\omega_3^{11} &= -2\omega_1^{10} \wedge \omega_2 \,. \end{aligned}$$

Using this we obtain

$$\psi = \psi^{03} + \psi^{12} \,.$$

Moreover we compute

$$\begin{aligned}
\psi^{12} &= \omega_1^{10} \wedge d\omega_1 + \omega_2 \wedge d\omega_2^{11} + \omega_3 \wedge d\omega_3^{11} + 4\omega_1^{10} \wedge \omega_2 \wedge \omega_3 \\
&= \omega_1^{10} \wedge d\omega_1 = \omega_1^{10} \wedge d\omega_1^{10} + \omega_1^{10} \wedge d\omega_1^{01}\,, \\
\psi^{03} &= \omega_1^{01} \wedge d\omega_1 + \omega_2 \wedge d\omega_2^{02} + \omega_3 \wedge d\omega_3^{02} + 4\omega_1^{01} \wedge \omega_2 \wedge \omega_3 \\
&= \omega_1^{01} \wedge d\omega_1 + \omega_2 \wedge \pi^*\rho_2 + \omega_3 \wedge \pi^*\rho_3\,.
\end{aligned}$$

Using these formulas we have

$$\begin{aligned}
\Omega &= d\psi = d\psi^{12} + d\psi^{03} \\
&= d(\omega_1^{10} \wedge d\omega_1^{10} + \omega_1^{10} \wedge d\omega_1^{01}) + d\psi^{03} \\
&= d\omega_1^{10} \wedge d\omega_1^{10} + d(d\omega_1^{10} \wedge \omega_1^{01} + \psi^{03})\,.
\end{aligned}$$

According to 1), 2) $d\omega_1^{10} \wedge \omega_1^{01} + \psi^{03} \in \wedge^{03}$ is Ad_H-invariant and hence defines an invariant 3-form τ on M. Hence, on the manifold M

$$\Omega = \sigma \wedge \sigma + d\tau\,. \quad \Box$$

As a corollary we obtain

Proposition 3.4 *σ is an invariant symplectic form on M and $M = G/H$ is identified with the universal covering $G/Z_G^0(a)$ of the adjoint orbit $Ad_G a = G/Z_G(a)$. Moreover, the group G is simple.*

Proof. It is clear that the form σ is closed and invariant. Moreover, the form σ^{2n} is cohomological to Ω^n. Since Ω^n is not cohomological to zero by Koszul's theorem, the invariant form $\sigma^{2n} \neq 0$. Hence, σ is non-degenerate, that is σ is a symplectic form. The second statement follows now from the Kirillov-Kostant description of homogeneous symplectic manifolds. Suppose now that the semisimple group G is not simple. Without restriction of generality we may assume that $G = G_1 \times G_2$. Then the homogeneous manifold G/H is G-isomorphic to the direct product $G_1/H_1 \times G_2/H_2$ of homogeneous manifolds, where $H = Z_G^0(a) = H_1 \times H_2$. Any invariant metric on such a manifold is reducible. On the other hand, it is known that a quaternionic Kähler metric of non zero scalar curvature is irreducible. This contradiction shows that the group G is simple. $\Box$

3.4. Type 2 The proof of the theorem for type 2 manifolds is based on the following two lemmas.

Lemma 3.5 *Assume that G/H is a quaternionic Kähler manifold of type 2 and $\mathrm{rk}\,\mathfrak{g} > 2$. Then there exists $h \in \bar{\mathfrak{h}}$ such that $\mathfrak{z}_{\mathfrak{g}}(h)$ is non-compact.*

Proof. Consider the root system $\mathcal{R}$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, where $\mathfrak{t} = \mathbb{R}a + \bar{\mathfrak{t}}$, $\bar{\mathfrak{t}} \subset \bar{\mathfrak{h}}$, is a compact Cartan subalgebra of $\mathfrak{h}$ and hence of $\mathfrak{g}$. Any root $\alpha \in \mathcal{R}$ generates a 3-dimensional subalgebra $\mathfrak{g}(\alpha) = span_{\mathbb{C}}\{h_\alpha, e_\alpha, e_{-\alpha}\} \cap \mathfrak{g}$, which is isomorphic to $\mathfrak{su}(2)$ or to $\mathfrak{sl}(2, \mathbb{R})$. The root α is called **compact** respectively **non-compact**, if $\mathfrak{g}(\alpha) \cong \mathfrak{su}(2)$ respectively $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{R})$. If $\mathfrak{g}$ is non-compact, then there exists a non-compact root β, s. [He]. Choose $0 \neq h \in \bar{\mathfrak{t}} \cap ker\beta$. Then $\mathfrak{z}_{\mathfrak{g}}(h) \supset \mathfrak{g}(\beta) \cong \mathfrak{sl}(2, \mathbb{R})$. □

Lemma 3.6 *Let $M = G/H$ be a homogeneous manifold, where G is a real simple Lie group of rank 2 and H a compact subgroup of the form $H = Z^0_G(a)$, $a \in \mathfrak{h}$. Assume that the isotropy representation of H preserves a quaternionic structure on $\mathfrak{m} \cong T_H M$. Then $G/H = SU(3)/U(2) \cong \mathbb{C}P^2$ or $= SU(1,2)/U(2) \cong \mathbb{C}H^2$.*

Proof. According to the theory of semisimple Lie algebras $\mathfrak{g}$ is of type A_2, B_2 or G_2 and $\mathfrak{h}$ is isomorphic to $\mathfrak{t}^2$ or to $\mathfrak{t}^1 \oplus \mathfrak{su}(2)$, where $\mathfrak{t}^n$ denotes the Lie algebra of the n-dimensional torus. Assume that the isotropy representation of M preserves some quaternionic structure. Then $\dim G/H \equiv 0\ (4)$ and $(\mathfrak{g}, \mathfrak{h})$ can only be of type $(A_2, \mathfrak{t}^1 \oplus \mathfrak{su}(2))$, $(B_2, \mathfrak{t}^2)$ or $(G_2, \mathfrak{t}^2)$. Checking the real Lie algebras of Type A_2, we conclude that the first pair gives exactly the two manifolds G/H described in Lemma 3.6. Let now $\mathfrak{g}$ be a real simple Lie algebra of type B_2 or G_2 with a compact Cartan subalgebra $\mathfrak{t} = \mathfrak{t}^2$. To prove the lemma, it is sufficient to check that the isotropy representation $ad_{\mathfrak{t}}|\mathfrak{m}$ of $\mathfrak{t}$ on $\mathfrak{m} = [\mathfrak{t}, \mathfrak{g}]$ does not preserve any quaternionic structure $\mathfrak{q}$. Suppose that such a quaternionic structure $\mathfrak{q}$ exists. Then

$$ad_{\mathfrak{t}}|\mathfrak{m} \subset \mathfrak{n}_{\mathfrak{so}(\mathfrak{m})}(\mathfrak{q}) = \mathfrak{sp}(1) \oplus \mathfrak{gl}(n, \mathbb{H}),$$

where $n = 2$ (resp. 3) if $\mathfrak{g}$ has type B_2 (resp. G_2). There exists an element $0 \neq b \in \mathfrak{t}$ such that $A = ab_b|\mathfrak{m} \in \mathfrak{gl}(n, \mathbb{H})$. Since for any $A \in \mathfrak{gl}(n, \mathbb{H})$ the multiplicity of an eigenvalue of A is even, the root system $\mathcal{R}$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ must satisfy the following condition for any $\alpha \in \mathcal{R}$:

$$\#\{\beta \in R |\ \beta(b) = \alpha(b)\} \equiv 0\ (2).$$

From the picture of the root systems of type B_2 and G_2 one sees that this is impossible. □

Now we prove that there is no homogeneous quaternonic Kähler manifold $M = G/H$ of type 2 with an unimodular group G. By Prop. 3.4 we may assume that G is simple. We will use induction on the rank of G. First we remark that there is no quaternionic Kähler manifold $M = G/H$ of type 2 and $\operatorname{rk} G \leq 2$. Indeed, if $\operatorname{rk} G = 1$, then $\dim G = 3$. If $\operatorname{rk} G = 2$, the only quaternionic Kähler manifolds are the symmetric manifold $SU(3)/U(2)$ and its non-compact dual, which are not of type 2. Applying induction, we assume that there is no quaternionic Kähler manifold G/H of type 2 and $\mathrm{rk}\, G < k$. Let now $M = G/H$ be a quaternionic Kähler manifold of type 2 with an unimodular and hence simple group G of $\mathrm{rk}\, G = k$. Let $\mathfrak{g} = (\mathbb{R}a + \bar{\mathfrak{h}}) + \mathfrak{m}$ be the corresponding reductive decomposition. We may assume that $\mathrm{rk}\, \mathfrak{g} > 2$ and hence $\bar{\mathfrak{h}} \neq 0$. By Lemma 3.5 there exists $b \in \bar{\mathfrak{h}}$ with non-compact centralizer $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(b)$. Remark that $\mathfrak{g}_0$ is a reductive and hence unimodular Lie algebra and $\mathfrak{g} \neq \mathfrak{g}_0 \not\subset \mathfrak{h}$. According to Prop. 3.2 1) the orbit N of the corresponding connected Lie group $Z^0_G(b)$ is a quaternionic Kähler submanifold of type 2. The corresponding reductive and hence unimodular isometry group G_N of $(N, g|N)$ is the quotient of $Z^0_G(b)$ by the kernel of non-effectivity, which contains $\{\exp tb|\ t \in \mathbb{R}\}$. Hence, $\mathrm{rk}\, G_N < \mathrm{rk}\, Z^0_G(b) = \mathrm{rk}\, G = k$. This contradicts the inductive assumption. □

4 Proof of the theorem for type 3 manifolds

Now we consider a homogeneous quaternionic Kähler manifold $M = G/H$ of type 3 with semisimple Lie group G. We will consider the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{h}$ with respect to the Cartan-Killing form B. Moreover, $\mathfrak{h} = \mathfrak{a} + \bar{\mathfrak{h}}$, where $\bar{\mathfrak{h}}$ is the kernel of the homomorphism $\phi : \mathfrak{h} \to \mathfrak{q} \cong \mathfrak{sp}(1)$ and $\mathfrak{a}$ is the B-orthogonal complementary ideal to $\bar{\mathfrak{h}}$ in $\mathfrak{h}$, s. 2.2. With respect to a standard basis $(J_\alpha)_\alpha$ of $\mathfrak{q}$ the isomorphism $\phi|\mathfrak{a} : \mathfrak{a} \xrightarrow{\sim} \mathfrak{q} \cong \mathfrak{sp}(1)$ is given by $\phi(h) = \sum_{\alpha=1}^{3} \omega_\alpha(h) J_\alpha$, in particular, the forms $\omega_\alpha|\mathfrak{a}$ are linearly independent.

Proposition 4.1 *For any $a \in \mathfrak{a} - \{0\}$, $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(a) \subset \mathfrak{h}$.*

Proof. Without restriction of generality we may assume that $\omega_1(a) = 1$, $\omega_2(a) = \omega_3(a) = 0$. According to Prop. 3.2 2)

$$\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(a) = \mathfrak{h}_0 + \mathfrak{n} = \mathbb{R}a + \bar{\mathfrak{h}} + \mathfrak{n}$$

defines a totally geodesic Kähler submanifold and $\omega_2|\mathfrak{g}_0 = \omega_3|\mathfrak{g}_0 = 0$. Remark that $\mathfrak{g}_0$ (and any quotient of $\mathfrak{g}_0$) is reductive and hence unimodular. By the structure equations (1) $d\omega_1 = \nu\pi^*\rho_1$ on $\mathfrak{g}_0$. Consider the decomposition of $\omega_1|\mathfrak{g}_0$

$$\omega_1 = \omega_1^{10} + \omega_1^{01} \in \mathfrak{h}_0^* + \mathfrak{n}^*$$

as before. Since ω_1 is $ad_{\mathfrak{h}_0}$-invariant, the 1-form ω_1^{01} is invariant, vanishes on $\mathfrak{h}_0$ and hence defines some invariant form on the homogeneous Kähler manifold $N = G_0/H_0$, where G_0 and H_0 are the connected Lie subgroups of G with Lie algebra $\mathfrak{g}_0$ and $\mathfrak{h}_0$ respectively. ρ_1 defines the Kähler form σ on N and $d\omega_1^{10} = d\omega_1 - d\omega_1^{01}$ defines an invariant form on N, which is cohomological to σ (up to the factor $\nu \neq 0$). Since σ^{2k}, $k = \dim_{\mathbb{C}} N$, is a volume form, the cohomological form $(d\omega_1^{10})^{2k}$ is not zero on N by Koszul's theorem. In other words, $d\omega_1^{10}$ defines an invariant symplectic form on N.

Remark now that the 1-form ω_1^{10} equals

$$\omega_1^{10} = \lambda B(a, \cdot) \in \mathfrak{g}_0^*, \quad \lambda \in \mathbb{R}^-,$$

since $\omega_1^{10}(\bar{\mathfrak{h}} + \mathfrak{n}) = 0$ and $\omega_1^{10}(a) = 1$ and $\bar{\mathfrak{h}} + \mathfrak{n}$ is the orthogonal complement of $\mathbb{R}a$ in $\mathfrak{g}_0$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. This implies $d\omega_1^{10} = 0$ on $\mathfrak{g}_0$:

$$d\omega_1^{10}(x, y) = -\omega_1^{10}([x, y]) = -\lambda B(a, [x, y]) = \lambda B([x, a], y) = 0$$

for $x, y \in \mathfrak{g}_0$. On the other hand we proved that $d\omega_1^{10}$ defines a non-degenerate form on N, hence $N = pt$ and $\mathfrak{g}_0 \subset \mathfrak{h}$. □

Corollary 4.2 *1) For all $a \in \mathfrak{a}$ we have $\mathfrak{z}_{\mathfrak{g}}(a) = \mathbb{R}a + \bar{\mathfrak{h}}$.*

2) $\mathfrak{h} = \mathfrak{a} + \bar{\mathfrak{h}} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$.

3) Any Cartan subalgebra of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and has the form $\mathfrak{t} = \mathbb{R}a + \bar{\mathfrak{t}}$, where $\bar{\mathfrak{t}}$ is a Cartan subalgebra of $\bar{\mathfrak{h}}$.

Proposition 4.3 *1) $\mathfrak{a}$ is a compact regular 3-dimensional subalgebra associated to a long root α of $(\mathfrak{g}, \mathfrak{t})$.*

2) $\mathfrak{g}$ is simple.

Proof. By Cor. 4.2 3) there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ of the form $\mathfrak{t} = \mathbb{R}a + \bar{\mathfrak{t}} \subset \mathfrak{h}$. Obviously it normalizes $\mathfrak{a}$, hence $\mathfrak{a}^{\mathbb{C}}$ is a regular 3-dimensional subalgebra associated with some root α of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Since any 3-dimensional regular subalgebra is contained in some simple ideal and its normalizer contains all other simple ideals, from Cor. 4.2 2) and from the effectivity of G statement 2) follows. It remains only to prove that α is long. It was proved in [A] (s. Lemma 5 2)) that under our assumptions α is long, if $\mathfrak{g}$ is not of type G_2. In the latter case the normalizer $\mathfrak{n}_\alpha$ of the regular 3-dimensional subalgebra associated to (any root) α is of the form $\mathfrak{n}_\alpha^{\mathbb{C}} = \mathfrak{a}_{long}^{\mathbb{C}} + \mathfrak{a}_{short}^{\mathbb{C}}$, where $\mathfrak{a}_{long}$ (resp. $\mathfrak{a}_{short}$) is a regular 3-dimensional subalgebra associated to a long (resp. short) root. Moreover, $(\mathfrak{g}_2/\mathfrak{n}_\alpha)^{\mathbb{C}} \cong \mathbb{C}^4 \otimes \mathbb{C}^2$, where $\mathfrak{a}_{short}^{\mathbb{C}}$ (resp. $\mathfrak{a}_{long}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^4$ (resp. $\mathbb{C}^2$) and trivially on $\mathbb{C}^2$ (resp. $\mathbb{C}^4$). This shows that $\mathfrak{a} = \mathfrak{a}_{short}$ is impossible, hence $\mathfrak{a} = \mathfrak{a}_{long}$. □

The proof of the main theorem follows immediately from the following proposition.

Proposition 4.4 *Let $\mathfrak{a}_\alpha$ be a compact regular 3-dimensional subalgebra associated with a long root α of a simple non-compact real Lie algebra $\mathfrak{g}$. If its normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}_\alpha)$ is compact, then it is maximal compact and hence the corresponding homogeneous space $G/N_G(\mathfrak{a}_\alpha)$ is a non-compact symmetric quaternionic Kähler manifold (dual to a Wolf space).*

The proof of Prop. 4.4 is based on the following lemma.

Lemma 4.5 *Let σ, σ_0 be two involutive automorphisms of a simple complex Lie algebra $\mathfrak{g}$, with fix point sets $\mathfrak{g}^\sigma$, $\mathfrak{g}^{\sigma_0}$. Assume $\mathfrak{g}^{\sigma_0} \subset \mathfrak{g}^\sigma$, then $\sigma = \sigma_0$.*

Proof. Let $\mathfrak{g} = \mathfrak{g}^{\sigma_0} + \mathfrak{g}^{\sigma_0}_-$ and $\mathfrak{g} = \mathfrak{g}^\sigma + \mathfrak{g}^\sigma_-$ denote the corresponding symmetric decompositions. They are orthogonal with respect to the Cartan-Killing form. Moreover, since σ preserves $\mathfrak{g}^{\sigma_0}$, it preserves also the orthogonal complement $\mathfrak{g}^{\sigma_0}_- = \mathfrak{a}_+ + \mathfrak{a}_-$, $\mathfrak{a}_+ = \mathfrak{g}^\sigma \cap \mathfrak{g}^{\sigma_0}_-$, $\mathfrak{a}_- = \mathfrak{g}^\sigma_-$. Then

$$[\mathfrak{a}_+, \mathfrak{a}_-] \subset [\mathfrak{g}^\sigma, \mathfrak{g}^\sigma_-] \subset \mathfrak{g}^\sigma_- \subset \mathfrak{g}^{\sigma_0}_- .$$

On the other hand

$$[\mathfrak{a}_+, \mathfrak{a}_-] \subset [\mathfrak{g}^{\sigma_0}_-, \mathfrak{g}^{\sigma_0}_-] \subset \mathfrak{g}^{\sigma_0} .$$

Hence $[\mathfrak{a}_+, \mathfrak{a}_-] = [\mathfrak{a}_+, \mathfrak{g}^\sigma_-] = 0$. Therefore the kernel $\mathfrak{k}$ of the isotropy representation of $\mathfrak{g}^\sigma$ on $\mathfrak{g}^\sigma_-$, which is an ideal of $\mathfrak{g}$, contains $\mathfrak{a}_+$. Since $\mathfrak{g}$ is simple, $0 = \mathfrak{k} = \mathfrak{a}_+$ and $\sigma = \sigma_0$. □

Corollary 4.6 *Let $\mathfrak{l}$ be a simple complex Lie algebra. There is no inclusion between maximal compact subalgebras of different real forms $\mathfrak{g}$, $\mathfrak{g}' \subset \mathfrak{l}$ of $\mathfrak{l} = \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}'^{\mathbb{C}}$.*

Proof. It is sufficient to consider the Cartan involutions of the real forms and apply the lemma to their complex linear extensions. □

Proof (of Prop. 4.4). Let $\mathfrak{k} \supset \mathfrak{n}_\alpha = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}_\alpha)$ be a maximal compact subalgebra of $\mathfrak{g}$. There exists some real form $\mathfrak{g}'$ of $\mathfrak{l} = \mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{n}_\alpha$ is maximally compact in $\mathfrak{g}'$. This real form corresponds to the non-compact dual of the Wolf space $G_c/N_{G_c}(\mathfrak{a}_\alpha)$, where $Lie\, G_c$ is the compact real form of $\mathfrak{l}$. Cor. 4.6 implies $\mathfrak{k} = \mathfrak{n}_\alpha$. □

References

[A] D.V. Alekseevskiĭ: *Compact quaternion spaces*, Functional Anal. Appl. **2** (1968), 106–114.

[A-K] D.V. Alekseevskiĭ, B.N. Kimel'fel'd: *Structure of homogeneous Riemannian spaces with zero Ricci curvature*, Functional Anal. Appl. **9** (1975), 97–102.

[Do] I. Dotti Miatello: *Transitive group actions and Ricci curvature properties*, Michigan Math. J. **35** (1988), 427–434.

[Ha] J.-I. Hano: *On Kählerian homogeneous spaces of unimodular Lie groups*, Amer. J. Math. **79** (1957), 885–900.

[He] S. Helgason: *Differential geometry, Lie groups and symmetric spaces*, Academic Press, Orlando, 1978.

[Ko] J.L. Koszul: *Homologie et cohomologie des algèbres de Lie*, Bulletin Soc. Math. France vol. 78 (1950), 65–127.

[T] K. Tsukada: *Parallel submanifolds in a quaternion projective space*, Osaka J. Math. **22** (1985), 187–241.