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# Special Geometry of Euclidean Supersymmetry III: the local r-map, instantons and black holes

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ABSTRACT: We define and study projective special para-Kähler manifolds and show that they appear as target manifolds when reducing five-dimensional vector multiplets coupled to supergravity with respect to time. The dimensional reductions with respect to time and space are carried out in a uniform way using an $\epsilon$-complex notation. We explain the relation of our formalism to other formalisms of special geometry used in the literature. In the second part of the paper we investigate instanton solutions and their dimensional lifting to black holes. We show that the instanton action, which can be defined after dualising axions into tensor fields, agrees with the ADM mass of the corresponding black hole. The relation between actions via Wick rotation, Hodge dualisation and analytic continuation of axions is discussed.

KEYWORDS: special geometry, para-complex manifolds, vector multiplets, instantons, black holes.

Contents

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## 1. Introduction

This is the third in a series of papers on the special geometry of Euclidean supersymmetry. The first two papers [1, 2] explored the geometries of rigid vector and hypermultiplets, respectively. This paper is devoted to vector multiplets coupled to Euclidean supergravity. We address three main topics: scalar geometry, dimensional reduction of five-dimensional supergravity, and instanton solutions for vector multiplets in four dimensions.

In the first part of the paper we introduce projective special para-Kähler manifolds as quotients of conical (affine) special para-Kähler manifolds. These will turn out later to be the target geometries of Euclidean vector multiplets coupled to supergravity. Affine special para-Kähler manifolds were introduced in [1], where it was shown that they are precisely the target spaces for rigid Euclidean vector multiplets. A *conical* special para-Kähler manifold is an affine special para-Kähler manifold together with a vector field $\xi$, such that

$$\nabla \xi = D\xi = \mathrm{Id} ,$$

where $D$ is the Levi-Civita connection, and $\nabla$ is the flat special connection. The main result of the first part is Theorem 2, which provides a canonical realisation of (simply connected) conical special para-Kähler manifolds as certain Lagrangian cones. As a corollary we obtain that the geometry of any conical special para-Kähler manifold and, hence, of any projective special Kähler manifold is locally encoded in a para-holomorphic function which is homogenous of degree 2. Throughout the paper we use a notation involving $\epsilon = \pm 1$, which allows to treat the scalar geometries of Euclidean ($\epsilon = +1$) and Minkowskian ($\epsilon = -1$) supergravity in parallel.

In the second part we work out the dimensional reduction of the bosonic part of the Lagrangian of vector multiplets coupled to five-dimensional supergravity. We find that the resulting scalar manifold of the four-dimensional theory is projective special Kähler for reduction over a space-like direction, and projective special *para*-Kähler for reduction over time. The projective special $\epsilon$-Kähler manifolds obtained in this way are not generic, because they are fully captured by the homogenous *cubic* polynomial which defines the five-dimensional theory. In the case $\epsilon = -1$, it is known that *any* choice of a holomorphic prepotential which is homogenous of degree two and gives rise to a non-degenerate metric defines a consistent Minkowskian supergravity theory [3, 4]. Starting from a general homogeneous para-holomorphic prepotential, we derive the corresponding bosonic Euclidean Lagrangian, which is then found to be related to the Minkowskian Lagrangian through replacing special holomorphic coordinates by special para-holomorphic coordinates and the holomorphic prepotential by a para-holomorphic prepotential. We then show that a non-linear sigma model with projective special $\epsilon$-Kähler target is equivalent to a gauged sigma model with conical special $\epsilon$-Kähler target. For the case $\epsilon = -1$ this construction is part of the superconformal quotient which we expect to have a counterpart for Euclidean theories. Finally we reformulate our constructions in the language of line bundles. This allows to compare our formulae, which hold for both $\epsilon = 1$ and $\epsilon = -1$ to formulae obtained in the supergravity literature for $\epsilon = -1$.

In the third part we investigate solutions of the Euclidean field equations for the scalars and the metric in four dimensions. We start by a general analysis which is valid for any projective special $\epsilon$-Kähler target. The field equations consist of the harmonic map equation for the scalars, and the Einstein equation with the energy-momentum tensor of the scalars as source. We discuss the relation between the harmonic map equation and totally geodesic submanifolds of the target and derive some consequences of the Einstein equation. For symmetric target manifolds the description of totally geodesic submanifolds reduces to an algebraic problem. We illustrate this method for the projective special para-Kähler manifold

$$\frac{SL_2(\mathbb{R})}{SO_0(1,1)} \times \frac{SO_0(p+1,q+1)}{SO_0(1,1)\times SO_0(p,q)} \; .$$

For the rest of the paper we specialise to the case $p = q = 1$, which is the Euclidean STU model [5, 6]. As the simplest example for our method we construct a solution involving only the four-dimensional heterotic dilaton-axion field. This solution is used to explore features of vector multiplet instanton solutions. We find that vector multiplet instantons are quite similar to instanton solutions for hypermultiplets [7, 8, 9, 10]. The most pronounced feature is that the action obtained by dimensional reduction vanishes when evaluated on instanton solutions. A non-zero finite action, is found after dualising the axion into an antisymmetric tensor field. Instanton solutions are charged under the axion and, hence, under the dual antisymmetric tensor field. The instanton action is proportional to the absolute value of the instanton charge, and inversely proportional to the square of the coupling constant. Moreover, the action of our instanton solution is the minimal action for given charge.

When dualising the antisymmetric tensor field back into an axion, one obtains a boundary term, which we keep as part of the action. When this boundary is evaluated on instanton solutions, it gives precisely the instanton action found in the scalar-tensor formulation of the theory. We show that the instanton solution lifts to a five-dimensional extremal black hole, and we find that the ADM mass of this black hole equals the action of the corresponding instanton. The ADM mass is a boundary term, which is different from the boundary term obtained by dualising the antisymmetric tensor field, but which takes the same value when evaluated on solutions.

The Euclidean action obtained by dimensional reduction is not positive definite, while the dual Euclidean action, where the axion has been dualised into an antisymmetric tensor field is positive definite. We determine all Euclidean and Minkowskian actions which can be obtained by composing the operations of dimensional reduction, Wick rotation and Hodge dualisation. A detailed discussion of the properties and physical interpretation of these actions is given.

Finally we show that our explicit instanton solution can be lifted to a five-brane solution in ten dimensions. Therefore this solution is relevant for five-brane instanton effects in heterotic string theory compactified on $K3 \times T^2$.

## 2. Affine special $\epsilon$-Kähler manifolds

In this section we briefly review affine special pseudo-Kähler manifolds and affine special para-Kähler manifolds, see [11, 1] and references therein for more details. We will use the following unified terminology:

**Definition 1** *An* $\epsilon$-Kähler manifold $(M, J, g)$ *is a pseudo-Riemannian manifold* $(M, g)$ *endowed with a parallel skew-symmetric endomorphism field* $J \in \Gamma(\mathrm{End}\, TM)$ *such that* $J^2 = \epsilon \mathrm{Id}$*, where* $\epsilon \in \{-1, 1\}$*.*

$-1$-Kähler manifolds are usually called *pseudo-Kähler manifolds*, whereas $+1$-Kähler manifolds are known as *para-Kähler manifolds*. The signature of the pseudo-Riemannian metric $g$ is of the form $(2p, 2q)$, in the former case and is $(n, n)$ in the latter case, where $2n = \dim M$. In both cases, we have a symplectic form $\omega$, which is defined by

$$g = \omega(J\cdot, \cdot)\,, \text{ i.e. } \quad \omega = \epsilon g(J\cdot, \cdot)\,. \tag{2.1}$$

It is called the *Kähler form*. The endomorphism field $J$ has vanishing Nijenhuis tensor and defines on $M$ the structure of an $\epsilon$*-complex manifold*, i.e. complex or para-complex manifold for $\epsilon = \pm 1$, respectively. In both cases, we can define the notion of a holomorphic function $f : M \to \mathbb{C}_\epsilon$ with values in the ring of $\epsilon$*-complex numbers*

$$\mathbb{C}_\epsilon := \mathbb{R}[i_\epsilon]\,, \quad i_\epsilon^2 = \epsilon\,, \tag{2.2}$$

(complex or para-complex numbers for $\epsilon = \pm 1$, respectively). A function $f : M \to \mathbb{C}_\epsilon$ is called *$\epsilon$-holomorphic*, or simply *holomorphic*, if $df J = i_\epsilon df$. More generally, a differentiable map $f : (M, J) \to (M', J')$ between $\epsilon$-complex manifolds is called *holomorphic* if $df J = J' df$.

**Definition 2** *An* affine special $\epsilon$-Kähler manifold $(M, J, g, \nabla)$ *is an $\epsilon$-Kähler manifold $(M, J, g)$ endowed with a flat torsion-free connection $\nabla$ such that*

*(i) $\nabla$ is symplectic with respect to the $\epsilon$-Kähler form, i.e. $\nabla\omega = 0$ and*

*(ii) $\nabla J$ is a symmetric (1,2)-tensor field, i.e. $(\nabla_X J)Y = (\nabla_Y J)X$ for all $X, Y$.*

Let us now recall how such manifolds can be constructed from suitable immersions into $V = \mathbb{C}_\epsilon^{2n}$. Here $V$ is endowed with:

(i) the standard holomorphic symplectic form

$$\Omega = \sum dz^i \wedge dw_i \,, \tag{2.3}$$

where

$$(z^i, w_i) = (x^i + i_\epsilon u^i, y_i + i_\epsilon v_i) \tag{2.4}$$

are the standard linear holomorphic coordinates, and

(ii) the standard real structure, i.e. anti-linear involution $\tau : V \to V$, $v \mapsto \tau v = \bar{v}$, for which $V^\tau = \mathbb{R}^{2n} \subset \mathbb{C}_\epsilon^{2n}$ is the subset of real points, i.e. fixed points of $\tau$.

Combining these two data one obtains the sesquilinear form

$$\gamma := i_\epsilon \Omega(\cdot, \tau\cdot) \tag{2.5}$$

which is Hermitian-symmetric, i.e.

$$\gamma(Y, X) = \overline{\gamma(X, Y)} \,, \tag{2.6}$$

where the overline stands for the $\epsilon$-complex conjugation:

$$\overline{a + i_\epsilon b} = a - i_\epsilon b \,, \quad a, b \in \mathbb{R} \,. \tag{2.7}$$

Its real part $g_V := \mathrm{Re}\,\gamma$ is an $\epsilon$-Kähler metric of split signature $(2n, 2n)$.

**Definition 3** *Let $(M, J)$ be a connected $\epsilon$-complex manifold of real dimension $2n$. A holomorphic immersion $\phi : M \to V$ is called* $\epsilon$-Kählerian *(respectively,* Lagrangian*) if $\phi^*\gamma_V$ is non-degenerate (respectively, if $\phi^*\Omega = 0$).*

The following results are proven in [11, 1]:

**Proposition 1** *Let $\phi : M \to V$ be an $\epsilon$-Kählerian Lagrangian immersion. It induces the following data on the $\epsilon$-complex manifold $(M, J)$:*

*(i) an $\epsilon$-Kähler metric $g := \phi^* g_V$ with the Kähler form*

$$\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i \,, \tag{2.8}$$

*where*

$$\tilde{x}^i := x^i \circ \phi \,, \quad \tilde{y}_i := y_i \circ \phi \,, \tag{2.9}$$

*see (2.4), and*

*(ii) a flat torsion-free connection $\nabla$ such that the globally defined functions $(\tilde{x}^i, \tilde{y}_i)$ form a system of $\nabla$-affine local coordinates near any point of $M$.*

**Theorem 1** *Let $\phi : M \to V$ be an $\epsilon$-Kählerian Lagrangian immersion of a connected $\epsilon$-complex manifold $(M, J)$ with induced data $(g, \nabla)$. Then $(M, J, g, \nabla)$ is an affine special $\epsilon$-Kähler manifold. Conversely, let $(M, J, g, \nabla)$ be a simply connected affine special $\epsilon$-Kähler manifold. Then there exists an $\epsilon$-Kählerian Lagrangian immersion $\phi : M \to V$ which induces the special geometric structures on $M$. Moreover, the immersion $\phi$ is unique up to an affine transformation of $\mathbb{C}_\epsilon^{2n}$ with linear part in the real symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$.

Given a simply connected affine special $\epsilon$-Kähler manifold $(M, J, g, \nabla)$ and a point $p \in M$, one can choose the $\epsilon$-Kählerian Lagrangian immersion $\phi : M \to V$ in such a way that the image $\phi(U)$ of some neighborhood $U \subset M$ of $p$ is defined by a system of equations of the form

$$w_i = F_i := \frac{\partial F}{\partial z^i} \,, \tag{2.10}$$

where $F = F(z^1, \ldots, z^n)$ is a (locally defined) $\epsilon$-holomorphic function of $n$ $\epsilon$-complex variables. $F$ is called the holomorphic *prepotential*. The holomorphic functions

$$\tilde{z}^i := z^i \circ \phi|_U : U \to \mathbb{C}_\epsilon \,, \quad i = 1, \ldots, n \,, \tag{2.11}$$

form a system of local holomorphic coordinates. Such coordinates are called *special holomorphic coordinates*, whereas the $\nabla$-affine local coordinates $(\tilde{x}^i, \tilde{y}_i)$ are called *special affine coordinates.*

**Proposition 2** *Let $(M, J, g, \nabla)$ be an affine special $\epsilon$-Kähler manifold. Then $(M, J, g, \nabla^J)$ is an affine special $\epsilon$-Kähler manifold, where the connection $\nabla^J$ is defined by*

$$\nabla^J := J \circ \nabla \circ J^{-1} \,. \tag{2.12}$$

*Moreover, given an $\epsilon$-Kählerian Lagrangian immersion $\phi : M \to V$, which induces the special geometric data on $M$, the functions*

$$\tilde{u}^i := u^i \circ \phi\,, \quad \tilde{v}_i := v_i \circ \phi \tag{2.13}$$

*are special affine coordinates for the affine special $\epsilon$-Kähler manifold $(M, J, g, \nabla^J)$.*

## 3. Conical special $\epsilon$-Kähler manifolds

**Definition 4** *A* conical *affine special $\epsilon$-Kähler manifold $(M, J, g, \nabla, \xi)$ is an affine special $\epsilon$-Kähler manifold $(M, J, g, \nabla)$ endowed with a vector field $\xi$ such that*

$$\nabla\xi = D\xi = \mathrm{Id}\,, \tag{3.1}$$

*where $D$ is the Levi-Civita connection.*

**Proposition 3** *Let $(M, J, g, \nabla, \xi)$ be a conical affine special $\epsilon$-Kähler manifold. Then the following holds:*

*(i) $L_\xi X = -X$ and $L_\xi(JX) = -JX$ for all $\nabla$-parallel local vector fields $X$,*

*(ii) $L_\xi \alpha = \alpha$ and $L_\xi(J^*\alpha) = J^*\alpha$ for all $\nabla$-parallel local 1-forms $\alpha$,*

*(iii) $L_\xi \omega = 2\omega$, $L_\xi g = 2g$ and $L_\xi J = 0$.*

*(iv) $L_{J\xi}\omega = 0$, $L_{J\xi} g = 0$ and $L_{J\xi} J = 0$.*

*Proof:* To prove the first part of (i), we calculate

$$L_\xi X = \nabla_\xi X - \nabla_X \xi = -\nabla_X \xi = -X\,.$$

For the second part, we observe that the flat torsionfree connection $\nabla^J = J \circ \nabla \circ J^{-1}$ is related to the connection $D$ by the equation

$$\nabla^J = D - S\,, \tag{3.2}$$

where $S = D - \nabla^J = \nabla - D$. This shows that

$$L_\xi(JX) = \nabla^J_\xi(JX) - \nabla^J_{JX}\xi = -\nabla^J_{JX}\xi = -D_{JX}\xi + S_{JX}\xi = -JX\,.$$

Here we have used that $S\xi = \nabla\xi - D\xi = 0$. Item (ii) follows immediately from (i), by calculating the Lie derivative of the constant functions $\alpha(X)$ and $(J^*\alpha)(JX)$, e.g.

$$0 = L_\xi(\alpha(X)) = (L_\xi\alpha)(X) + \alpha(L_\xi X) = (L_\xi\alpha)(X) - \alpha(X)\,.$$

This shows that $L_\xi \alpha = \alpha$ for all $\nabla$-parallel 1-forms $\alpha$. In particular,

$$L_\xi d\tilde{x}^i = d\tilde{x}^i \quad \text{and} \quad L_\xi d\tilde{y}_i = d\tilde{y}_i \,. \tag{3.3}$$

Using (2.8), we obtain

$$L_\xi \omega = 2 \sum L_\xi (d\tilde{x}^i \wedge d\tilde{y}_i) = 2 \sum L_\xi (d\tilde{x}^i) \wedge d\tilde{y}_i + 2 \sum d\tilde{x}^i \wedge L_\xi d\tilde{y}_i = 2\omega \,.$$

Next we calculate $(L_\xi g)(X, Y)$, with the help of (i) and (ii), for two $\nabla$-parallel vector fields $X$ and $Y$:

$$\begin{aligned}(L_\xi g)(X,Y) &= L_\xi(g(X,Y)) - g(L_\xi X, Y) - g(X, L_\xi Y) = L_\xi(\omega(JX,Y)) + 2g(X,Y) \\ &= (2-1-1)\omega(JX,Y) + 2g(X,Y) = 2g(X,Y)\,.\end{aligned}$$

This proves (iii), since the Lie derivative of $J = \omega^{-1} g$ is determined by that of $g$ and $\omega$: $L_\xi J = L_\xi(\omega^{-1}) g + \omega^{-1} L_\xi g = -2J + 2J = 0$. To prove (iv), we observe that the vector field $J\xi$ satisfies

$$D(J\xi) = JD\xi = J$$

and is therefore a Killing field, i.e. $L_{J\xi} g = 0$. Similarly,

$$\nabla(J\xi) = (D+S)(J\xi) = JD\xi - JS\xi = J$$

implies that $L_{J\xi}\omega = 0$ and, hence, $L_{J\xi} J = 0$. □

**Proposition 4** *Let $(M, J, g, \nabla, \xi)$ be a conical affine special $\epsilon$-Kähler manifold. Then $(M, J, g, \nabla^J, \xi)$ is a conical affine special $\epsilon$-Kähler manifold.*

*Proof:* It is sufficient to check that $\nabla^J \xi = \mathrm{Id}$. This follows from (3.2). □

**Proposition 5** *Let $(M, J, g, \nabla, \xi)$ be a conical affine special $\epsilon$-Kähler manifold. Then near any point $p \in M$ there exists a system of special affine coordinates $(q^a) = (\tilde{x}^i, \tilde{y}_i)$, $a = 1, \ldots, 2n$, such that $\xi$ takes the form*

$$\xi = \sum q^a \frac{\partial}{\partial q^a} = \sum \tilde{x}^i \frac{\partial}{\partial \tilde{x}^i} + \sum \tilde{y}_i \frac{\partial}{\partial \tilde{y}_i}\,. \tag{3.4}$$

*The special affine coordinates $(q^a)$ are unique up to a linear symplectic transformation.*

*Proof:* Let $\xi = \sum \xi^a \partial/\partial q^a$ be the expression for $\xi$ with respect to some system of special affine coordinates $(q^a)$. From Proposition 3 (i), we have that

$$\sum \frac{\partial \xi^a}{\partial q^b} \frac{\partial}{\partial q^a} = \left[\frac{\partial}{\partial q^b}, \xi\right] = \frac{\partial}{\partial q^b}\,.$$

Therefore, $\xi^a = q^a + c^a$ for some constants $c^a \in \mathbb{R}$ and putting $q'^a := q^a + c^a$ yields special affine coordinates such that $\xi = \sum q'^a \partial/\partial q'^a$. The uniqueness statement is clear, since, in virtue of Theorem 1, the special affine coordinates $(q^a)$ we started with are unique up to an affine transformation of $\mathbb{R}^{2n}$ with linear part in the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. □

**Definition 5** *Special affine coordinates* $(q^a) = (\tilde{x}^i, \tilde{y}_i)$ *as in Proposition 5 are called* conical special affine coordinates.

Let us denote by $\xi^V$ the position vector field in the vector space $V = \mathbb{C}_\epsilon^{2n}$:

$$\xi_p^V = p \in V \cong T_p V\,. \tag{3.5}$$

**Definition 6** *Let* $(M, J)$ *be a connected* $\epsilon$*-complex manifold of real dimension* $2n$*. A holomorphic immersion is called* conical *if the vector field* $\xi^V$ *is tangent along* $\phi$*, i.e. if*

$$\xi_{\phi(p)}^V \in d\phi_p T_p M \tag{3.6}$$

*for all* $p \in M$*.*

A conical $\epsilon$-Kählerian Lagrangian immersion $\phi : M \to V$ induces a smooth vector field $\xi$ on $M$ such that

$$d\phi_p \xi_p = \xi_{\phi(p)}^V\,. \tag{3.7}$$

**Lemma 1** *Let* $(M, J, g, \nabla)$ *be an affine special* $\epsilon$*-Kähler manifold and* $\phi : M \to V$ *an* $\epsilon$*-Kählerian Lagrangian immersion inducing the data* $(g, \nabla)$ *on* $M$*. If* $\phi$ *is conical and* $\xi$ *is the induced vector field on* $M$*, then* $\xi = \sum \tilde{x}^i \partial/\partial \tilde{x}^i + \sum \tilde{y}_i \partial/\partial \tilde{y}_i$*, in the special affine coordinates* $(\tilde{x}^i, \tilde{y}_i)$ *and*

$$\xi = \sum \tilde{u}^i \frac{\partial}{\partial \tilde{u}^i} + \sum \tilde{v}_i \frac{\partial}{\partial \tilde{v}_i}\,, \tag{3.8}$$

*in the special* $\nabla^J$*-affine coordinates* $(\tilde{u}^i, \tilde{v}_i)$*, see Proposition 2.*

*Proof:* These expressions for the induced vector field $\xi$ follow from (3.7). □

**Theorem 2** *Let* $\phi : M \to V$ *be a conical* $\epsilon$*-Kählerian Lagrangian immersion of a connected* $\epsilon$*-complex manifold* $(M, J)$ *with induced data* $(g, \nabla, \xi)$*. Then* $(M, J, g, \nabla, \xi)$ *is a conical affine special* $\epsilon$*-Kähler manifold. Moreover, the special affine coordinates* $(\tilde{x}^i, \tilde{y}_i)$*, defined in (2.9), are conical and the special* $\nabla^J$*-affine coordinates* $(\tilde{u}^i, \tilde{v}_i)$ *are also conical, cf. Proposition 4. Conversely, let* $(M, J, g, \nabla, \xi)$ *be a simply connected conical affine special* $\epsilon$*-Kähler manifold. Then there exists a conical* $\epsilon$*-Kählerian Lagrangian immersion* $\phi : M \to V$ *which induces the special geometric structures on* $M$*. Moreover, the immersion* $\phi$ *is unique up to a linear transformation from the group* $\mathrm{Sp}(2n, \mathbb{R})$*.*

*Proof:* Let $\phi : M \to V$ be a conical $\epsilon$-Kählerian Lagrangian immersion of a connected manifold with induced data $(g, \nabla, \xi)$. According to Theorem 1, $(M, J, g, \nabla)$ is an affine special $\epsilon$-Kähler manifold. By Lemma 1, we have that $\xi = \sum \tilde{x}^i \partial/\partial \tilde{x}^i + \sum \tilde{y}_i \partial/\partial \tilde{y}_i$ with respect to the $\nabla$-affine coordinates $(\tilde{x}^i, \tilde{y}_i)$. This shows that $\nabla\xi = \mathrm{Id}$. Similarly, (3.8) shows that $\nabla^J\xi = \mathrm{Id}$ and, hence, by (3.2),

$$D\xi = \frac{1}{2}(\nabla\xi + \nabla^J\xi) = \mathrm{Id}\,.$$

This proves that $(M, J, g, \nabla, \xi)$ is a conical affine special $\epsilon$-Kähler manifold, that $(\tilde{x}^i, \tilde{y}_i)$ are conical special affine coordinates and that $(\tilde{u}^i, \tilde{v}_i)$ are conical $\nabla^J$-special affine coordinates.

To prove the converse, let $(M, J, g, \nabla, \xi)$ be a simply connected conical affine special $\epsilon$-Kähler manifold. By Theorem 1, there exists an $\epsilon$-Kählerian Lagrangian immersion $\phi : M \to V$ which induces the special geometric structures on $M$. Moreover, the immersion $\phi$ is unique up to an affine transformation of $\mathbb{C}_\epsilon^{2n}$ with linear part in the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The argument in the proof of Proposition 5, shows that there exists a translation $t_v : V \to V$ by a vector $v \in V$ such that the special affine coordinates $(x^i \circ \phi_v$, $y_i \circ \phi_v)$ associated with the $\epsilon$-Kählerian Lagrangian immersion $\phi_v = t_v \circ \phi = \phi + v$ are conical. Moreover, the real part

$$\mathrm{Re}\, v = \frac{1}{2}(v + \bar{v}) \tag{3.9}$$

of $v$ is uniquely determined, whereas the imaginary part

$$\mathrm{Im}\, v = \frac{1}{2i_\epsilon}(v - \bar{v}) \tag{3.10}$$

is arbitrary. By the same argument, there is a unique choice of the imaginary part $\mathrm{Im}\, v$ for which the $\nabla^J$-affine functions $(u^i \circ \phi_v,\ v_i \circ \phi_v)$ are conical special affine coordinates for the conical affine special $\epsilon$-Kähler manifold $(M, J, g, \nabla^J, \xi)$. These conditions mean precisely that the vector field $d(\phi_v)\xi$ along $\phi_v$ has the components

$$(x^i \circ \phi_v, y_i \circ \phi_v, u^i \circ \phi_v, v_i \circ \phi_v)$$

with respect to the standard basis of the real vector space $V = \mathbb{C}_\epsilon^{2n} = \mathbb{R}^{4n}$, i.e. $d(\phi_v)\xi = \xi^V \circ \phi_v$. In other words, there is a unique vector $v \in V$ such that $\phi_v : M \to V$ is a conical $\epsilon$-Kählerian Lagrangian immersion. This shows that a conical $\epsilon$-Kählerian Lagrangian immersion exists and is unique up to a linear transformation in $\mathrm{Sp}(2n, \mathbb{R})$. $\square$

Special holomorphic coordinates $\tilde{z}^i := z^i \circ \phi$ associated to a conical $\epsilon$-Kählerian Lagrangian immersion $\phi : U \to V$ of some connected open subset $U \subset M$ will be called *conical* special holomorphic coordinates, cf. (2.11). Let us denote by $\tilde{U} \subset \mathbb{C}_\epsilon^n$ the open subset which corresponds to $U \subset M$ under a system of special holomorphic coordinates $(z^i)$ and and let $F : \tilde{U} \to \mathbb{C}_\epsilon$ be a corresponding holomorphic prepotential such that

$$\phi(U) = \{(z, w) \in \mathbb{C}_\epsilon^{2n} \mid z \in \tilde{U} \quad \text{and} \quad w_i = F_i(z) \quad \text{for} \quad i = 1, \ldots, n\}\,, \tag{3.11}$$

where $z = (z^1, \ldots, z^n)$ and $w = (w_1, \ldots, w_n)$. Notice that $F$ is determined only up to an additive constant.

**Proposition 6** *The holomorphic prepotential $F : \tilde{U} \to \mathbb{C}_\epsilon$ associated to a system of special holomorphic coordinates $(\tilde{z}^i)$ can be chosen homogeneous of degree 2 if and only if the special holomorphic coordinates are conical.*

*Proof:* It is easy to see that an $\epsilon$-Kählerian Lagrangian immersion $\phi : U \to V$ is conical if and only if for all $(z, w) \in \phi(U)$ there exists a neighborhood $W \subset \mathbb{C}_\epsilon$ of $1 \in \mathbb{C}_\epsilon$ such that $(\lambda z, \lambda w) \in \phi(U)$ for all $\lambda \in W$. This is true if and only if $F_i(\lambda z) = \lambda F_i(z)$ for all $\lambda \in W$, see (3.11), which means that $F_i$ is homogeneous of degree 1. In that case,

$$\tilde{F} := \frac{1}{2} \sum z^i F_i \tag{3.12}$$

is homogeneous of degree 2 and differs from $F$ by a constant. In fact,

$$\frac{\partial}{\partial z^j}(F - \tilde{F}) = F_j - \frac{1}{2}(F_j + \sum z^i F_{ij}) = F_j - \frac{1}{2}(F_j + F_j) = 0\,. \tag{3.13}$$

So $\tilde{F}$ is a prepotential which is homogeneous of degree 2. Conversely, if $F$ is homogeneous of degree 2 then the $F_i$ are homogeneous of degree 1 and $\phi : U \to V$ is conical. □

## 4. Projective special $\epsilon$-Kähler manifolds

Let $(M, J, g, \nabla, \xi)$ be a conical affine special $\epsilon$-Kähler manifold of real dimension 2n+2. At any point $p \in M$ we consider the subspace

$$\mathcal{D}_p = \mathrm{span}\{\xi_p, J\xi_p\} \subset T_pM\,. \tag{4.1}$$

The vector fields $\xi$ and $J\xi$ commute:

$$[\xi, J\xi] = L_\xi(J)\xi = 0\,, \tag{4.2}$$

see Proposition 3 (iii). Therefore $\mathcal{D} \subset TM$ is an integrable distribution of $\epsilon$-complex subspaces, provided that $\dim \mathcal{D}_p = 2$ for all $p \in M$. In that case, we consider the space of leaves (i.e. the space of integral surfaces) $\bar{M}$ of $\mathcal{D}$ endowed with the topology induced by the canonical quotient map $\pi : M \to \bar{M}$. We will assume that $\pi : M \to \bar{M}$ is a holomorphic submersion onto a *Hausdorff* $\epsilon$-complex manifold of real dimension $2n$. The $\epsilon$-complex structure of $\bar{M}$ is again denoted by $J$. The following definition will ensure that $\mathcal{D}$ is a two-dimensional distribution and that $\bar{M}$ inherits an $\epsilon$-Kähler metric $\bar{g}$ from the affine special Kähler metric $g$.

**Definition 7** *A conical special $\epsilon$-Kähler manifold $(M, J, g, \nabla, \xi)$ is called* regular *if the function $g(\xi, \xi)$ does not vanish on $M$ and $\pi : M \to \bar{M}$ is a holomorphic submersion (onto a Hausdorff manifold).*

The regularity condition implies the orthogonal decomposition $T_pM = \mathcal{D}_p \oplus \mathcal{D}_p^\perp$ for all $p \in M$. In particular, $d\pi_p$ maps $\mathcal{D}_p^\perp$ isomorphically onto $T_{\pi(p)}\bar{M}$.

**Proposition 7** *The* $(0,2)$*-tensor field*

$$h = \frac{g}{g(\xi,\xi)} - \frac{g(\cdot,\xi) \otimes g(\cdot,\xi) - \epsilon g(\cdot, J\xi) \otimes g(\cdot, J\xi)}{g(\xi,\xi)^2} \tag{4.3}$$

*on* $M$ *induces an* $\epsilon$*-Kähler metric* $\bar{g}$ *on* $\bar{M}$*, such that* $\pi^*\bar{g} = h$.

*Proof:* The Proposition 3 easily implies that $L_\xi h = L_{J\xi} h = 0$. This shows that $h = \pi^*\bar{g}$ for a pseudo-Riemannian scalar product $\bar{g}$ on $\bar{M}$. Since $J$ is skew-symmetric with respect to $h$, the induced $\epsilon$-complex structure $J$ on $\bar{M}$ is skew-symmetric with respect to the induced metric $\bar{g}$ on $\bar{M}$. To prove that $(\bar{M}, \bar{g})$ is $\epsilon$-Kähler it suffices to check that the two-form $\bar{\omega} = \epsilon\bar{g}(J\cdot,\cdot)$ is closed. Let $c \in \mathbb{R}^*$ be a value of the function $g(\xi,\xi)$. The equation $g(\xi,\xi) = c$ defines a smooth hypersurface $S \subset M$, as we see from

$$dg(\xi,\xi) = 2g(D\xi,\xi) = 2g(\cdot,\xi).$$

Since $TS = \xi^\perp \supset \mathcal{D}^\perp$, it is sufficient to check that $\pi^*\bar{\omega} = \epsilon h(J\cdot,\cdot)$ restricts to a closed form on $S$. The restriction of $\epsilon h(J\cdot,\cdot)$ to a two-form on $S$ coincides with the restriction of $\frac{1}{c}\omega$, which is closed since $\omega$ is the Kählerform of $M$. □

The $\epsilon$-Kähler manifold $(\bar{M}, J, \bar{g})$ is called a *projective special* $\epsilon$*-Kähler manifold.*

## 5. The universal bundle of a projective special $\epsilon$-Kähler manifold

### 5.1 The Chern connection of the universal bundle $\mathcal{U} \to P(V')$

Let us consider the $\epsilon$-complex symplectic vector space $V = T^*\mathbb{C}_\epsilon^{n+1}$ endowed with the $\epsilon$-Hermitian metric (2.5). We denote by $V' := \{v \in V | \gamma(v,v) \neq 0\} \subset V$ the open subset of non-isotropic vectors and by $P(V')$ the set of $\epsilon$-complex lines $\mathbb{C}_\epsilon v$, $v \in V'$.

Let us first discuss the universal bundle $\pi_{\mathcal{U}} : \mathcal{U} \to P(V')$. The fiber $\mathcal{U}_p$ over $p = \mathbb{C}_\epsilon v \in P(V')$ is given by the line $\mathbb{C}_\epsilon v \subset V$. This defines a line subbundle $\mathcal{U} \subset \underline{V}$ of the trivial bundle $\underline{V} := P(V') \times V \to P(V')$. The $\epsilon$-Hermitian metric $\gamma$ on $V$ induces an $\epsilon$-Hermitian metric on $\mathcal{U}$.

**Lemma 2** *There exists a unique connection* $\mathcal{D}$ *on* $\mathcal{U}$ *which satisfies the following constraints:*

*(i) $\mathcal{D}$ is metric, that is*

$$X\gamma(v,w) = \gamma(\mathcal{D}_X v, w) + \gamma(v, \mathcal{D}_{\overline{X}} w),$$

*for all sections $v, w \in \Gamma(\mathcal{U})$ of $\mathcal{U}$ and all $\epsilon$-complex valued vector fields $X \in \Gamma(TP(V')\otimes \mathbb{C}_\epsilon)$ on $P(V')$.*

*(ii) For all $\epsilon$-holomorphic sections $v \in \mathcal{O}(\mathcal{U})$ and all $Z \in T^{1,0}M$ we have*

$$\mathcal{D}_{\overline{Z}} v = 0.$$

The above connection will be called the *Chern connection.*

*Proof:* We give a geometric description of the connection $\mathcal{D}$. Let us denote by $d_X v$ the ordinary derivative of a section $v$ of the trivial bundle $\underline{V}$ and by $\pi^V_{\mathcal{U}}$ the orthogonal projection $\underline{V} \to \mathcal{U} \subset \underline{V}$ with respect to the $\epsilon$-Hermitian scalar product $\gamma$ on $V$. Then $\mathcal{D}$ is given by

$$\mathcal{D}_X v := \pi^V_{\mathcal{U}} d_X v, \tag{5.1}$$

where $X$ is a vector field on $P(V')$ and $v$ is a section of $\mathcal{U} \subset \underline{V}$. Let us check that $\mathcal{D}$ satisfies (i-ii).

(i) For all $v, w \in \Gamma(\underline{V})$ and all $X \in \Gamma(TP(V') \otimes \mathbb{C}_\epsilon)$ we have

$$d_X\gamma(v,w) = \gamma(d_X v, w) + \gamma(v, d_{\overline{X}} w).$$

For $v, w \in \Gamma(\mathcal{U})$ we may replace $d$ by $\mathcal{D}$ in that formula. This proves (i).

(ii) For all $v \in \mathcal{O}(\underline{V})$ and $Z \in T^{1,0}M$ we have $d_{\overline{Z}} v = 0$. In particular, $\mathcal{D}_{\overline{Z}} v = \pi^V_{\mathcal{U}} d_{\overline{Z}} v = 0$ for all $v \in \mathcal{O}(\mathcal{U})$.

To prove the uniqueness we consider the difference $\Theta := \mathcal{D} - \mathcal{D}'$ of two connections $\mathcal{D}, \mathcal{D}'$ satisfying (i-ii). The tensor field $\Theta$ verifies

$$\gamma(\Theta(Z)v, w) = -\gamma(v, \Theta(\overline{Z})w)$$

for all $Z \in T^{1,0}M$ and $v, w \in \Gamma(\mathcal{U})$ and

$$\Theta(\overline{Z})u = 0$$

for all $Z \in T^{1,0}M$ and $u \in \mathcal{O}(\mathcal{U})$. The second condition implies $\Theta(\overline{Z}) = 0$, since $\Theta(\overline{Z})$ is tensorial. Then the first condition implies $\Theta = 0$. $\square$

### 5.2 The pull back of $(\mathcal{U}, \mathcal{D})$ to $M$

Now let $(M, J, g, \nabla, \xi)$ be a regular conical affine special $\epsilon$-Kähler manifold. Then we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & V' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_V} \\ \bar{M} & \xrightarrow{\bar{\phi}} & P(V') \end{array} \tag{5.2}$$

where $\phi$ is a conical $\epsilon$-Kählerian Lagrangian immersion inducing the special geometric data on $M$ and $\overline{\phi}$ is the corresponding $\epsilon$-holomorphic Legendrian immersion.

We denote by $\mathcal{U}^M := (\overline{\phi} \circ \pi)^* \mathcal{U} = (\pi_V \circ \phi)^* \mathcal{U}$ the pull back of the universal bundle under the map $M \to P(V')$. Let us recall that given a smooth map $f : M \to N$ between smooth manifolds $M$ and $N$ we can pull back any vector bundle $\pi_E : E \to N$ on $N$ to a vector bundle $f^*E$ on $M$. The total space of $f^*E$ is defined by

$$f^*E := \{(e, m) \in E \times M | \pi_E(e) = f(m)\}$$

and the bundle projection $f^*E \to M$ is the restriction of the canonical projection $E \times M \to M$ to $f^*E \subset E \times M$. Any section $s \in \Gamma(E)$ gives rise to a section $f^*s \in \Gamma(f^*E)$ defined by

$$(f^*s)(m) = s(f(m)).$$

In particular, the pull back of any trivial bundle is again trivial. Given a connection $D$ in $E$, the pull back connection $f^*D$ in $f^*E$ is defined by

$$(f^*D)_X f^*s := D_{dfX} s.$$

Notice that $(f^*E)_m = E_{f(m)} \times \{m\} \cong E_{f(m)}$ for all $m \in M$.

We can consider $\phi : M \to V$ as an $\epsilon$-holomorphic section of $\mathcal{U}^M$. This follows from

$$\phi(m) \in V', \quad \pi_V \phi(m) = \overline{\phi}(\pi(m)),$$

since $\pi_{\mathcal{U}}$ coincides with $\pi_V$ on the complement $V' = \mathcal{U} \setminus \mathbf{0}$ of the zero section in $\pi_{\mathcal{U}} : \mathcal{U} \to P(V')$. ($\mathcal{U}$ is precisely the blow up of the open cone $V'$ at the origin.)

Next we consider the pull back via $\pi_V \circ \phi = \overline{\phi} \circ \pi : M \to P(V')$ of the connection $\mathcal{D}$ on $\mathcal{U} \to P(V')$ to a connection on $\mathcal{U}^M \to M$. We shall denote all pull backs of $\mathcal{D}$ by the same letter $\mathcal{D}$. Since $\phi$ is an $\epsilon$-holomorphic section the pull back connection satisfies

$$\mathcal{D}_i \phi = i_\epsilon A_i^h \phi, \quad \mathcal{D}_{\bar{i}} \phi = 0,$$

where $\mathcal{D}_i := \mathcal{D}_{\partial_i}$ and $\mathcal{D}_{\bar{i}} := \mathcal{D}_{\partial_{\bar{i}}}$ are derivatives with respect to holomorphic and anti-holomorphic coordinates.

**Proposition 8** *The connection one-form $i_\epsilon \sum A^h_i dz^i$ of the pull back connection on $\mathcal{U}^M$ with respect to the $\epsilon$-holomorphic section $\phi$ is given by*

$$i_\epsilon A^h_i = \frac{\gamma(\partial_i \phi, \phi)}{\gamma(\phi, \phi)} = \frac{\sum(\partial_i z^j \overline{F}_j - \partial_i F_j \overline{z}^j)}{\sum(z^j \overline{F}_j - F_j \overline{z}^j)}, \quad A_{\bar{i}} = 0,$$

*where $(z^i)$, $i = 1, 2, \ldots, n+1$, are conical special $\epsilon$-holomorphic coordinates and $F$ is the corresponding prepotential.*

*Proof:* This follows from (5.1). $\square$

For future use we express the above pullback connection also with respect to the unit section $\phi_1 := \frac{\phi}{\|\phi\|}$, where $\|\phi\| := \sqrt{|\gamma(\phi, \phi)|}$.

**Proposition 9** *The connection one-form $i_\epsilon \sum A_i dz^i + i_\epsilon \sum A_{\bar{i}} d\overline{z}^i$ of the pull back connection on $\mathcal{U}^M$ with respect to the unitary section $\phi_1$ is given by*

$$A_i = \frac{1}{2} A^h_i, \quad A_{\bar{i}} = \frac{1}{2} \overline{A^h_i}.$$

*Proof:* We compute

$$\mathcal{D}_i \phi_1 = \partial_i \left( \frac{1}{\|\phi\|} \right) \phi + \frac{1}{\|\phi\|} \mathcal{D}_i \phi = -\frac{i_\epsilon}{2} A^h_i \phi_1 + i_\epsilon A^h_i \phi_1 = \frac{i_\epsilon}{2} A^h_i \phi_1,$$

$$\mathcal{D}_{\bar{i}} \phi_1 = \partial_{\bar{i}} \left( \frac{1}{\|\phi\|} \right) \phi + \frac{1}{\|\phi\|} \mathcal{D}_{\bar{i}} \phi = -\overline{\frac{i_\epsilon}{2} A^h_i} \phi_1 + 0 = \frac{i_\epsilon}{2} \overline{A^h_i} \phi_1.$$

$\square$

### 5.3 The pull back of $(\mathcal{U}^M, \mathcal{D})$ under a smooth map $f : N \to M$

Let $N$ be a smooth manifold with local coordinates $(x^\mu)$ and $f : N \to M$ a smooth map into the regular conical affine special $\epsilon$-Kähler manifold $M$.

**Proposition 10** *(i) The connection one-form $i_\epsilon \sum A^h_\mu dx^\mu$ of the pull back connection on $f^* \mathcal{U}^M$ with respect to the pull back $\phi^N = f^* \phi$ of the $\epsilon$-holomorphic section $\phi$ is given by*

$$A^h_\mu = \sum \partial_\mu z^i A^h_i.$$

*Here $\partial_\mu z^j$ stands for $\partial_\mu (z^j \circ \phi \circ f)$ and $A^h_i$ is evaluated along $f$.*

*(ii) The connection one-form $i_\epsilon \sum A_\mu dx^\mu$ of the pull back connection on $f^* \mathcal{U}^M$ with respect to the pull back $\phi^N_1 = f^* \phi_1$ of the unitary section $\phi_1$ is given by*

$$A_\mu = \sum \partial_\mu z^i A_i + \sum \partial_\mu z^{\bar{i}} A_{\bar{i}}.$$

Next we consider the special case $N = \bar{M}$ and $f = s : \bar{M} \to M$ a section of $\pi : M \to \bar{M}$. Let us first observe that $\mathcal{U}^{\bar{M}} := \overline{\phi}^* \mathcal{U} = s^* \mathcal{U}^M$, since $\overline{\phi} = \pi_V \circ \phi \circ s$, see (5.2).

**Corollary 1** *The pull back connection $\mathcal{D}$ on $\mathcal{U}^{\bar{M}}$ satisfies:*

*(i) For every holomorphic section $s$*

$$\mathcal{D}_a s = i_\epsilon \underbrace{\partial_a z^i A_i^h}_{A_a^h :=} s = \frac{\gamma(\partial_a \phi, \phi)}{\gamma(\phi, \phi)} s = \frac{\sum(\partial_a z^j \overline{F}_j - \partial_a F_j \overline{z}^j)}{\sum(z^j \overline{F}_j - F_j \overline{z}^j)} s, \quad \mathcal{D}_{\overline{a}} s = 0, \tag{5.3}$$

*where the derivative $\partial_a = \frac{\partial}{\partial \zeta^a}$ is with respect to local $\epsilon$-holomorphic coordinates on $\bar{M}$, $\phi$ and $A_i^h$ are evaluated on $s$ and the functions $z^j$ and $F_j$ are evaluated on $\phi \circ s$.*

*(ii) For every unitary section $s_1$ we have*

$$\mathcal{D}_a s_1 =: i_\epsilon A_a s_1 = \frac{\gamma(\partial_a \phi, \phi) - \gamma(\phi, \partial_{\overline{a}} \phi)}{2\gamma(\phi, \phi)} s_1, \quad \mathcal{D}_{\overline{a}} s_1 =: i_\epsilon A_{\overline{a}} s_1 = \frac{\gamma(\partial_{\overline{a}} \phi, \phi) - \gamma(\phi, \partial_a \phi)}{2\gamma(\phi, \phi)} s_1, \tag{5.4}$$

*where $\gamma(\phi, \phi) = \gamma(\phi(s_1), \phi(s_1)) = \pm 1$ and*

$$\begin{aligned} \gamma(\partial_a \phi, \phi) - \gamma(\phi, \partial_{\overline{a}} \phi) &= -\overline{(\gamma(\partial_{\overline{a}} \phi, \phi) - \gamma(\phi, \partial_a \phi))} \\ = i_\epsilon \sum(\partial_a z^j \overline{F}_j - \partial_a F_j \overline{z}^j - z^j \partial_a \overline{F}_j + F_j \partial_a \overline{z}^j) &= -i_\epsilon \sum(z^j \overset{\leftrightarrow}{\partial_a} \overline{F}_j - F_j \overset{\leftrightarrow}{\partial_a} \overline{z}^j) \, , \end{aligned}$$

*Here we use the notation $a \overset{\leftrightarrow}{\partial_\mu} b := a \partial_\mu b - (\partial_\mu a) b$. (Notice that $A_{\bar{a}} = \overline{A_a}$ and that these formulas can be rewritten in various ways using that for a unitary section $\gamma(\partial_a \phi, \phi) = -\gamma(\phi, \partial_{\overline{a}} \phi)$.)*

Now let $f : N \to \bar{M}$ be any smooth map from a manifold $N$ with local coordinates $(x^\mu)$ into the projective special $\epsilon$-Kähler manifold $\bar{M}$. Pulling back the connection $\mathcal{D}$ on $\mathcal{U}^{\bar{M}}$ we get with the above notation

$$\mathcal{D}_\mu f^* s =: i_\epsilon A_\mu^h f^* s = i_\epsilon \partial_\mu \zeta^a A_a^h f^* s \tag{5.5}$$

$$\mathcal{D}_\mu f^* s_1 =: i_\epsilon A_\mu f^* s_1 = i_\epsilon (\partial_\mu \zeta^a A_a + \partial_\mu \overline{\zeta}^a A_{\overline{a}}) f^* s_1. \tag{5.6}$$

## 6. Dimensional reduction of five-dimensional supergravity

### 6.1 The five-dimensional theory

In [12] the general Lagrangian for vector multiplets coupled to five-dimensional supergravity was derived. By dimensional reduction on a space-like circle they obtained four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to supergravity. Whereas the five-dimensional couplings are determined by very special real geometry, the four-dimensional couplings are determined by projective special Kähler geometry. We will generalise the analysis of [12] to the case where the compactification circle is time-like, which leads to a theory with Euclidean space-time signature. To compare the effects of space-like and time-like dimensional reduction we perform both types of reduction in parallel. Then, it is convenient to introduce a parameter $\epsilon$, which takes the value $\epsilon = -1$ for reduction over space and $\epsilon = 1$ for reduction over time. As we will see in due course, the geometry of the scalar target space of the four-dimensional theory is (projective special) $\epsilon$-Kähler, and the $\epsilon$ introduced above will turn out to be identical to the one defined in section 2.

The fields of the five-dimensional theory organise themselves into the following supermultiplets:

- The gravity supermultiplet $(e_{\hat{\mu}}{}^{\hat{m}}, \psi^A_{\hat{\mu}}, \mathcal{A}_{\hat{\mu}})$ contains the fünfbein (graviton), two gravitini and the graviphoton.

- A vector multiplet $(\mathcal{A}_{\hat{\mu}}, \Lambda^A, \phi)$ consists of a gauge field, a pair of symplectic Majorana spinors and a real scalar field. We consider a theory with an arbitrary number of vector multiplets, labeled by the index $x = 1, \ldots, n_V^{(5)}$.

The other indices have the following ranges: $\hat{\mu}, \hat{\nu}, \ldots = 0, \ldots, 4$ are five-dimensional world indices, $\hat{m}, \hat{n}, \ldots = 0, \ldots, 4$ are five-dimensional tangent space indices and $A = 1, 2$ is the index of R-symmetry group $SU(2)_R$. Since the gravity multiplet contributes an additional gauge field, there are $n_V^{(5)} + 1$ gauge fields, which we denote by $\mathcal{A}^i_{\hat{\mu}}$, with $i = 0, \ldots, n_V^{(5)}$. The corresponding field strengths are $\mathcal{F}^i_{\hat{\mu}\hat{\nu}}$.

The full Lagrangian is completely determined by the choice of the scalar manifold $\hat{M}$, which must be a so-called very special real manifold, i.e., a cubic hypersurface [12]. The hypersurface is characterised by a cubic function, the prepotential $\mathcal{V}$:

$$\mathcal{V} := c_{ijk} h^i h^j h^k \; = \; 1 \; , \tag{6.1}$$

where $c_{ijk}$ is a real symmetric constant tensor and $h^i$ are embedding coordinates of the scalar manifold. The physical scalars $\phi^x$ are obtained by solving the hypersurface constraint (6.1). It turns out to be convenient to work with constrained fields $h^i$. When we refer to them as 'five-dimensional scalars', the constraint (6.1) is understood.

In order to identify the scalar geometry of the four-dimensional theories obtained by dimensional reduction, we only need to reduce the bosonic terms. Therefore we start from the bosonic part of the five-dimensional Lagrangian for supergravity coupled to an arbitrary number of vector multiplets [12]:

$$\hat{\mathbf{e}}^{-1}\hat{\mathcal{L}} = \frac{1}{2}\hat{R} - \frac{3}{4}a_{ij}\partial_{\hat{\mu}} h^i \partial^{\hat{\mu}} h^j - \frac{1}{4}a_{ij}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j\hat{\mu}\hat{\nu}} + \frac{\hat{\mathbf{e}}^{-1}}{6\sqrt{6}}\, c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}} \; . \tag{6.2}$$

Here $\hat{\mathbf{e}}$ is the determinant of the fünfbein and $\hat{R}$ the space-time Ricci scalar. The terms quadratic in the matter fields contain the field dependent coupling matrix $a_{ij}$, which is determined by the prepotential through

$$a_{ij} \;=\; -\frac{1}{3}\partial_{h^i}\partial_{h^j} \ln \mathcal{V}_{|\mathcal{V}=1} \; . \tag{6.3}$$

The explicit expression is:

$$a_{ij} = -2\left(\frac{(ch)_{ij}}{chhh} - \frac{3}{2}\frac{(chh)_i(chh)_j}{(chhh)^2}\right) \; , \tag{6.4}$$

where we introduced the following notation:

$$chhh := c_{ijk}h^ih^jh^k \; , \quad (chh)_i := c_{ijk}h^jh^k \; , \quad (ch)_{ij} := c_{ijk}h^k \; . \tag{6.5}$$

The coefficients $c_{ijk}$ of the Chern-Simons terms are proportional to the third derivatives of the prepotential. Note that the sigma model metric for the physical scalars $\phi^x$ is the pullback of the tensor field $a_{ij}dh^idh^j$ to the hypersurface $\mathcal{V} = 1$. However, for the purpose of dimensional reduction it turns out to be convenient to work with the constrained scalars $h^i$.

While the scalar manifold is determined by the constants $c_{ijk}$, it is understood that the range of the scalars $h^i$ has been chosen such that both $a_{ij}$ and its pull back onto $\hat{M}$ are positive definite. This is needed in order to ensure that the scalars and gauge fields have well defined (positive definite) kinetic terms.

We close this section by pointing out that an interpretation of very special real geometry in the framework of affine differential geometry has been given in [13]. In that construction the metric and very special real structure on $\hat{M}$ are induced through a centroaffine embedding into $\mathbb{R}^{n_V^{(5)}+1}$, equipped with its standard affine structure. The embedding is encoded in the real prepotential $\mathcal{V}$, which plays a similar role as the holomorphic prepotential in the $\epsilon$-complex case. We refer to [13] for more details.

### 6.2 Dimensional reduction of the bosonic terms

We now perform the dimensional reduction of the bosonic Lagrangian (6.2) with respect to a time-like ($\epsilon = 1$) or space-like ($\epsilon = -1$) direction. A standard Ansatz for the fünfbein is:

$$\hat{e}_{\hat{\mu}}{}^{\hat{m}} \;=\; \begin{pmatrix} e^{\sigma} & 0 \\ e^{\sigma}\mathcal{A}^0_{\mu} & e^{-\sigma/2}e_{\mu}{}^{m} \end{pmatrix} \; , \quad \eta_{\hat{m}\hat{n}} = \mathrm{diag}\big(-\epsilon, \eta_{mn} = (+,+,+,\epsilon)\big) \; . \tag{6.6}$$

We introduced four-dimensional world indices $\mu, \nu, \ldots = 1, \ldots, 4$ and four-dimensional tangent space indices $m, n, \ldots = 1, \ldots, 4$. The compactified direction is taken to be the

0-direction, for both $\epsilon = \pm 1$. $\mathcal{A}^0_\mu$ is the Kaluza-Klein gauge field, $\sigma$ is the Kaluza-Klein scalar.

The four-dimensional epsilon tensor is:

$$\epsilon_{mnpq} \ := \ \epsilon_{0\hat{m}\hat{n}\hat{p}\hat{q}} \ , \quad \text{with} \quad \epsilon^{mnpq}\epsilon_{mnpq} \ = \ 4! \ \epsilon \ . \tag{6.7}$$

The 0-components of the five-dimensional gauge fields are four-dimensional scalar fields, $m^i := \mathcal{A}^i_0$.

We obtain the following bosonic Lagrangian:

$$\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^\epsilon = & \ \frac{1}{2}R - \frac{3}{4}(\partial_\mu\sigma)^2 - \frac{3}{4}a_{ij}\partial_\mu h^i\partial^\mu h^j + \epsilon\frac{1}{2}e^{-2\sigma}a_{ij}\partial_\mu m^i\partial^\mu m^j \\
& + \epsilon\frac{1}{8}e^{3\sigma}(\mathcal{F}^0_{\mu\nu})^2 - \frac{1}{4}e^\sigma a_{ij}\mathcal{F}^i_{\mu\nu}\mathcal{F}^{j\mu\nu} - e^\sigma\,\mathcal{A}^{0\mu}\partial^\nu a_{ij}m^i\,\mathcal{F}^j_{\mu\nu} - \frac{1}{2}e^\sigma a_{ij}\partial^\mu m^i\partial_\mu m^j\mathcal{A}^{0\mu}\mathcal{A}^0_\mu \\
& + \frac{1}{2}e^\sigma a_{ij}\partial^\mu m^i\partial^\nu m^j\,\mathcal{A}^0_\mu\mathcal{A}^0_\nu - \epsilon\frac{\mathbf{e}^{-1}}{2\sqrt{6}}c_{ijk}m^k\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^i_{\mu\nu}\mathcal{F}^j_{\rho\sigma} \ .
\end{aligned} \tag{6.8}$$

Here $\mathbf{e}$ is the determinant of the vierbein $e_\mu{}^m$, and $R$ is the four-dimensional Ricci scalar. We remark that, as expected, the metric of the scalar manifold has split signature for $\epsilon = 1$. This is due to the fact that the scalars $m^i$ come from the time-like components of the five-dimensional gauge fields.

The reduced Lagrangian contains terms in which bare gauge fields appear. Therefore the gauge invariances of the four-dimensional Lagrangian are not manifest. Of course, gauge invariance has not been broken by the Kaluza-Klein reduction, but it is not manifest in terms of the gauge fields $\mathcal{A}^i_\mu$. Also note that through dimensional reduction the reparametrisation symmetry of the fifth direction has become an additional internal Abelian gauge symmetry. The corresponding gauge field is the Kaluza-Klein gauge field $\mathcal{A}^0_\mu$. Therefore the number of vector multiplets is increased by one in dimensional reduction: $n_V^{(4)} = n_V^{(5)} + 1$. Since the four-dimensional $\mathcal{N} = 2$ supergravity multiplet contains one gauge field, the graviphoton, we expect to find $n_V^{(4)} + 1$ abelian gauge symmetries. To make the four-dimensional gauge symmetries manifest we introduce redefined gauge fields:

$$A^i_\mu \ := \ \mathcal{A}^i_\mu - m^i\mathcal{A}^0_\mu \ , \quad A^0_\mu \ := \ -\mathcal{A}^0_\mu \ . \tag{6.9}$$

We also introduce a new index $I = (0, i)$ and denote the four-dimensional gauge fields by $A^I_\mu$. The corresponding field strength $F^I_{\mu\nu}$ are invariant under all the $n_V^{(4)} + 1$ four-dimensional gauge transformations. The Hodge dual field strengths are defined as $\tilde{F}_{\mu\nu} = \frac{1}{2}\mathbf{e}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, such that $\tilde{\tilde{F}}_{\mu\nu} = \epsilon F_{\mu\nu}$.

Inserting the new field strengths into the Lagrangian we obtain the following terms for the gauge fields:

$$
\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^{\epsilon}_{\text{gauge}} &= e^{3\sigma}\big(\tfrac{1}{8}(\epsilon - 2\,e^{-2\sigma}(amm))F^0\cdot F^0 + \tfrac{1}{2}e^{-2\sigma}(am)_iF^i\cdot F^0 - \tfrac{1}{4}e^{-2\sigma}\,a_{ij}F^i\cdot F^j\big) \\
&\quad -\epsilon\frac{1}{\sqrt{6}}\big((cm)_{ij}F^i\cdot\tilde{F}^j - (cmm)_iF^i\cdot\tilde{F}^0 + \frac{1}{3}\,(cmmm)F^0\cdot\tilde{F}^0\big)\ , \qquad (6.10)
\end{aligned}
$$

where we suppressed contracted Lorentz indices on the field strengths, i.e., $F^I\cdot G^J := F^I_{\mu\nu}G^{J\,\mu\nu}$. Note that only field strengths appear, so that the four-dimensional gauge symmetries are manifest. There are two types of terms, generalised Maxwell terms in the first line and generalised $\theta$-terms in the second line.

In order to make contact with the conventions of four-dimensional special geometry, we now perform the following rescaling:

$$
\begin{aligned}
h^i &=: 6^{-1/3}\,e^{-\sigma}y^i\ , \quad m^i =: \frac{6^{1/6}}{2}\,x^i\ , \quad a_{ij} =: -\epsilon g_{ij}\cdot 8\cdot 6^{-1/3}\,e^{2\sigma}\ , \\
F^i_{\mu\nu} &=: \frac{6^{1/6}}{\sqrt{2}}\,F^i_{(\text{new})\mu\nu}\ , \quad F^0_{\mu\nu} =: \sqrt{2}\,F^0_{(\text{new})\mu\nu}\ , \\
\tilde{F}^i_{\mu\nu} &=: \epsilon\frac{6^{1/6}}{\sqrt{2}}\,\tilde{F}^i_{(\text{new})\mu\nu}\ , \quad \tilde{F}^0_{\mu\nu} =: \epsilon\sqrt{2}\,\tilde{F}^0_{(\text{new})\mu\nu}\ . \qquad (6.11)
\end{aligned}
$$

In four dimensions we adapt the range of our indices to the usual conventions, i.e., $\mu,\nu = 0,\ldots,3$ for $\epsilon = -1$ and $\mu,\nu = 1,\ldots,4$ for $\epsilon = 1$. Since we prefer to normalise the four-dimensional $\epsilon$-tensor such that $\epsilon^{0123} = 1$, we had to redefine the dual field strength by an extra factor $\epsilon$ to compensate for this redefinition. We note that $chhh = 1$ implies $Cyyy = 6e^{3\sigma}$. Therefore the fields $y^i$ are unconstrained, in contrast to the $h^i$, because the Kaluza Klein scalar has been scaled in. To avoid cluttered notation, we drop the subscript on the new field strength, $F^I_{(\text{new})\mu\nu} =: F^I_{\mu\nu}$.

The four-dimensional bosonic Lagrangian takes the following form in terms of the rescaled fields:

$$
\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^{\epsilon} &= \frac{1}{2}R - g_{ij}\left(\partial_\mu x^i\partial^\mu x^j - \epsilon\partial_\mu y^i\partial^\mu y^j\right) \\
&\quad +\epsilon\left(\frac{1}{4}cyyy\left(\frac{1}{6}+\frac{2}{3}\,gxx\right)F^0\cdot F^0 - \frac{1}{3}\,cyyy\,(gx)_iF^0\cdot F^i + \frac{1}{6}\,cyyy\,g_{ij}\,F^i\cdot F^j\right) \\
&\quad -\frac{1}{12}\left(cxxx\,F^0\cdot\tilde{F}^0 - 3(cxx)_i\,F^i\cdot\tilde{F}^0 + 3(cx)_{ij}\,F^i\cdot\tilde{F}^j\right)\ . \qquad (6.12)
\end{aligned}
$$

The explicit form of $g_{ij}$ is

$$
g_{ij} = \epsilon\,\frac{3}{2}\left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\frac{(cyy)_i(cyy)_j}{(cyyy)^2}\right)\ , \qquad (6.13)
$$

and the metric of the scalar manifold of the four-dimensional theory is $g_{ij}\oplus(-\epsilon)g_{ij}$. Introducing $\epsilon$-holomorphic coordinates $z^j = x^j + i_\epsilon y^j$ we observe that the metric is $\epsilon$-Kähler with $\epsilon$-Kähler potential $K = -\ln \mathcal{V}(y)$. The signature is determined by the signature of the five-dimensional scalar metric $a_{ij}$. To have standard kinetic terms in the five-dimensional theory, $a_{ij}$ needs to be positive definite, and then (6.4) implies that $g_{ij}$ is positive (negative) definite for $\epsilon = -1$ ($\epsilon = 1$). Thus for space-like reduction ($\epsilon = -1$) the scalar metric $g_{ij} \oplus (-\epsilon) g_{ij}$ is positive definite, while for time-like reduction ($\epsilon = 1$) it has split signature. In the latter case the scalars $x^i$, which descend from five-dimensional gauge fields, have a non-standard negative definite kinetic term. We will investigate and comment on this feature in due course.

## 7. The four-dimensional Lagrangian and its special $\epsilon$-Kähler geometry

We will now show that the scalar geometry of the dimensionally reduced Lagrangian is projective special $\epsilon$-Kähler. Moreover, we will show that for space-like dimensional reduction it agrees with the standard form [3] of a four-dimensional vector multiplet Lagrangian, and that the Euclidean vector multiplet Lagrangian is obtained from this by replacing the complex structure by a para-complex structure. We will work in local $\epsilon$-complex coordinates and write all formulae such that they apply simultanously to both cases $\epsilon = \pm 1$. The $\epsilon$-complex unit is denoted $i_\epsilon$ and has the property that $i_\epsilon^2 = \epsilon$. Thus $i_\epsilon = i$ with $i^2 = -1$ for $\epsilon = -1$, and $i_\epsilon = e$ with $e^2 = 1$ for $\epsilon = 1$.

Given the form of the scalar term in (6.12), it is natural to introduce $\epsilon$-complex scalar fields $Z^i = x^i + i_\epsilon y^i$. Then the scalar term takes the form

$$\mathbf{e}^{-1} \mathcal{L}^{\epsilon}_{\text{scalar}} = -\bar{g}_{ij} \partial_\mu Z^i \partial^\mu \overline{Z}^j \ , \tag{7.1}$$

and we see that the scalar metric is $\epsilon$-Hermitean. We will now elaborate on this observation and make the geometry underlying (6.12) manifest.

This section is organised as follows. In subsection 7.1 we generalise various standard formulae used in the physics literature on special geometry to the $\epsilon$-complex case. We work in local coordinates, but mention the geometrical interpretation of various objects, where helpful. The details are postponed to subsection 7.3. The main result of subsection 7.1 is the $\epsilon$-complex generalisation of the bosonic part of the Lagrangian for four-dimensional $\mathcal{N} = 2$ vector multiplets. The prepotential is required to be $\epsilon$-holomorphic and homogenous of degree 2, but unconstrained otherwise. For $\epsilon = -1$ we show that we recover the bosonic part of the $\mathcal{N} = 2$ vector multiplet Lagrangian, as given in [14].[1]

[1]This reference uses the so-called 'new conventions', which differ from the conventions used in [3], [4]. Most of the recent supergravity and string theory literature uses the new conventions (or closely related conventions).

In subsection 7.2 we specialise to the case of so-called very special prepotentials and show that the resulting Lagrangian agrees with the one obtained by dimensional reduction over time (for $\epsilon = 1$) and space (for $\epsilon = -1$), respectively.

In subsections 7.3 and 7.4 we return to the case of a general prepotential and relate the formalism of subsection 7.1 to the results of sections 2 - 5, thus providing the geometrical interpretation. In section 7.3 we show that the scalar term of the four-dimensional Lagrangian has two gauge-equivalent formulations: one as a gauged sigma models with scalars $X^I$ taking values in $M$, the other as a sigma model with scalars $Z^i$ taking values in $\bar{M}$. The second formulation is obtained by gauge-fixing the local $\mathbb{C}_\epsilon^*$ symmetry of the gauged sigma model. For $\epsilon = -1$ this is of course part of the well known construction of $\mathcal{N} = 2$ vector multiplet based on the superconformal calculus. This constuction makes use of the gauge equivalence between $n + 1$ superconformal vector multiplets coupled to conformal supergravity (the Weyl multiplet)[2] with $n$ vector multiplets coupled to Poincaré supergravity. While we do not fully develop the superconformal calculus for $\epsilon = 1$, we cover its most relevant aspect for vector multiplets, namely the underlying geometry. As we will see in detail, the respective scalar manifolds $M$ and $\bar{M}$ are precisely related by the geometrical construction of section 4.

### 7.1 The four-dimensional Lagrangian for general prepotentials

We start from a prepotential $F(X)$, which is $\epsilon$-holomorphic and homogenous of degree 2 in its $\epsilon$-complex variables $X^I$, where $I = 0, \ldots, n_V^{(4)}$. The supergravity variables $X^I$ are scalar fields which take values in the conical special $\epsilon$-Kähler manifold $M$, as we will see in more detail in section 7.3. They are the components of a map $\mathcal{X}$ from space-time $N$ into $M$, which is parametrised in terms of the (conical holomorphic) special coordinates introduced used in section 3:

$$X^I : N \xrightarrow{\mathcal{X}} M \xrightarrow{\phi^I} \mathbb{C}_\epsilon \; . \tag{7.2}$$

Here $\phi^I$ denotes the $I$-th coordinate map with respect to a system of (local conical holomorphic) special coordinates on $M$. For convenience we will follow common usage in the physics literature and refer to the fields $X^I$ simply as 'special coordinates on $M$'.

Derivatives of the prepotential with respect to the variables $X^I$ are denoted $F_I, F_{IJ}, \ldots$, and the $\epsilon$-complex conjugated quantities are denoted by $\bar{F}, \bar{F}_I, \ldots$. We define

$$Z^I = \frac{X^I}{X^0} \; , \tag{7.3}$$

[2] For completeness we mention that one further 'compensating' multiplet needs to be added, which, however, is not relevant for the purpose of this paper.

so that $Z^0 = 1$, while $Z^i, i = 1, \ldots, n_V^{(4)}$ 'are' special coordinates on the projective special $\epsilon$-Kähler manifold $\bar{M}$ defined by the prepotential.[3] The real and imaginary parts of $Z^i$ are denoted by $x^i$ and $y^i$ respectively:

$$Z^i = x^i + i_\epsilon y^i \ . \tag{7.4}$$

Using that $F$ is homogenous of degree 2 we define a 'rescaled, non-homogeneous prepotential' $\mathcal{F}(Z)$ by

$$F(X^0, X^1, \ldots) = (X^0)^2 F\left(1, \frac{X^1}{X^0}, \ldots\right) = (X^0)^2 \mathcal{F}(Z^1, \ldots, Z^n) \ .$$

Now we can rewrite $F$ and its derivatives in terms of special coordinates $Z^i$:

$$\begin{array}{lll} F(X) = (X^0)^2\mathcal{F}(Z) \ , & F_0(X) = X^0(2\mathcal{F} - Z^i\mathcal{F}_i) \ , & F_i(X) = X^0\mathcal{F}_i \ , \\ F_{ij}(X) = \mathcal{F}_{ij} \ , & F_{0i}(X) = \mathcal{F}_i - Z^j\mathcal{F}_{ij} \ , & F_{00}(X) = 2\mathcal{F} - 2Z^i\mathcal{F}_i + Z^iZ^j\mathcal{F}_{ij} \ . \end{array} \tag{7.5}$$

We use a notation where $\mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial Z^i}$, etc.

The metric $\bar{g}$ on $\bar{M}$ is given by

$$\bar{g}_{ij} = \frac{\partial^2 K}{\partial Z^i \partial \bar{Z}^j} \ ,$$

where

$$K = -\log Y \ , \quad Y = i_\epsilon \left( 2(\mathcal{F} - \bar{\mathcal{F}}) - (Z^i - \bar{Z}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i) \right) \tag{7.6}$$

is the $\epsilon$-Kähler potential. For $\epsilon = -1$ this is the standard formula for the Kähler potential of the metric on $\bar{M}$ in terms of special coordinates. We will verify in subsection 7.3 that this is an $\epsilon$-Kähler potential for the metric defined in section 4.

Following supergravity conventions, the metric $g$ of $M$ is given by the matrix

$$N_{IJ} = -i_\epsilon(F_{IJ} - \bar{F}_{IJ}) \ . \tag{7.7}$$

This quantity enters into the definition of the gauge field coupling matrix

$$\bar{\mathcal{N}}_{IJ} = F_{IJ}(X) + i_\epsilon \epsilon \frac{(N\bar{Z})_I (N\bar{Z})_J}{\bar{Z} N \bar{Z}} \ . \tag{7.8}$$

For $\epsilon = -1$ this agrees with the standard definition of $\mathcal{N}_{IJ}$ in the 'new conventions' of [14].[4] Now consider the following four-dimensional bosonic Lagrangian:

$$\mathbf{e}^{-1}\mathcal{L}^{(4)} = \frac{1}{2}R - \bar{g}_{ij}\partial_\mu Z^i \partial^\mu \bar{Z}^j + \frac{1}{4}\mathrm{Im}\mathcal{N}_{IJ}F^I \cdot F^J + \frac{1}{4}\mathrm{Re}\mathcal{N}_{IJ}F^I \cdot \tilde{F}^J \ , \tag{7.9}$$

[3] Here the same terminological simplification is applied as for the $X^I$.

[4] The matrices $\mathcal{N}_{IJ}$ and $\bar{\mathcal{N}}_{IJ}$ are related by $\epsilon$-complex conjugation.

where $F^I_{\mu\nu}$ are field strengths, and we suppressed the Lorentz indices in the Lagrangian. For $\epsilon = -1$ this is the bosonic part of the standard four-dimensional Lagrangian of $\mathcal{N} = 2$ supergravity coupled to vector multiplets [3], written in terms of the 'new conventions' of [14]. The bosonic Lagrangian for $\epsilon = -1$ can be found, for example, in [15] or [16].[5] For $\epsilon = 1$ we get the para-complex version of the standard Lagrangian. While we have only derived a bosonic Lagrangian here, it is known for $\epsilon = -1$, and expected for $\epsilon = 1$, that this is the bosonic part of an $\mathcal{N} = 2$ supersymmetric Lagrangian. The explicit study of fermionic terms for $\epsilon = 1$ is left to future work. Since for rigid Euclidean $\mathcal{N} = 2$ vector multiplets the full Lagrangian and supersymmetry rules were constructed in [1], it is clear that this is a straightforward task. Moreover, for prepotentials which can be obtained by dimensional reduction, the supersymmetry of the corresponding Lagrangian holds by construction.

### 7.2 Very special prepotentials and comparison to the dimensionally reduced Lagrangian

We will now show that for a suitable choice of prepotential the Lagrangian (7.9) takes the form of the Lagrangian (6.12), which we obtained by dimensional reduction. It is know from [12] that a space-like dimensional reduction from five to four dimension gives rise to a 'very special prepotential':

$$F(X) = \frac{1}{6} C_{ijk} \frac{X^i X^j X^k}{X^0} \,, \tag{7.10}$$

where $C_{ijk}$ are real. Such prepotentials are sometimes referred to as 'cubic', which alludes to the fact that they are in one-to-one correspondence with the cubic prepotentials of five-dimensional vector multiplets. We anticipate that the case of time-like reduction can be obtained by replacing the holomorphic coordinates $X^I$ by para-holomorphic coordinates.

To compare the Lagrangians (6.12) and (7.9) to one another, we need to compute the derivatives of a very special prepotential (7.10) with respect to the $X^I$:

$$\begin{aligned} &F_0 = -\tfrac{1}{6} C_{ijk} \tfrac{X^i X^j X^k}{(X^0)^2} \,, \quad F_i = \tfrac{1}{2} C_{ijk} \tfrac{X^j X^k}{X^0} \,, \quad F_{00} = \tfrac{1}{3} C_{ijk} \tfrac{X^i X^j X^k}{(X^0)^3} \,, \\ &F_{0i} = -\tfrac{1}{2} C_{ijk} \tfrac{X^j X^k}{(X^0)^2} \,, \qquad F_{ij} = C_{ijk} \tfrac{X^k}{X^0} \,. \end{aligned} \tag{7.11}$$

Using (7.5) we can replace the $X^I$ by the special coordinates $Z^i$ and obtain:

$$\begin{aligned} &\mathcal{F} = \tfrac{1}{6} CZZZ \,, \qquad \mathcal{F}_i = \tfrac{1}{2}(CZZ)_i \,, \quad \mathcal{F}_{ij} = (CZ)_{ij} \,, \\ &\mathcal{F}_{0i} = -\tfrac{1}{2}(CZZ)_i \,, \quad \mathcal{F}_{00} = \tfrac{1}{3} CZZZ \,, \end{aligned} \tag{7.12}$$

[5]Note that in these references the space-time Riemann tensor is defined with a relative minus sign compared to the definition used in this paper.

where we suppressed indices which are summed over. To compute the scalar metric, we need $Y$, where $K = -\log Y$ is the $\epsilon$-Kähler potential. The explicit expression for $Y$ is:

$$Y = \frac{i_\epsilon}{3}(CZZZ - C\bar{Z}\bar{Z}\bar{Z}) - \frac{i_\epsilon}{2}(Z-\bar{Z})^i(CZZ + C\bar{Z}\bar{Z})_i = -\frac{4}{3}Cyyy \ , \tag{7.13}$$

where $y^i$ is the imaginary part of $Z^i$. To compute the metric, the following form of $Y$ is convenient:

$$Y = -\frac{i_\epsilon}{6}C(Z-\bar{Z})(Z-\bar{Z})(Z-\bar{Z}) \ . \tag{7.14}$$

The resulting projective special $\epsilon$-Kähler metric is

$$\begin{aligned} g_{ij} &= \frac{\partial^2 K}{\partial Z^i \partial \bar{Z}^j} = \frac{6}{Y}C(Z-\bar{Z})_{ij} - \frac{9}{Y^2}C(Z-\bar{Z})(Z-\bar{Z})_i C(Z-\bar{Z})(Z-\bar{Z})_j \\ &= \epsilon\left(\frac{3}{2}\frac{Cy_{ij}}{Cyyy} - \frac{9}{4}\frac{Cyy_i Cyy_j}{(Cyyy)^2}\right) \ . \end{aligned} \tag{7.15}$$

Next, we evaluate the components of $N_{IJ} = -i_\epsilon(F_{IJ} - \bar{F}_{IJ})$:

$$N_{00} = -\frac{i_\epsilon}{3}(CZZZ - C\bar{Z}\bar{Z}\bar{Z}) \ , \quad N_{0i} = \frac{i_\epsilon}{2}(CZZ_i - C\bar{Z}\bar{Z}_i) \ , \quad N_{ij} = -i_\epsilon(CZ_{ij} - C\bar{Z}_i) \ . \tag{7.16}$$

For later use we compute

$$\begin{aligned} N_{0I}Z^I &= \frac{i_\epsilon}{6}CZZZ - \frac{i_\epsilon}{2}CZ\bar{Z}\bar{Z} + \frac{i_\epsilon}{3}C\bar{Z}\bar{Z}\bar{Z} \ , \\ N_{iI}Z^I &= -\frac{i_\epsilon}{2}CZZ_i + i_\epsilon CZ\bar{Z}_i - \frac{i_\epsilon}{2}C\bar{Z}\bar{Z}_i \ , \\ ZNZ &= -\frac{i_\epsilon}{3}\left(CZZZ - 3CZZ\bar{Z} + 3CZ\bar{Z}\bar{Z} - C\bar{Z}\bar{Z}\bar{Z}\right) \ . \end{aligned} \tag{7.17}$$

Note that for very special prepotentials we have

$$ZNZ = \bar{Z}N\bar{Z} = -2ZN\bar{Z} = 2Y \ . \tag{7.18}$$

Finally, we use our results to evaluate $\bar{\mathcal{N}}_{IJ}$:

$$\begin{aligned} &\bar{\mathcal{N}}_{00} = \tfrac{1}{3}Cxxx + i_\epsilon\epsilon\, Cyyy\left(\tfrac{2}{3}\bar{g}xx + \tfrac{1}{6}\right) \ , \quad \bar{\mathcal{N}}_{0i} = -\tfrac{1}{2}(Cxx)_i - \tfrac{2}{3}i_\epsilon\,\epsilon\, Cyyy\,(\bar{g}x)_i \ , \\ &\bar{\mathcal{N}}_{ij} = Cx_{ij} + \tfrac{2}{3}i_\epsilon\epsilon\, Cyyy\,\bar{g}_{ij} \ , \end{aligned} \tag{7.19}$$

where $\bar{g}_{ij}$ is the metric (7.15) and $\bar{g}xx$ and $(\bar{g}x)_i$ denote the obvious contractions.

We now have all the data required to compare (6.12) and (7.9) to one another. Using (6.13) and (7.15) together with (7.4) we see that the scalar terms agree for $C_{ijk} = \pm c_{ijk}$. To compare the gauge field terms we have to substitute the components of $\bar{\mathcal{N}}_{IJ}$ into (7.9):

$$\begin{aligned} \mathbf{e}^{-1}\mathcal{L}^{(4)}_{\text{gauge}} &= \frac{1}{4}\text{Im}\mathcal{N}_{00}F^0F^0 + \frac{1}{2}\text{Im}\mathcal{N}_{i0}F^iF^0 + \frac{1}{4}\text{Im}\mathcal{N}_{ij}F^iF^j \\ &\quad + \frac{1}{4}\text{Re}\mathcal{N}_{00}F^0\tilde{F}^0 + \frac{1}{2}\text{Re}\mathcal{N}_{i0}F^i\tilde{F}^0 + \frac{1}{4}\text{Re}\mathcal{N}_{ij}F^i\tilde{F}^j \\ &= -\epsilon\left(\frac{1}{4}Cyyy\left(\frac{1}{6} + \frac{2}{3}gxx\right)F^0F^0 - \frac{1}{3}\,Cyyy\,(gx)_iF^0F^i + \frac{1}{6}\,Cyyy\,g_{ij}\,F^iF^j\right) \\ &\quad + \frac{1}{12}\left(Cxxx\,F^0\tilde{F}^0 - 3(Cxx)_i\,F^i\tilde{F}^0 + 3(Cx)_{ij}\,F^i\tilde{F}^j\right) \ . \end{aligned} \tag{7.20}$$

Comparing this to the gauge field part of (6.12) we see that the gauge field terms match if we set $C_{ijk} = -c_{ijk}$.

### 7.3 Reformulation of the scalar sector as a gauged sigma-model

We now return to the case of a general ($\epsilon$-holomorphic and homogenous) prepotential and relate the formalism presented in subsection 7.1 to the geometrical construction of sections 2-5. At the same time we adapt those parts of the superconformal construction of vector multiplets which are relevant for the scalar term to the $\epsilon$-complex framework. For an introduction to the superconformal calculus and its use in constructing supergravity Lagrangians we refer the reader to [17]. A detailed review of the construction of the vector multiplet Lagrangian in this formalism is contained in [16], which also contains extensive references. A short summary of the relevant material can be found in [18].

The following diagram is useful in summarising the relevant spaces and maps:

$$\begin{tikzcd} & M \arrow[r, "\phi"] \arrow[d, shift left, "\pi"] & V' \arrow[r, "\subset"] \arrow[d, "\pi_V"] & \mathcal{U} \arrow[dl, "\pi_U"] \\ N \arrow[ur, "\mathcal{X}"] \arrow[r, "\mathcal{Z}"] & \bar{M} \arrow[r, "\bar{\phi}"] \arrow[u, shift left, "s"] & P(V') & \end{tikzcd} \tag{7.21}$$

Here $N$ is space-time, which is Riemannian or Lorentzian depending on $\epsilon$. $M$ is a regular conical special $\epsilon$-Kähler manifold (see Definition 7) and $\phi : M \to V' \subset V = T^*\mathbb{C}^{n+1} \simeq \mathbb{C}_\epsilon^{2n+2}$ is the conical holomorphic immersion of Theorem 2, which induces the holomorphic immersion $\bar{\phi} : \bar{M} \to P(V')$. The map $\mathcal{X} : N \to M$ is locally described by the $n+1$ $\epsilon$-complex scalar fields $X^I \circ \mathcal{X} : N \to \mathbb{C}_\epsilon$, where $X^I$, $I = 0, \ldots, n$, are special coordinates on $M$. Similarly, the induced map $\mathcal{Z} : N \to \bar{M}$ in the above commutative diagram is locally described by the $n$ scalar fields $Z^i \circ \mathcal{Z}$, where $Z^i$, $i = 1, \ldots, n$, are special coordinates on $\bar{M}$. As usual in the physical literature, we shall use a simplified notation where the scalar fields on $N$ are simply denoted by $X^I$ and $Z^i$, instead of $X^I \circ \mathcal{X}$ and $Z^i \circ \mathcal{Z}$. For a generic choice of the immersion, $\phi$ comes from a prepotential $F$, which is $\epsilon$-holomorphic and homogenous of degree two. The homogeneity condition is needed within the superconformal framework in order to couple the corresponding $n+1$ vector multiplets to conformal supergravity. Geometrically, it implies that locally $\phi$ maps $M$ into a Lagrangian cone in the $\epsilon$-complex symplectic vector space $V$. By dividing out the local group action generated by the commuting vector fields $\xi$ and $J\xi$ on $M$ (which corresponds to the $\mathbb{C}_\epsilon^*$-action in $V$ via the immersion $\phi$) one arrives at the projective special $\epsilon$-Kähler manifold $\bar{M}$. In supergravity $\mathbb{C}_\epsilon^*$ is a local gauge symmetry which is part of the superconformal group.[6] As we will see, the projection $\pi : M \to \bar{M}$ corresponds to

[6]This is known for $\epsilon = -1$, and we expect it to be true for $\epsilon = 1$ as well.

gauge-fixing this symmetry. Further, we have included in our diagram that the space of non-isotropic vectors $V'$ projects similary to the corresponding projective space $P(V')$ of non-isotropic lines, into which $\bar{M}$ is immersed by $\bar{\phi}$.

In this subsection we start from $M$ and obtain $\bar{M}$ and the corresponding sigma model by projection. For simplicity (and without restriction of generality), we shall assume that the immersion $\phi : M \to V'$ is an embedding of $M$ into a Lagrangian cone. In particular, this implies that the local group action generated by $\xi$ and $J\xi$ is induced from a global action of the group $\mathbb{C}_\epsilon^*$ on $M$. $M$ can be regarded as the total space of a $\mathbb{C}_\epsilon^*$-bundle over $\bar{M}$, and we can go from $\bar{M}$ to $M$ by choosing a section $s : \bar{M} \to M$ of this bundle. Moreover, there is a corresponding line bundle[7] $\pi_U : \mathcal{U} \to P(V')$ over $P(V')$. This is the so-called canonical line bundle introduced in section 5, which coincides with $V' \to P(V')$ on the image of $\phi$. This allows us to reinterpret various maps as sections of line bundles obtained as pull-backs of the universal bundle. We will come back to this fact in subsection 7.4, where we briefly relate our construction to an alternative formulation of special geometry, which makes extensive use of these sections.

We start by constructing a gauged sigma model with target space $M$, adapting the standard procedure used in the superconformal formalism to the $\epsilon$-complex framework. The $\epsilon$-complex scalars $X^I$ are subject to $\epsilon$-complex scale transformations, under which they transform as follows:

$$X^I \to \lambda X^I \; , \quad \lambda \in \mathbb{C}_\epsilon^* \; .$$

The group $\mathbb{C}_\epsilon^* = GL(1, \mathbb{C}_\epsilon)$ contains real dilatations, where $\lambda \in \mathbb{R}^{>0}$, and $U(1)_\epsilon$ gauge transformations, where $U(1)_\epsilon := \{z \in \mathbb{C}_\epsilon | z\bar{z} = 1\}$. The latter are chiral $U(1) = SO(2)$-transformations for $\epsilon = -1$ and chiral $\mathbb{R}^* = SO(1,1)$-transformations for $\epsilon = +1$. For $\epsilon = 1$ the group $\mathbb{C}_\epsilon^* = GL(1, \mathbb{C}_\epsilon) = \mathbb{R}^{>0} \times O(1,1) \supset GL^+(1, \mathbb{C}_\epsilon) = \mathbb{R}^{>0} \times SO(1,1) = \mathbb{R}^{>0} \times U(1)_\epsilon$ is obtained by removing all isotropic elements (i.e. the lightcone of the origin) from $\mathbb{C}_\epsilon$. It has four connected components. (The '+'-index stands for positive determinant of the representing real $2 \times 2$-matrix.) Comparing to section 3, we see that the dilatations are the homotheties generated by the vector field $\xi$, whereas the Killing vector field $J\xi$ generates the maximal connected subgroup in the group $U(1)_\epsilon$. In special holomorphic coordinates the homothety $\xi$ takes the form

$$\xi = X^I \frac{\partial}{\partial X^I} + \bar{X}^I \frac{\partial}{\partial \bar{X}^I} \; . \tag{7.22}$$

This expression follows from the one given in Lemma 1, section 3 by going from special affine to special holomorphic coordinates, while using that the prepotential is homogenous

[7] Here and in the following it is understood that 'line bundle' means '$\epsilon$-complex line bundle'.

of degree 2. By applying the $\epsilon$-complex structure tensor $J$ to (7.22) we obtain the following expression for the Killing vector field $J\xi$ in special holomorphic coordinates:

$$J\xi = i_\epsilon X^I \frac{\partial}{\partial X^I} - i_\epsilon \bar{X}^I \frac{\partial}{\partial \bar{X}^I} \; .$$

The conical affine special $\epsilon$-Kähler metric $g$ on $M$ is obtained from the prepotential $F(X)$ by

$$N_{IJ} = 2\mathrm{Im} F_{IJ} = -i_\epsilon (F_{IJ} - \bar{F}_{IJ}) \; . \tag{7.23}$$

Here and in the following we follow supergravity conventions and denote the matrix representing the metric $g$ in terms of holomorphic special coordinates by $N_{IJ}$. (More precisely, $g$ is the real part of the sesquilinear form $N_{IJ} dX^I \otimes d\bar{X}^J$.)

To write down a Lagrangian which is invariant under local $\mathbb{C}^*_\epsilon$-transformations, we introduce gauge fields $b_\mu$ for dilatations and $\mathcal{A}_\mu$ for $U(1)_\epsilon$ gauge transformations. The covariant derivatives of scalars are

$$\mathcal{D}_\mu X^I = (\partial_\mu - b_\mu + i_\epsilon \mathcal{A}_\mu) X^I \; , \quad \mathcal{D}_\mu \bar{X}^I = (\partial_\mu - b_\mu - i_\epsilon \mathcal{A}_\mu) \bar{X}^I \; . \tag{7.24}$$

Notice that homogeneous coordinates on projective space are not functions but are sections of the line bundle $\mathcal{U}^*$ which is dual to the universal bundle $\mathcal{U}$, discussed in 5. Correspondingly, the scalar fields $X^I$ are sections of the pull back of $\mathcal{U}^*$ to space-time $N$. It follows from this remark that $\mathcal{A}_\mu = -A_\mu$, where $A_\mu$ is the $U(1)_\epsilon$-connection one-form of the pull back of the universal bundle $\mathcal{U}$ to $N$ with respect to the section $(X^I, F_I)$, see 7.4 for a detailed discussion. Then the gauged non-linear sigma model is

$$\mathbf{e}^{-1} \mathcal{L}_{\text{scalar}} = -N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J \; .$$

It is instructive to consider the Einstein-Hilbert term in (7.9) alongside the scalar sigma model. The space-time metric is invariant under $U(1)_\epsilon$-transformations, but carries weight $-2$ under dilatations.[8] The Einstein-Hilbert action can be made invariant under dilatations by multiplying the Ricci scalar by a scalar field which acts as a compensator.[9] Adapting standard results from the superconformal calculus, we take the following locally $\mathbb{C}^*_\epsilon$-invariant Lagrangian $\mathcal{L}_{\text{grav+scalar}}$ as our starting point:

$$\mathbf{e}^{-1} \mathcal{L}_{\text{grav+scalar}} = -\frac{i_\epsilon}{2} (X^I \bar{F}_I - F_I \bar{X}^I) R - N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J \; . \tag{7.25}$$

[8] In the superconformal formalism, all fields transform under dilatations according to their weight. Here we use that the vielbein $e^a_\mu$ has weight $-1$ and that space-time coordinates have weight 0, see [17, 16, 18].

[9] This is a variant of the Stückelberg mechanism, which is an essential part of the superconformal formalism. See for example [17, 18].

Here the composite scalar $i_\epsilon(X^I\bar{F}_I - F_I\bar{X}^I)$ plays the role of the compensating field for the dilatations. We will show that we recover the scalar and gravitational terms of (7.9) by gauge fixing the $\mathbb{C}^*_\epsilon$ symmetry, which in turn amounts to implementing the quotient described in section 4.

At this point it is convenient to use a fact which is well known from the superconformal calculus: it is consistent to set $b_\mu = 0$ in (7.25), because the terms containing $b_\mu$ have to cancel anyway.[10] Next, the $U(1)_\epsilon$-gauge field $\mathcal{A}_\mu$ is non-dynamical and can be eliminated by its algebraic equation of motion,

$$\mathcal{A}_\mu = -A_\mu = \frac{1}{2}\frac{\bar{F}_I \overleftrightarrow{\partial}_\mu X^I - \bar{X}^I \overleftrightarrow{\partial}_\mu F_I}{i_\epsilon(F_I\bar{X}^I - \bar{F}_I X^I)}. \tag{7.26}$$

Notice that this coincides with the formula (5.4) for the Chern connection $A_\mu$ with respect to a unitary frame (for which automatically $b_\mu = 0$). For us it is useful to rewrite (7.26) in the form

$$i_\epsilon \mathcal{A}_\mu = -\frac{1}{2}\frac{N_{IJ}X^I \overleftrightarrow{\partial}_\mu \bar{X}^J}{-XN\bar{X}} .$$

Substituting this back into the Lagrangian, the scalar part becomes an 'ordinary' (rather than gauged) non-linear sigma model. For our purposes the following form of the result is convenient

$$\begin{aligned}-N_{IJ}\mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J &= -\left(N_{IJ} + \frac{(N\bar{X})_I (NX)_J}{-XN\bar{X}}\right)\partial_\mu X^I \partial^\mu \bar{X}^J \\ &\quad + \frac{1}{4}\frac{[\partial_\mu(XN\bar{X}) - X(\partial_\mu N)\bar{X}][(\partial^\mu X)N\bar{X} + XN(\partial^\mu \bar{X})]}{-XN\bar{X}} .\end{aligned} \tag{7.27}$$

The expression for the metric simplifies, after imposing a gauge condition which fixes the local dilatation symmetry. The natural gauge condition is[11]

$$i_\epsilon(X^I\bar{F}_I - F_I\bar{X}^I) = -1 , \tag{7.28}$$

because this turns the first term of (7.25) into the standard Einstein-Hilbert term:

$$\mathbf{e}^{-1}\mathcal{L}_{\text{grav}} = -\frac{i_\epsilon}{2}(X^I\bar{F}_I - F_I\bar{X}^I)R = \frac{1}{2}R . \tag{7.29}$$

To analyze the scalar term, first note that (7.28) is equivalent to

$$-N_{IJ}X^I\bar{X}^J = 1 .$$

Since the scalar fields are constrained to the hypersurface (7.28), it follows that

$$\partial_\mu(N_{IJ}X^I\bar{X}^J) = 0 .$$

[10] In the superconformal framework, the condition $b_\mu = 0$ is known as the K-gauge. We refer to [16, 18] for details. In particular, local dilatation invariance is discussed in section 2 of [18].

[11] This is known as the D-gauge in the superconformal literature.

Moreover, homogeneity of degree two of the prepotential implies $F_{IJK}X^K = 0$, and therefore

$$(\partial_\mu N_{IJ})X^I\bar{X}^J = 0 \ .$$

As a consequence the second line of (7.27) vanishes, and the scalar sigma model takes the following form after imposing the gauge condition (7.28):

$$\mathbf{e}^{-1}\mathcal{L}_{\rm scal} = -(N_{IJ} + (N\bar{X})_I(NX)_J)\partial_\mu X^I\partial^\mu\bar{X}^J =: -M_{IJ}\partial_\mu X^I\partial^\mu\bar{X}^J \ . \tag{7.30}$$

This is a sigma model with 'metric' $M_{IJ}$, which we need to relate to a sigma model with values in $\bar{M}$ and with metric $\bar{g}_{ij}$, as it occurs in (7.9).

At this point it is useful to connect our discussion with the construction of $\bar{M}$ used in section 4. First note that

$$i_\epsilon(X^I\bar{F}_I - F_I\bar{X}^I) = N_{IJ}X^I\bar{X}^J = g(\xi,\xi)$$

is the length-squared of the homothetic vector $\xi$. The gauge condition (7.28) sets $g(\xi,\xi) = -1$, which, according to section 4, defines a smooth hypersurface $S \subset M$. Moreover, since (7.28) is $U(1)_\epsilon$ invariant, it is manifest that the isometries generated by the Killing vector field $J\xi$ act on $S$, and we know from section 4 that the projective special $\epsilon$-Kähler manifold $\bar{M}$ is obtained by taking the quotient of $S$ by this isometry. We should therefore expect that $M_{IJ}$ is related to the tensor field $h$ defined in (4.3), which induces the $\epsilon$-Kähler metric on $\bar{M}$.

The following observation turns out to be helpful. The tensor field $h$ is defined on $M$, and while $M_{IJ}$ is originally defined on $S$, we can extend it to a tensor field on $M$ in the following way. Take the function

$$K = -\log\left(-i_\epsilon(X^I\bar{F}_I - F_I\bar{X}^I)\right) \ , \tag{7.31}$$

and define

$$\begin{aligned} M_{IJ} = \frac{\partial^2 K}{\partial X^I\partial\bar{X}^J} &= \frac{-i_\epsilon(F_{IJ}-\bar{F}_{IJ})}{i_\epsilon(F_K\bar{X}^K - X^K\bar{F}_K)} + \frac{i_\epsilon(F_{IL}-\bar{F}_{IJ})\bar{X}^L i_\epsilon(F_{JK}-\bar{F}_{JK})X^K}{[i_\epsilon(F_K\bar{X}^K - X^K\bar{F}_K)]^2} \\ &= \frac{N_{IJ}}{-XN\bar{X}} + \frac{(N\bar{X})_I(NX)_J}{[-XN\bar{X}]^2} \ . \end{aligned} \tag{7.32}$$

This coincides with the original $M_{IJ}$ defined in (7.30) when restricting to $S$, and can be shown to be proportional to the tensor field (4.3). In order to verify this we only have to use that the scalar product $g(U,V)$ of two vectors $U,V$ on $M$ is given by

$$g(U,V) = \frac{1}{2}\left(U^I N_{IJ}\bar{V}^J + V^I N_{IJ}\bar{U}^J\right) = \frac{1}{2}\left(UN\bar{V} + VN\bar{U}\right) \ .$$

Then it is straightforward to show that

$$h(U,V) = -\frac{1}{2}\left(UM\bar{V} + VM\bar{U}\right) \ ,$$

and therefore, up to an overall sign, $M_{IJ}$ is the representative of $h$ in special holomorphic coordinates. While $M_{IJ}$ can be obtained by taking the second derivatives of the '$\epsilon$-Kähler potential' (7.31), this tensor field is not a metric on $M$ because it is degenerate along the directions generated by the vector fields $\xi, J\xi$. This can be shown either by evaluating (4.3) on $\xi$ and $J\xi$ with the result

$$h(\xi,\xi) = h(\xi, J\xi) = h(J\xi, J\xi) = 0 \ ,$$

or by an equivalent calculation in local coordinates, using that

$$X^I M_{IJ} = 0 = M_{IJ}\bar{X}^J \ . \tag{7.33}$$

However, according to section 4 the tensor field $h$ projects onto a non-degenerate metric on $\bar{M}$, and therefore $M_{IJ}$ must be non-degenerate on the horizontal space of the submersion $\pi : M \to \bar{M}$. These directions are spanned by vectors which are orthogonal to the plane span$\{\xi, J\xi\}$ with respect to the (non-degenerate) metric $g$ on $M$. In local coordinates, vectors $W$ orthogonal to span$\{\xi, J\xi\}$ satisfy:

$$WN\bar{X} + XN\bar{W} = 0 \ ,$$

which implies

$$WM\bar{W} = \frac{WN\bar{W}}{-XN\bar{X}} \ . \tag{7.34}$$

Since $N_{IJ}$ is non-degenerate and $X^I N_{IJ}\bar{X}^J$ is non-vanishing, it is clear that $M_{IJ}$ is non-degenerate on the horizontal space. In fact from (7.32) and (7.34) we can easily read off the signature of $M_{IJ}$ on the horizontal space. $M_{IJ}$ is invariant under $N_{IJ} \to -N_{IJ}$, so that the signature of $M_{IJ}$ is independent of the overall sign of $N_{IJ}$. Now consider first $\epsilon = -1$, where $N_{IJ}$ is either positive definite or negative definite along the complex direction spanned by $\xi, J\xi$. Then, by inspection of (7.32) and (7.34), if $N_{IJ}$ is either positive or negative definite, then $M_{IJ}$ is negative definite on the horizontal space. However, for a supergravity theory in Lorentzian space-time we want $M_{IJ}$ to be positive definite along these directions, which can be arranged by taking $N_{IJ}$ to have signature $(2, 2n)$ or $(2n, 2)$.[12] Next, consider the case $\epsilon = 1$, where $N_{IJ}$ has always split signature $(n + 1, n + 1)$. Since the direction spanned by

[12]Thus the Kähler metric $N_{IJ}$ on $M$ must have indefinite signature. Such metrics are usually called pseudo-Kähler in the literature. In this paper we suppress the prefix 'pseudo-' most of the time, but we stress that all the results obtained for $\epsilon = -1$ apply irrespective of the metric being definite or indefinite.

$\xi, J\xi$ is para-complex, it has signature $(1,1)$, and therefore $M_{IJ}$ must have split signature $(n,n)$ on the horizontal space. Of course, this already follows from $\bar{M}$ being para-Kähler.

Imposing the gauge (7.28) has brought us from $M$ to the real hypersurface $S$ (a level set of the moment map $g(\xi,\xi)$), on which $U(1)_\epsilon$ acts isometrically. $\bar{M}$ is then obtained by taking the quotient of $S$ with respect to $U(1)_\epsilon$. This is precisely the $\epsilon$-Kähler quotient of $M$ with respect to the isometric and holomorphic $U(1)_\epsilon$-action. The submersion $M \to \bar{M}$ is $\epsilon$-holomorphic and a homothety on horizontal spaces, whereas $S \to \bar{M}$ is even a Riemannian submersion. The crucial point is that the vector field $J\xi$ on $M$ is not only Hamiltonian, which is sufficient to induce a symplectic structure on $\bar{M}$ (that is to perform the symplectic quotient), but that it is also a Killing vector field with respect to the $\epsilon$-Kähler metric. Therefore, $\bar{M}$ inherits not only a symplectic structure but also a pseudo-Riemannian metric. Combining the two yields the $\epsilon$-complex structure.

To descend from $S$ to $\bar{M}$ we could impose a condition which gauge-fixes the $U(1)_\epsilon$ transformations. However, it is more convenient to express everything in terms of $U(1)_\epsilon$-invariant objects. Therefore we introduce para-complex scalar fields,

$$Z^I = \frac{X^I}{X^0} , \qquad (7.35)$$

which are invariant under $\mathbb{C}^*_\epsilon$ and therefore in particular under $U(1)_\epsilon$. The $Z^I$ are defined on the open set where $X^0 \neq 0$. Note that $Z^0 = 1$, so that there are $n$ independent fields $Z^i$, which we will show to be the scalar fields in the Lagrangian (7.9). We remark that $X^0, Z^i$ provide local coordinates on $M$.

Using the homogeneity properties of the prepotential and the formulae (7.5) from section 7.1, we can rewrite (7.31) as a function of $X^0$ and $Z^i$:

$$K = -\log\left(2i_\epsilon(\mathcal{F} - \bar{\mathcal{F}}) - i_\epsilon(Z^i - \bar{Z}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)\right) - \log\left(X^0\bar{X}^0\right) . \qquad (7.36)$$

We now observe that the second term can be removed by a Kähler transformation. Therefore $M_{IJ}$ only depends on the $\mathbb{C}^*_\epsilon$-invariant variables $Z^i$. To obtain the metric $\bar{g}_{ij}$ we need to project $M_{IJ}$ onto $\bar{M}$. We take the $Z^i$ as coordinates on $\bar{M}$, and interprete the $X^I$ as functions of the $Z^i$, by picking a holomorphic non-vanishing function $h(Z)$ and setting $X^0 = h(Z)$. We can now pull back $M_{IJ}$ to $\bar{M}$, and the result does not depend on our choice of $h(Z)$, because changing this function amounts to a Kähler transformation. The resulting scalar Lagrangian is

$$\mathbf{e}^{-1}\mathcal{L}_{\text{scal}} = -\bar{g}_{ij}\partial_\mu Z^i \partial^\mu \bar{Z}^j , \quad \text{where} \quad \bar{g}_{ij} = \frac{\partial^2 K}{\partial Z^I \partial \bar{Z}^j} , \qquad (7.37)$$

with $K$ given by

$$K = -\log\left(2i_\epsilon(\mathcal{F} - \bar{\mathcal{F}}) - i_\epsilon(Z^i - \bar{Z}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)\right) . \qquad (7.38)$$

This agrees with the scalar term in (7.9), and completes the proof that the scalar and gravitational part of the Lagrangian (7.9) is gauge-equivalent to the Lagrangian (7.25). Moreover, it is clear that the signature of $\bar{g}_{ij}$ is the same as the signature of $M_{IJ}$ on the horizontal space. For $\epsilon = -1$, we have a theory with Lorentzian space-time, and therefore impose that $\bar{g}_{ij}$ is positive definite. Thus we need to choose the metric $g$ of $M$ such that it has signature $(2n, 2)$ or $(2, 2n)$. For $\epsilon = 1$ the metrics of both $M$ and $\bar{M}$ necessarily have split signature. The relevance of this feature will become clear when we discuss instanton solutions.

### 7.4 Reformulation in terms of line bundles

In the previous subsection we presented the field-theoretic implementation of the projection $\pi : M \to \bar{M}$ by adapting methods taken from the superconformal calculus. The special geometry of vector multiplets can be reformulated in various ways. One such reformulation, which is frequently used in the literature, focusses on $\bar{M}$ rather than $M$, and reinterprets various quantities which we already encountered as sections of a line bundle over $\bar{M}$ [19, 20, 21]. We refer the reader to [22] for a detailed review of $\mathcal{N} = 2$ supergravity in this formalism. In the following we will briefly indicate how our results can be expressed from this alternative point of view. Moreover, we will also provide a geometrical interpretation for the $U(1)_\epsilon$ gauge field $\mathcal{A}_\mu$ and of the associated covariant derivative $\mathcal{D}_\mu X^I = \partial_\mu X^I + i_\epsilon \mathcal{A}_\mu X^I$ introduced in subsection 7.3.

In order to proceed, it is useful to summarise the results of section 5 in the following diagram:

$$\begin{tikzcd}
\mathcal{U}^N \arrow[r] \arrow[d, shift left] & \mathcal{U}^{\bar{M}} \arrow[r] \arrow[d, shift left] & \mathcal{U}^M \arrow[r] \arrow[d, shift left] & \mathcal{U} \arrow[d, "\pi_U"] \\
N \arrow[u, shift left, "\mathcal{X}^*\phi"] \arrow[r, "\mathcal{Z}"] \arrow[rr, bend right, "\mathcal{X}"'] & \bar{M} \arrow[u, shift left, "s^*\phi"] \arrow[r, "s"] \arrow[rr, bend right, "\bar{\phi}"'] & M \arrow[u, shift left, "\phi"] \arrow[r, "\bar{\phi}\circ\pi=\pi_V\circ\phi"] & P(V')
\end{tikzcd}$$

Here $\pi_U : \mathcal{U} \to P(V')$ is the universal line bundle introduced in section 5. Since $M$ and $\bar{M}$ are mapped into $P(V')$ by $\bar{\phi} \circ \pi$ and by $\bar{\phi}$ respectively, one obtaines line bundles $\mathcal{U}^M$ over $M$ and $\mathcal{U}^{\bar{M}}$ over $\bar{M}$ by pulling back the universal line bundle. Space-time $N$ is mapped into $M$ and $\bar{M}$ by $\mathcal{X}$ and by $\mathcal{Z}$, respectively, so that one also obtains a line bundle $\mathcal{U}^N$ over $N$. The immersion $\phi : M \to V'$ can be interpreted as a section of $\mathcal{U}^M$, and sections of the line bundles $\mathcal{U}^{\bar{M}}$ and $\mathcal{U}^N$ are obtained by pull back. Finally, the universal line bundle comes equipped with the Chern connection $\mathcal{D}$ described in Lemma 2. Connections on the other lines bundles are obtained by pull back and are likewise denoted by $\mathcal{D}$. Notice that the canonical maps $\mathcal{U}^N \to \mathcal{U}^{\bar{M}} \to \mathcal{U}^M \to \mathcal{U}$ restrict to isomorphisms on the fibers.

To make contact with the supergravity formalism, we note that $\phi$, which can be interpreted as an $\epsilon$-holomorphic section of $\mathcal{U}^M$, takes the following form in terms of special coordinates:

$$\phi : (X^I) \to (X^I, F_I(X)) \ .$$

If we take a non-vanishing $\epsilon$-holomorphic section $s : \bar{M} \to M$ of the $\mathbb{C}^*_\epsilon$-bundle $\pi : M \to \bar{M}$, we can pull back $\phi$ to an $\epsilon$-holomorphic section $s^*\phi$ of $\mathcal{U}^{\bar{M}}$. If $\zeta^a$, $a = 1, \ldots, n$ are $\epsilon$-holomorphic coordinates on $\bar{M}$, then

$$\begin{aligned} s : (\zeta^a) &\to (X^I(\zeta)) \ , \\ s^*\phi : (\zeta^a) &\to (X^I(\zeta), F_I(\zeta)) \ . \end{aligned} \tag{7.39}$$

Finally, this pulls back to a section of $\mathcal{U}^N$, which takes the form

$$(s \circ \mathcal{Z})^*\phi : (x^\mu) \to (X^I(\zeta(x)), F_I(\zeta(x))) \ ,$$

where $x^\mu$ are coordinates on space-time $N$.

One particular choice of $\epsilon$-holomorphic coordinates on $\bar{M}$ are the special coordinates $Z^i = \frac{X^i}{X^0}$. In the previous subsection we found the expression (7.38) for the $\epsilon$-Kähler potential of the metric $\bar{g}$ of $\bar{M}$ in terms of special coordinates. We also noted that the $X^I$ could be interpreted as functions on $\bar{M}$ by picking (locally) a smooth non-vanishing function $h$ on $\bar{M}$ and setting $X^0 = h(Z)$. If we take this function to be $\epsilon$-holomorphic, then $s : (Z^i) \to X^I(Z)$ is an $\epsilon$-holomorphic section of $\pi : M \to \bar{M}$, expressed in terms of special coordinates. By an $\epsilon$-holomorphic change of coordinates we can go from the special coordinates $Z^i$ to general $\epsilon$-holomorphic coordinates $\zeta^a$. In terms of these the Kähler potential (7.38) takes the form

$$K = -\log\left(-i_\epsilon(X^I(\zeta)\bar{F}_I(\bar{\zeta}) - F_I(\zeta)\bar{X}^I(\bar{\zeta}))\right) \ , \tag{7.40}$$

where $s^*(\phi) : (\zeta^a) \to (X^I(\zeta), F_I(\zeta))$ is the $\epsilon$-holomorphic section of $U^{\bar{M}}$, which is obtained by pulling back $\phi$ using $s$. This can be rewritten in a coordinate free way as

$$K = -\log|\gamma(\phi(s), \phi(s))| \ ,$$

where $\gamma$ is the $\epsilon$-Hermitean form on $V = T^*\mathbb{C}^{n+1}$. Note that the resulting metric $\bar{g}$ on $\bar{M}$, which is given by

$$g_{ab} = \frac{\partial^2 K}{\partial\zeta^a \partial\bar{\zeta}^b} \ ,$$

does not depend on the choice of the section $s$. Locally, any other non-vanishing $\epsilon$-holomorphic section is of the form $e^f s$, where $f$ is an $\epsilon$-holomorphic function. Replacing $s$ by $e^f s$ changes the $\epsilon$-Kähler potential by a Kähler transformation, and therefore the metric is invariant.

Another useful quantity is

$$\partial_a K = -\frac{\partial_a X^I(\zeta)\bar{F}_I(\bar{\zeta}) - \partial_a F_I(\zeta)\bar{X}^I(\bar{\zeta})}{X^I(\zeta)\bar{F}_I(\bar{\zeta}) - F_I(\zeta)\bar{X}^I(\bar{\zeta})} \ . \tag{7.41}$$

By comparing to (5.3) we see that $\partial_a K = -i_\epsilon A^h_a$, where $i_\epsilon A^h_a$ is the connection one-form $i_\epsilon A^h_a$ of the Chern connection on $\mathcal{U}^{\bar{M}}$, evaluated on a holomorphic section. Therefore $\partial_a K$ is the connection one-form of the dual connection in the dual bundle with respect to the dual section $s^*$. In terms of coordinates the equivalent statement is that

$$\begin{aligned} \mathcal{D}_a X^I(\zeta) &= (\partial_a + (\partial_a K))X^I(\zeta) \ , \\ \mathcal{D}_{\overline{a}} X^I(\zeta) &= \partial_{\overline{a}} X^I(\zeta) (= 0) \ , \end{aligned} \tag{7.42}$$

is a covariant derivative with respect to $\epsilon$-holomorphic transformations $X^I \to e^f X^I$, where $f$ is an $\epsilon$-holomorphic function on $\bar{M}$. Here covariant derivative means that $\mathcal{D}_a X^I$ transforms homogenously, i.e.,

$$\mathcal{D}_a X^I \to e^f \mathcal{D}_a X^I \ .$$

The formulae (7.40), (7.41), (7.42) are the key formulae for expressing special geometry in terms of holomorphic sections of $\mathcal{U}^{\bar{M}}$. In particular, note that our expression (7.42) for the covariant derivative on the holomorphic line bundle agrees with the formula (4.17) of [22].

Another, closely related reformulation of special geometry is obtained by rewriting these formulae in terms of unitary sections. Given the holomorphic section $\phi : M \to \mathcal{U}^M$, we can obtain a unitary section $\phi_1$ by normalising it:

$$\phi_1 = \frac{\phi}{||\phi||} \ ,$$

where $||\phi|| = \sqrt{|\gamma(\phi,\phi)|}$. Since the D-gauge can be expressed as $\gamma(\phi,\phi) = -1$, the formalism based on unitary sections is closely related to the gauged sigma model discussed in subsection 7.1.

In terms of coordinates, a unitary section of $\mathcal{U}^{\bar{M}}$ is obtained from the holomorphic section $(\zeta^a) \to (X^I(\zeta), F_I(\zeta))$ by

$$(\zeta^a) \to (X^I, F_I) \ ,$$

where

$$X^I = e^{\frac{1}{2}K} X^I(\zeta) \ , \quad F_I = e^{\frac{1}{2}K} F_I(\zeta) \ ,$$

and where $K$ is the $\epsilon$-Kähler potential. Under holomorphic transformations $X^I(\zeta) \to e^f X^I(\zeta)$ the components of the unitary section transform by a $U(1)_\epsilon$ transformation:

$$X^I \to e^{i_\epsilon \mathrm{Im} f} X^I \ .$$

Therefore $(X^I, F_I)$ can also be interpreted as a section of the principal $U(1)_\epsilon$ bundle associated to the line bundle $\mathcal{U}^{\bar{M}}$. The covariant derivative (7.42) induces the $U(1)_\epsilon$-covariant derivative given by

$$\begin{aligned} \mathcal{D}_a X^I &= (\partial_a + (\frac{1}{2}\partial_a K))X^I \ , \\ \mathcal{D}_{\overline{a}} X^I &= (\partial_{\overline{a}} - (\frac{1}{2}\partial_{\overline{a}} K))X^I \ . \end{aligned} \tag{7.43}$$

By comparing to (5.4) we see that, up to sign, $\frac{1}{2}(\partial_a K, -\partial_{\overline{a}} K)$ is equal to the connection one-form $i_\epsilon A_a, i_\epsilon A_{\overline{a}}$ of the Chern connection evaluated on a unitary section of $\mathcal{U}^{\bar{M}}$. Therefore we find that $\frac{1}{2}(\partial_a K, -\partial_{\overline{a}} K)$ is again the connection one-form of the dual connection. This shows that the formulation of special geometry in terms of unitary sections can be obtained by replacing holomorphic sections of the pulled back universal bundle by the corresponding unitary sections. In particular, note that our formula (7.43) for $U(1)_\epsilon$-covariant derivatives agrees with the formula (4.15) of [22].

Finally, we would like to interprete the $U(1)_\epsilon$ gauge field $A_\mu$ of the gauged sigma model discussed in subsection 7.1 within this framework. Since $A_\mu$ is defined on space-time $N$, we need to consider the pullback $\mathcal{U}^N$ of the universal bundle to space-time $N$. Equation (7.26) expresses $A_\mu$ in terms of the pull back of the section $\phi$ of $\mathcal{U}^M$ to space-time $N$. Imposing the D-gauge amounts to taking a unitary section, which is equivalent to working with the associated $U(1)_\epsilon$-principal bundle. The pull back of the Chern connection to $\mathcal{U}^N$ evaluated on a unitary section is given by (5.6). Comparing this to (7.26), evaluated in the D-gauge, we see that the pull back of the Chern connection to $\mathcal{U}^N$ is dual to the $U(1)_\epsilon$-connection used in the gauged sigma model.

## 8. Scalar solutions of the Euclidean field equations

In this section we will discuss solutions of $\mathcal{N} = 2$ supergravity coupled to vector multiplets in four dimensions. The action is completely determined by the projective special $\epsilon$-Kähler target, which for simplicity from now on is denoted by $(M, g)$ instead of $(\bar{M}, \overline{g})$. We will restrict ourselves to solutions where all field strengths and all fermions are set to zero. The remaining fields are the metric and the scalar fields. If the action can be obtained from a five-dimensional action by dimensional reduction over time, then solutions of the Euclidean action lift to stationary solutions of the five-dimensional theory which involve the metric, the five-dimensional scalars, and the electric components of the five-dimensional gauge fields. The use of dimensional reduction over time as a solution generating technique dates back to [23], where it was applied to four-dimensional Einstein-Maxwell theory. Later, the method was adapted to construct four-dimensional black hole solutions in Kaluza-Klein theories [24]. Then this was extended to $p$-brane solutions [25], and it was realised, as reviewed in [26], that dimensional reduction and lifting provided a viable approach to generating and classifying solitonic solutions in string theory. More recently dimensional reduction over time has been used to explore extremal black holes (both supersymmetric and non-supersymmetric) [28, 6, 29].

In this section we give a self-contained account of the structure of the Euclidean field equations of scalars coupled to gravity, its relation to harmonic maps, and provide an overview of the classes of solutions which can be constructed through harmonic maps onto totally geodesic submanifolds of the scalar manifold. We give a coordinate-free definition of the relevant maps, which applies to the case where the totally geodesic submanifold is totally isotropic, and we analyse one family of symmetric spaces in detail. A concrete example chosen form this family is worked out in the two following sections.

After truncating out the gauge fields and the fermions of the four-dimensional Euclidean theory, the remaining field equations are the Euler-Lagrange equations of the following truncated action:

$$\begin{aligned} S &= \int d^4x \mathcal{L} = \int \mathrm{dvol}(h) \left( \frac{1}{2} R - \langle df, df \rangle \right) \\ &= \int d^4x \sqrt{|\det h|} \left( \frac{1}{2} R(h) - \sum g_{ab} \partial_\mu \zeta^a \partial^\mu \overline{\zeta}^b \right), \end{aligned} \tag{8.1}$$

where $R = R(h)$ stands for the scalar curvature of the space-time metric $h$ and the projective special $\epsilon$-Kähler metric $g = (g_{ab})$ is evaluated along the map $f : N \to M$. $\zeta^a$ are holomorphic coordinates on $M$.

**Proposition 11** *The Euler-Lagrange equations of (8.1) are given by the harmonic map equation for $f$*

$$\mathrm{tr} D df = 0$$

*and the Einstein equation*

$$Ric - \frac{1}{2} R h = T, \quad T = 2 f^* g - \langle df, df \rangle h,$$

*where $D$ is the covariant derivative induced by the Levi-Civita connections of the source and target manifolds of $f : N \to M$.*

In components, the harmonic map equations reads

$$\Delta_h \zeta^a + \sum \Gamma^a_{bc} \partial_\mu \zeta^b \partial^\mu \zeta^c = 0$$

and the energy momentum tensor

$$T_{\mu\nu} = \frac{-2}{\sqrt{|\det h|}} \frac{\delta \mathcal{L}}{\delta h^{\mu\nu}} = 2 \sum g_{ab} \partial_\mu \zeta^a \partial_\nu \overline{\zeta}^b - h_{\mu\nu} \sum g_{ab} \partial_\mu \zeta^a \partial^\mu \overline{\zeta}^b .$$

### 8.1 Analysis of the field equations

The harmonic map equation can be simplified if the target manifold possesses totally geodesic submanifolds. Let $\iota : M' \to (M, D)$ be an embedding of $M'$ into $M$, where $M$ is equipped with a connection $D$.

**Definition 8** *The embedding $\iota : M' \to (M, D)$ is called* totally geodesic *if for any two vector fields $X, Y$ which are tangent to $M'$ the covariant derivative $D_X Y$ is again tangent to $M'$.*

In this case the embedded submanifold $M'$ is called totally geodesic. Let $X_1, \ldots, X_n$ be a local frame for $M$ defined on a neighbourhood of a point $p \in M'$, such that the restriction of the vector fields $X_1, \ldots, X_m$ to $M'$ is a local frame for $M'$. Here $m$ and $n$ are the dimensions of $M'$ and $M$, respectively.

Then $M'$ is totally geodesic if the equation

$$D_{X_i} X_j = \sum_{k=1}^{m} \Gamma^k_{ij} X_k$$

holds along $M'$ for all $i, j \in \{1, \ldots, m\}$. If $M'$ is a totally geodesic submanifold, then the connection $D$ on $M$ induces a connection $D$ on $M'$ such that $Dd\iota = 0$. This can be verified by noting that in terms of the local frame $X_i$ the differential of $\iota$ takes the form

$$d\iota = \sum_{i=1}^{m} X_i^* \otimes X_i$$

where $X_i^*$ is the dual frame. Using the relation between the connection coefficients of the connection $D$ on $TM'$ and the dual connection on $T^*M'$, we find

$$D_{X_j} d\iota = \sum_{i,k=1}^{m} \left[ (-\Gamma^k_{ji} X_k^*) \otimes X_i + X_i^* \otimes \Gamma^k_{ji} X_k \right] = 0 \, .$$

If $(M, g)$ is pseudo-Riemannian with Levi-Civita connection $D$ and if $M'$ is a non-degenerate submanifold, then the induced connection $D$ on $M'$ coincides with the Levi-Civita connection of the induced metric $g_{|M'}$. Note that we have formulated the notion of totally geodesic embedding in sufficient generality in order to include isotropic submanifolds.

**Definition 9** *A smooth map $f : N \to M$ from a pseudo-Riemannian manifold $(N, h)$ to a manifold $M$ endowed with a connection $D$ is called* harmonic*, if it satisfies the harmonic map equation*

$$trDdf = \sum_i \varepsilon_i (D_{e_i} df)(e_i) = 0 \ ,$$

*where $D$ stands for the connection on $T^*N \otimes f^*TM$ induced by the Levi-Civita connection on $N$ and the connection $D$ on $M$, and the summation is over an orthonormal basis, such that $h(e_i, e_i) = \varepsilon_i$.*

**Proposition 12** *Let $\iota : M' \to M$ be a totally geodesic embedding. Then a map $\varphi : N \to M'$ is harmonic if and only if $f = \iota \circ \varphi : N \to M$ is harmonic.*

*Proof:* To see this we first note that the chain rule implies that

$$df = d(\iota \circ \varphi) = d\iota \circ d\varphi \ .$$

Given that $\iota$ is totally geodesic, the connection $D$ of $M$ and the Levi-Civita connection of $N$ induce connections on $T^*N \otimes f^*TM$, $T^*N \otimes \varphi^*TM'$ and $\varphi^*T^*M' \otimes f^*TM$, which we also denote by $D$, and which are compatible with the composition of maps between the underlying manifolds:

$$Ddf = D(d\iota) \circ d\varphi + d\iota \circ Dd\varphi = d\iota \circ Dd\varphi \ ,$$

since $\iota$ is totally geodesic.

This implies

$$\mathrm{tr}Ddf = d\iota \left( \mathrm{tr}Dd\varphi \right) \ .$$

which, by the injectivity of $d\iota$, shows that $f : N \to M$ is harmonic if and only if $\varphi : N \to M'$ is harmonic. □

This means that we can reduce the problem of solving the harmonic map equation for $f : N \to M$ to the following two problems:

1. Find all totally geodesic embeddings $\iota : M' \subset M$.
2. Solve the harmonic map equation for $\varphi : N \to M'$.

For instance, any totally geodesic embedding $\iota : N \to M$ defines a particular solution with $M' = N$ and $\varphi = id$. Another special case is to consider flat totally geodesic submanifolds $M' \subset M$. In that case the harmonic map equation for $\varphi : N \to M'$ reduces to a system of linear equations for the components of $\varphi$ with respect to affine coordinates $\sigma^a$, $a = 1, \ldots, m$, on $M'$:

$$\Delta_h \sigma^a = 0.$$

In the simplest case, the projective special $\epsilon$-Kähler manifold $M$ is a pseudo-Riemannian symmetric space $M = G/K$. For instance, we can take $G = G_1 \times G_2 = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_0(p+1, q+1)$, $K = K_1 \times K_2 = \mathrm{SO}_0(1,1) \times \mathrm{SO}_0(1,1) \times \mathrm{SO}_0(p,q)$, $M = M_1 \times M_2 = G_1/K_1 \times G_2/K_2$. For any symmetric space we have a so-called symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k},$$

where $\mathfrak{g} = Lie\, G$, $\mathfrak{k} = Lie\, K$ and the subspace $\mathfrak{m} \subset \mathfrak{g}$ is complementary to $\mathfrak{k}$. The pseudo-Riemannian metric of $M = G/K$ is completely determined by an $Ad_K$-invariant scalar product on $\mathfrak{m} \cong T_o M$, where $o = eK$ is the canonical base point. The corresponding curvature tensor is given by

$$R(X,Y) = -ad_{[X,Y]} : \mathfrak{m} \to \mathfrak{m} \, . \tag{8.2}$$

There is a one-to-one correspondence between (complete) totally geodesic submanifolds $M' \subset M$ and *Lie triple systems*, that is subspaces $\mathfrak{m}' \subset \mathfrak{m}$ such that

$$[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'.$$

Putting $\mathfrak{k}' := [\mathfrak{m}', \mathfrak{m}']$ one can easily check that

$$\mathfrak{g}' := \mathfrak{k}' + \mathfrak{m}' \subset \mathfrak{g} \tag{8.3}$$

is a Lie subalgebra and that (8.3) is again a symmetric decomposition. The corresponding symmetric submanifold $M' = G'/K' \subset M = G/K$ is totally geodesic. The induced connection of $M'$ coincides with the Levi-Civita connection of the induced metric, provided that the restriction of the metric of $M$ to $M'$ is nondegenerate. $M'$ is flat with respect to the induced connection if and only if

$$[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] = 0, \tag{8.4}$$

as follows from (8.2). The latter statement holds even for isotropic submanifolds. For Riemannian symmetric spaces (that is those with a positive definite metric) the condition (8.4) is equivalent to

$$[\mathfrak{m}', \mathfrak{m}'] = 0.$$

In that case $G' = M'$ is an Abelian Liegroup.

We have the following examples of totally geodesic submanifolds of $M_2 := \frac{\mathrm{SO}_0(p+1,q+1)}{\mathrm{SO}_0(1,1)\times \mathrm{SO}_0(p,q)}$:

$$\frac{\mathrm{SO}_0(p'+1, q'+1)}{\mathrm{SO}_0(1,1) \times \mathrm{SO}_0(p', q')}, \quad \frac{\mathrm{SO}_0(p', q'+1)}{\mathrm{SO}_0(p', q')} \times \frac{\mathrm{SO}_0(p''+1, q'')}{\mathrm{SO}_0(p'', q'')},$$

where $p' + p'' \leq p$ and $q' + q'' \leq q$. In particular,

$$\frac{\mathrm{SO}_0(p, q+1)}{\mathrm{SO}_0(p, q)} \quad \text{and} \quad \frac{\mathrm{SO}_0(p+1, q)}{\mathrm{SO}_0(p, q)}$$

are maximal totally geodesic submanifolds of non-zero constant curvature of $M_2$ and we have a totally geodesic Riemannian sphere $S^r \subset M_2$ and hyperbolic space $H^r \subset M_2$ of maximal dimension $r = \max(p, q)$.

A flat Lorentzian totally geodesic surface $M' \subset M_2 \subset M$ is given by

$$\mathfrak{m}' = \mathrm{span}\{e'_1 \otimes e''_1, e'_2 \otimes e''_2\} \subset \mathfrak{m}_2 = E' \otimes E'' \ ,$$

where $(e'_1, e'_2)$ is an orthonormal basis of $E' = \mathbb{R}^{1,1}$ and $(e''_1, e''_2)$ is an orthonormal basis of a two-dimensional nondegenerate subspace of $E'' = \mathbb{R}^{p,q}$.

A flat totally isotropic and totally geodesic submanifold of $M_2$ of maximal dimension is associated to the Lie triple system

$$\mathfrak{m}' = e' \otimes E'',$$

where $e' \in E'$ is a non-zero null vector.

Similarly, a flat totally isotropic and totally geodesic curve $M' \subset M_1 = \mathrm{SO}_0(1, 2)/\mathrm{SO}_0(1, 1) \subset M$ is given by

$$\mathfrak{m}' = e' \otimes E'' \subset \mathfrak{m}_1 = E' \otimes E'' = \mathbb{R}^{1,1} \otimes \mathbb{R}^{0,1}$$

where $e'$ is a non-zero null vector in $E' = \mathbb{R}^{1,1}$ and $E'' = \mathbb{R}^{0,1}$. The example discussed in the next section is of this type.

Next we analyse the Einstein equation

$$2f^*g = Ric - \frac{1}{2}Rh + \langle df, df\rangle h. \tag{8.5}$$

In two dimensions $Ric - \frac{1}{2}Rh = 0$, and the Einstein equation reduces to the statement that $f$ is conformal with conformal factor $\frac{1}{2}\langle df\ , df\rangle$. If the dimension of $N$ is $n \neq 2$, and under the assumption that $h$ is an Einstein metric, i.e. $Ric = \frac{R}{n}h$, (8.5) simplifies to

$$f^*g = \frac{1}{2}\left(\langle df, df\rangle - \frac{n-2}{2n}R\right)h. \tag{8.6}$$

**Proposition 13** *Let $(N, h)$ be an Einstein manifold of dimension $n > 2$ and $f$ a solution of (8.6). Then either*

1. *$\langle df, df\rangle = \frac{n-2}{2n}R$, in which case $Ric = 0$ and $f$ is totally isotropic or*
2. *$\langle df, df\rangle \neq \frac{n-2}{2n}R$ and $f$ is a conformal immersion with conformal factor $\frac{1}{2}(\langle df, df\rangle - \frac{n-2}{2n}R)$.*

*Proof:* 1. This follows from $f^*g = 0 \Longrightarrow \langle df, df\rangle = \mathrm{tr}_h f^*g = 0 \Longrightarrow R = 0 \Longrightarrow Ric = 0$.
2. Equation (8.6) shows that $f^*g$ is nondegenerate, hence that $f$ is an immersion. □

## 9. Instanton solutions of the Euclidean STU model

In this section we consider explicit instanton solutions for a particular choice of the prepotential in detail. This does not only illustrate the general results of the previous section, but also allows us to discuss various physical properties of Euclidean actions and their instanton solutions.

### 9.1 The Euclidean STU model

The model which we consider is the Euclidean version of the so-called STU model.[13] This is a model with three vector multiplets which arises from dimensional reduction of the heterotic string on $K3 \times T^2$.[14] We only consider the classical limit of this model, which contains the leading (tree-level) part in both the expansion in the string coupling $g_S$ and in the string scale $\sqrt{\alpha'}$. The corresponding prepotential is of the very special form (7.10) and can be obtained by starting with the effective Lagrangian of the compactification on $K3 \times S^1$ and reducing further on a circle. We arrive at the Euclidean STU-model by taking this circle to be time-like.

The prepotential of the STU model is obtained by setting $c_{123} = -C_{123} = -1$ in (7.10), while all other independent $C_{ijk}$ vanish. Following conventions used in the supergravity literature, we parametrise the scalar fields as follows:

$$S = \epsilon i_\epsilon z^1 \ , \quad T = \epsilon i_\epsilon z^2 \ , \quad U = \epsilon i_\epsilon z^3 \ . \tag{9.1}$$

The resulting $\epsilon$-Kähler potential takes the form

$$K = -\log\left((S+\bar{S})(T+\bar{T})(U+\bar{U})\right) \ . \tag{9.2}$$

For space-like compactifications this is a Kähler potential for the projective special Kähler manifold

$$M_{(\epsilon=-1)} = \left(\frac{SU(1,1)}{U(1)}\right)^3 = \left(\frac{SL(2,\mathbb{R})}{SO(2)}\right)^3 \ . \tag{9.3}$$

For time-like compactifications this becomes the projective special para-Kähler manifold

$$M_{(\epsilon=1)} = \left(\frac{SL(2,\mathbb{R})}{SO_0(1,1)}\right)^3 \ . \tag{9.4}$$

In the notation of section 8, this is of the form $M_1 \times M_2$, with $M_1 = SL(2,\mathbb{R})/SO_0(1,1)$ and $M_2 = SO_0(2,2)/(SO_0(1,1) \times SO_0(1,1)) \simeq SL(2,\mathbb{R})/SO_0(1,1) \times SL(2,\mathbb{R})/SO_0(1,1)$.

[13] Part of our results on the Euclidean STU model were reported already in the proceedings contribution [5]. The Euclidean STU model has also been studied in [6].

[14] This model also has hypermultiplets, which are not relevant for the following discussion.

Since the scalar manifold factorises, we can focus on a single factor $SL(2,\mathbb{R})/SO(2)$ or $SL(2,\mathbb{R})/SO_0(1,1)$. This is parametrised by one $\epsilon$-complex scalar field, which we take to be the field $S$ for definiteness. The corresponding sigma-model takes the form

$$\mathbf{e}^{-1}\mathcal{L}_S = -g_{S\bar{S}}\partial_\mu S\partial^\mu \bar{S} = -\frac{\partial_\mu S\partial^\mu \bar{S}}{(S+\bar{S})^2} \ . \tag{9.5}$$

For space-like compactifications we immediately recognize that the sigma model metric is proportional to the Poincaré metric on the upper half plane by setting $\tau = iS$.

It will turn out to be useful to decompose $S$ into its real and imaginary part. The real part of $S$ must be non-vanishing, and choosing it to be positive we set:

$$S = e^{-2\phi} + i_\epsilon a \ , \tag{9.6}$$

where $\phi$ and $a$ are real scalar fields. In heterotic string theory the field $S$ is the four-dimensional complex dilaton. Its real part is related to the four-dimensional heterotic string coupling $g_S$ by

$$e^{\langle\phi\rangle} = g_S \ , \tag{9.7}$$

where $\langle\phi\rangle$ is the vacuum expectation value of the real dilaton $\phi$. The Lagrangian (9.5) is invariant under shifts in the imaginary part $a$, which is called the universal string axion. This shift symmetry is preserved in perturbation theory, but broken by non-perturbative corrections. We will see explicitly that instanton solutions break the continuous shift symmetry to a discrete one. The permutation symmetry between the three $\epsilon$-complex scalar fields $S$, $T$ and $U$ is already broken by perturbative corrections. This implies that in the full theory the relation of the field $S$ to the string coupling is unambigous.

For later use we rewrite the sigma model Lagrangian for $S$ in terms of the real fields:

$$\mathbf{e}^{-1}\mathcal{L}_S = -\partial_\mu\phi\partial^\mu\phi - (-\epsilon)\frac{1}{4}e^{4\phi}\partial_\mu a\partial^\mu a \ . \tag{9.8}$$

### 9.2 Instantons in the scalar picture

We would like to find instanton solutions of the same type as the ten-dimensional IIB D-instanton [30] and the hypermultiplet instantons in type-II Calabi-Yau compactifications [7, 31, 8, 9]. As solutions of the bosonic field equations, such instantons are characterised by the property that the scalar fields have a non-trivial profile, while the gauge fields vanish and the metric is flat (in the Einstein frame[15]). Moreover, they have four Killing spinors and preserve $\frac{1}{2}$ of the Euclidean supersymmetry.

[15]This is the frame where the Einstein Hilbert term takes its 'usual' form, as in the previous sections. Other frames, such as the so-called string frame are obtained by conformal rescalings of the metric, with the conformal factor being a function of the scalar fields (usually the dilaton). We will discuss such other frames later on.

In this paper we have focussed on the bosonic part of the theory, and we did not derive the Euclidean supersymmetry transformations. However, the supersymmetry transformations for rigid Euclidean vector multiplets have been derived in [1], and one can check that for purely scalar backgrounds with a flat Einstein frame metric the conditions for the existence of Killing spinors are the same for rigidly and for locally supersymmetric vector multiplets. In the following we use the formalim of [1], take the supersymmetry parameters to be symplectic Majorana spinors, and work with para-complex linear combinations of spinors. In this notation, the condition for a purely scalar field configuration to be invariant under Euclidean supersymmetry is

$$\gamma^m \partial_m Z^i (\epsilon^a + ie\gamma^0 \epsilon^a) = 0 \ , \tag{9.9}$$

where $Z^i$ are the para-complex scalar fields corresponding to special coordinates, $\epsilon^a$ are the supersymmetry transformation parameters, and $a = 1, 2$ is the $SU(2)_R$ index.[16] When taking the $\epsilon^a$ to be eigenvectors of $i\gamma^0$, $i\gamma^0\epsilon^a = \pm\epsilon^a$, then field configurations of the form

$$\partial_m \mathrm{Re} Z^i = \pm \partial_m \mathrm{Im} Z^i \tag{9.10}$$

are $\frac{1}{2}$-BPS, i.e. they admit four independent Killing spinors. These field configurations are 'isotropic' in the sense that the scalar fields vary along an isotropic submanifold $M'$, and we will see below this condition implies that the energy-momentum tensor vanishes, which makes the assumption of a flat space-time metric consistent. Furthermore, the 'bulk' action (7.9) vanishes when evaluated on such solutions, thus raising the question of how to obtain a non-vanishing instanton action. We will come back to this question later. In the following we will restrict ourselves to solutions involving one para-complex scalar field. A discussion of more general solutions will be given in [33].

The Lagrangian (7.9) can be truncated consistently by setting all gauge field strengths to zero and two of the scalar fields, say $T$ and $U$, to constant values. To get a consistent solution with a flat space-time metric we must impose that the energy-momentum tensor vanishes. Since only the field $S$ is non-trivial, the relevant part of the energy momentum tensor is:

$$T^{(S)}_{\mu\nu} = -\frac{2}{\mathbf{e}} \frac{\delta \mathcal{L}}{\delta h^{\mu\nu}} = 2\partial_\mu\phi\partial_\nu\phi - \epsilon\frac{1}{2}e^{4\phi}\partial_\mu a \partial_\nu a - h_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi - \epsilon\frac{1}{4}e^{4\phi}\partial_\alpha a\partial^\alpha a\right) \ . \tag{9.11}$$

Now we take (9.8) with $\epsilon = 1$, set $h_{\mu\nu} = \delta_{\mu\nu}$, and obtain the following flat-space Euclidean scalar action for the dilaton:

$$S^{(\mathrm{indef})}_{(0,4)}[\phi, a] = \int d^4x \left(\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{4\phi}\partial_\mu a\partial^\mu a\right) \ . \tag{9.12}$$

[16] One can verify that this condition is related by dimensional lifting with respect to time to the Killing spinor equations of [32], which characterise supersymmetric static black holes in five dimensions.

For later convenience we have taken the Euclidean action to be *minus* the integral of the Euclidean Lagrangian. In the following we use a notation for actions which specifies the space-time signature ($(0,4)$ for Euclidean space, $(1,3)$ for Minkowski space) and whether the action is positive definite or indefinite.[17] The relation between the various actions which we consider in the following is summarised in Figure 1.

The equations of motion obtained by variation of (9.12) are:

$$\Delta\phi = -\frac{1}{2}e^{4\phi}\partial_\mu a\partial^\mu a \;, \tag{9.13}$$

$$\Delta a = -4\partial_\mu\phi\partial^\mu a \;. \tag{9.14}$$

Here $\Delta$ is the four-dimensional Laplace operator. Solutions of these equations are only solutions of the full theory defined by (7.9) if we impose the vanishing of (9.11) as a constraint:

$$T^{(S)}_{\mu\nu}\Big|_{h_{\mu\nu}=\delta_{\mu\nu}} = 2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}e^{4\phi}\partial_\mu a\partial_\nu a - \delta_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{4}e^{4\phi}\partial_\alpha a\partial^\alpha a\right) = 0 \;. \tag{9.15}$$

Similar constraints appear in the literature on extremal black hole solutions, where they are usually referred to as Hamiltonian constraints. Equation (9.15) is equivalent to

$$\partial_\mu\phi = \pm\frac{1}{2}e^{2\phi}\partial_\mu a\,, \tag{9.16}$$

where we take the same sign for all $\mu$. To see that (9.15) implies (9.16), one takes the trace of $T^{(S)}_{\mu\nu}$ to show that $\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{4}e^{4\phi}\partial_\alpha a\partial^\alpha a = 0$, which implies that $\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}e^{4\phi}\partial_\mu a\partial_\nu a = 0$ for all $\mu,\nu$. This shows that the four two-component vectors $(\partial_\mu\phi, \partial_\mu a)$ are isotropic an colinear.

We refer to (9.16) as the instanton ansatz. Since $S = e^{-2\phi} + ea$, the instanton ansatz implies the Euclidean $\frac{1}{2}$-BPS condition

$$\partial_m \mathrm{Re} S = \pm\partial_m \mathrm{Im} S \;,$$

and the resulting field configurations are supersymmetric.

Note that the instanton ansatz does not work in Minkowski signature, $\epsilon = -1$. In this case one would have to set $\partial_\mu\phi = \pm\frac{i}{2}e^{2\phi}\partial_\mu a$, both to obtain a vanishing energy-momentum tensor, and to have a supersymmetric field configuration. For real fields $\phi$ and $a$ this forces one to set all scalars to constant values, resulting in a vacuum solution. Instanton solutions of the type considered here require target spaces of indefinite signature, which allow non-constant scalar supersymmetric field configurations with vanishing energy-momentum tensor. The indefinite signature of the target space is an automatic consequence

[17] In Euclidean signature the label definite/indefinite refers to the action itself, in Minkowski signature it refers to the kinetic terms (the terms quadratic in the time derivatives).

of Euclidean supersymmetry. More precisely, the existence of an action which is invariant under Euclidean supersymmetry transformations requires for vector multiplets that the target space is special para-Kähler and hence has indefinite signature [1]. The indefiniteness of the Euclidean action is an unusual feature, which we will further investigate below. We now continue with solving the field equations.

Given that we impose the instanton ansatz, the system (9.13), (9.14) is reduced to

$$\Delta\phi + 2\partial_\mu\phi\partial^\mu\phi = 0 \, , \tag{9.17}$$

which is equivalent to

$$\Delta e^{2\phi} = 0 \, . \tag{9.18}$$

Thus by imposing the instanton ansatz and performing the field redefinition $\phi \to e^{2\phi}$, we have reduced the non-linear harmonic map equation to an ordinary harmonic equation on $\mathbb{R}^4$. This corresponds to the fact that $e^{2\phi}$ is the affine coordinate on the null geodesic $M' \subset M$. This solution illustrates one of the cases discussed in section 8, namely harmonic maps into flat totally isotropic and totally geodesic submanifolds of $M_1 \subset M$.

We have seen that the field $e^{2\phi}$ must be harmonic, while $a$ is fixed in terms of $\phi$ up to an integration constant. Single-instanton solution are obtained by further imposing spherical symmetry, which implies

$$e^{2\phi} = e^{2\phi_\infty} + \frac{C}{r^2} \, . \tag{9.19}$$

Here we use four-dimensional spherical coordinates, with $r$ as the radial variable. The string coupling at infinity $g_S = e^{\phi_\infty}$ can take any value $0 \leq g_S < \infty$. To obtain solutions where the real part of the field $S = e^{-2\phi} + ea$ is positive for positive $r$, we need to impose that the constant $C$ is non-negative. A vanishing $C$ corresponds to the trivial special case where the field $S$ is constant. We will see later that $C$ is proportional to the absolute value of the instanton charge. Multi-instanton solutions are obtained by choosing multi-centered harmonic functions.

In the single-centred case, the axion takes the following form:

$$a = \mp e^{-2\phi} + D = \mp \left( e^{2\phi_\infty} + \frac{C}{r^2} \right)^{-1} + D \, . \tag{9.20}$$

We will argue later that the integration constant $D$ should be chosen to be zero.

The solution (9.19) is singular at $r = 0$ which we interpret as the position of a source for the field $S$. In string theory pointlike objects localised in space and (Euclidean) time are called $(-1)$-branes. The most prominent example is the interpretation of the D-instanton of IIB supergravity as a D-$(-1)$-brane in type-IIB string theory [30]. While the geometry is flat in the Einstein frame, it takes the form of a wormhole in the string frame:

$$ds^2_{\text{String}} = e^{2\phi} ds^2_{\text{Einstein}} = \left( e^{2\phi_\infty} + \frac{C}{r^2} \right) \delta_{\mu\nu} dx^\mu dx^\nu = \left( e^{2\phi_\infty} + \frac{C}{r^2} \right) \left( dr^2 + r^2 d\Omega^2_{(3)} \right) . \tag{9.21}$$

This is a semi-infinite wormhole with a throat approaching a finite size for $r \to 0$. The asymptotic three-sphere at $r \to 0$ has radius $R = \sqrt{C}$ and volume $2\pi^2 C^{3/2}$. In contrast, the ten-dimensional D-instanton is a finite-neck wormhole, which approaches flat space for $r \to 0$ and has a minimal size for an intermediate 'critical' value of $r$, which corresponds to the fixed point set of the discrete isometry which exchanges the two asymptotic regimes. This difference between the four-dimensional and the ten-dimensioal case has nothing to do with the dimensionality but is caused by the different coupling of the axion to the dilaton. In four dimensions we could obtain a finite neck wormhole by replacing $e^{4\phi}$ by $e^{2\phi}$ in the Lagrangian (9.8) [7]. Instanton solutions supported by hypermultiplet scalars involve axions with both types of couplings to the dilaton, and the corresponding wormholes can be finite(-neck), semi-infinite or have a more complicated structure [7, 9].

Let us now point out some remarkable features of the instanton solution (9.16, 9.18, 9.20) and of the underlying Euclidean action (9.12).

- For an instanton we expect that the action is non-zero and proportional to $\frac{1}{g_S^2}$ (or proportional to $\frac{1}{g_S}$ for D-instantons). However, if we evaluate the action (9.12) on the instanton solution, we get zero.

- The Euclidean action (9.12) is indefinite: while the kinetic term for $\phi$ is positive definite, the kinetic term for $a$ has a relative minus sign and is negative definite. This is necessary for the existence of scalar instanton solutions, since it allows the energy momentum tensor to vanish on a non-trivial scalar field configuration. But it also implies that the action is not bounded from below, so that the functional integral measure defined by $\exp(-S[\phi, a]_E)$ is not damped.[18]

- The Euclidean action (9.12), and, more generally, the scalar part of (7.9), is different from the Euclidean action obtained by a Wick rotation of the corresponding Lorentzian action. Both differ by an analytic continuation in field space. Restricting our attention to the case of a single scalar field $S$, the Wick rotation of the Lorentzian

[18]The Euclidean Einstein-Hilbert action exhibits the same feature. This is known as the 'conformal factor problem', and we refer to [34] for a discussion of the problem and proposals of its solution. Leaving the Einstein-Hilbert term aside, one expects that the matter action is positive definite, as this seems to be required for a well-defined functional integral in the limit where gravity is decoupled.

version of (9.5) yields:

$$S^{(\mathrm{def})}_{(0,4)}[\phi, a] = \int d^4x \left( \partial_\mu \phi \partial^\mu \phi + \frac{1}{4} e^{4\phi} \partial_\mu a \partial^\mu a \right) . \tag{9.22}$$

This action is positive definite.[19]

To obtain the action (9.12) one needs to combine the Wick rotation with the analytic continuation $a \to ia$ of the axion. For more complicated target space geometries one has to perform an analytic continuation of all the axionic scalars.

These observations give rise to the question whether the 'correct' Euclidean action is the indefinite action (9.12) or the positive definite action (9.22) with its standard, positive definite scalar kinetic term. The answer depends on which properties of the Euclidean action we decide to insist on. Note that the Euclidean action obtained by Wick rotation also has some undesirable features:

- The instanton solution (9.16, 9.18, 9.20) is not a solution of the field equations of the Wick rotated action (9.22). This is clear, because the energy-momentum tensor obtained from the definite Euclidean action (9.22) has the same form as in Minkowski signature, namely

  $$T^{(S)}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} e^{4\phi} \partial_\mu a \partial_\nu a - \frac{1}{2} \delta_{\mu\nu} \left( \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{4} e^{4\phi} \partial_\alpha a \partial^\alpha a \right) .$$

  Then $T_{\mu\nu} = 0$ cannot be achieved when $\phi$ and $a$ are (non-constant) real fields. In other words, the instanton can only be realised as a complex rather than a real saddle point of the Wick rotated action.

- The action (9.22) cannot be extended to an action invariant under Euclidean supersymmetry. The dimensional reduction from five Lorentzian to four Euclidean dimensions preserves supersymmetry and leads to a scalar sigma model with split signature. In the rigid case it was shown that the split signature and para-complex (rather than complex) structure of the scalar manifold is determined by the subgroup $SO_0(1,1)$ of the R-symmetry group of the Euclidean supersymmetry algebra [1]. The same reasoning applies to the supergravity case.

The difference between (9.12) and (9.22) illustrates the general fact that dimensional reduction over space followed by Wick rotation is different from dimensional reduction over

[19] We take the Wick rotation to be $t \to -it$. The Minkowskian action $S$ and the rotated action $S_{\mathrm{Wick}}$ are related by $i\, S|_{t \to -it} = -S_{\mathrm{Wick}}$. With this convention Minkowski signature matter actions continue into positive definite Euclidean actions.

time. Similarly, Wick rotation and (Hodge-)dualisation do not commute. This brings into play a third type of Euclidean action, which can be obtained by dualising the axion field $a$ into a two-form gauge field $B_{\mu\nu}$. We will see that this leads to a Euclidean action for $\phi$ and $B_{\mu\nu}$ which is positive definite and has instanton solutions.

## 9.3 Instantons in the scalar-tensor picture

We start with the following Euclidean action:

$$S^{(\mathrm{def})}_{(0,4)}[\phi, B] = \int d^4x \left( \partial_\mu \phi \partial^\mu \phi + \frac{1}{2 \cdot 3!} e^{-4\phi} H_{\mu\alpha\beta} H^{\mu\alpha\beta} \right) . \qquad (9.23)$$

This action can be obtained in two ways. One way is to start from (9.12) and to dualise the axion field $a$ into an antisymmetric tensor field $B_{\mu\nu}$. We will investigate the relation between (9.23) and (9.12) in detail below. The second way to obtain (9.23) is to start from an $N = 2$ vector-tensor multiplet, to truncate it to the two fields $\phi$ and $B_{\mu\nu}$, and then to perform a Wick rotation.

Since any supersymmetric string theory contains the ten-dimensional metric $G_{MN}$, dilaton $\Phi$ and tensor field $B_{MN}$, the dimensionally reduced theory always contains the four-dimensional metric $g_{\mu\nu}$, dilaton $\phi$ and tensor field $B_{\mu\nu}$. In four dimensions $B_{\mu\nu}$ can be dualised into the universal axion $a$. However, there are subtleties when one wants to perform this dualisation while preserving off-shell $N = 2$ supersymmetry. One expects that the vector multiplet containing the dilaton $\phi$ and axion $a$ can be dualised into an $N = 2$ vector-tensor supermultiplet containing $\phi$ and $B_{\mu\nu}$ [14]. But though an off-shell description for vector-tensor multiplets is known, vector-tensor multiplets are only dual to vector multiplets when certain conditions are met [35, 36]. The off-shell dualisation of the dilaton vector multiplets is not possible if the prepotential depends linearly on the dilaton. Under dualisation, the off-shell dilaton vector multiplet mixes with the gravitational multiplet, which prevents one from identifying a dual off-shell vector-tensor multiplet. However, one can at least identify an on-shell heterotic dilaton vector-tensor multiplet when going to the Einstein frame. This is the vector-tensor multiplet we take as our starting point. More precisely we take the string frame Lagrangian (5.40) of [35], transform it to the Einstein frame, truncate it to the two fields $\phi$ and $B_{\mu\nu}$, and perform a Wick rotation. Modulo constant rescalings, the result is (9.23). Later we will dualise this part of the action back into an action involving two scalars.

The action (9.23) is positive definite and therefore $\exp(-S[\phi, B])$ could be used to define a functional measure which is damped. We will now find instanton solutions of (9.23), and then, by dualising (9.23) into (9.12) we will show that these instantons are identical to the ones found in section 9.2.

Since we want the solution to be consistent with a flat Euclidean space-time metric, we need to impose that the energy-momentum tensor vanishes when evaluated on the solution. Therefore we compute the energy-momentum tensor[20]:

$$T_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{-4\phi}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \delta_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{2\cdot 3!}e^{-4\phi}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}\right) . \tag{9.24}$$

To obtain a field configuration with $T_{\mu\nu} = 0$, we make the instanton ansatz

$$H_{\mu\nu\rho} = Ae^{2\phi}\epsilon^\alpha{}_{\mu\nu\rho}\partial_\alpha\phi , \tag{9.25}$$

where $A$ is a real constant. By a straightforward calculation we find

$$T_{\mu\nu} = \left(1 - \frac{1}{2}A^2\right)\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\right) .$$

This implies that $T_{\mu\nu} = 0$ if we choose $A^2 = 2$, i.e. $A = \pm\sqrt{2}$.

The equations of motion resulting from the action (9.23) are

$$\Delta\phi + \frac{1}{3!}e^{-4\phi}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0 , \tag{9.26}$$

$$\partial^\mu\left(e^{-4\phi}H_{\mu\nu\rho}\right) = 0 . \tag{9.27}$$

The equation (9.27) for $H_{\mu\nu\rho}$ is satisfied identically if we impose the ansatz (9.25). The equation (9.26) leads to the condition

$$\Delta e^{2\phi} = 0 , \tag{9.28}$$

which is identical to the equation (9.18) that we found in the scalar picture. Note that (9.28) follows already from the instanton ansatz (9.25), because the tensor field $H_{\mu\nu\rho}$ must satisfy the Bianchi identity $\epsilon^{\sigma\mu\nu\rho}\partial_\sigma H_{\mu\nu\rho} = 0$. Substitution of the instanton ansatz (9.25) into the Bianchi identity implies (9.28) due to the identity

$$\epsilon^{\sigma\mu\nu\rho}\epsilon_{\alpha\mu\nu\rho} = 3!\delta^\sigma_\alpha .$$

Conversely, by the same identity, (9.28) and the instanton ansatz (9.25) imply the Bianchi identity for $H_{\mu\nu\rho}$. The fact that upon imposing the instanton ansatz an equation of motion becomes equivalent to an Bianchi identity is analogous to Yang-Mills instantons. Also note that the instanton ansatz (9.25) can be viewed as a variant of the (anti-)self-duality constraint of Yang-Mills instantons. Apparently, these analogies between scalar instantons and Yang-Mills instantons become manifest in the scalar-tensor picture, because $B_{\mu\nu}$ is a gauge field.

[20] This is done by re-installing the space-time metric and varying it.

To find explicit solutions for $H_{\mu\nu\rho}$ one can choose any harmonic function for $e^{2\phi}$ and inserts the result into (9.25). This fixes $B_{\mu\nu}$ up to a closed two-form. Later, we will compare this to the solution (9.20) for the axion $a$.

We now compute the instanton action by inserting the scalar-tensor instanton solution back into (9.23). For any field configurations satisfying the instanton ansatz (9.25) we have

$$S = 2 \int d^4x \partial_\mu \phi \partial^\mu \phi \ , \tag{9.29}$$

i.e. the contributions of the two terms in the action (9.23) are equal. We can express $\partial_\mu \phi$ in terms of $e^{2\phi}$:

$$\partial_\mu \phi = \frac{1}{2} e^{-2\phi} \partial_\mu e^{2\phi} \ . \tag{9.30}$$

Since we evaluate the action on instanton configurations we can use that $\Delta e^{2\phi} = 0$:[21]

$$S[\phi, B]_{\text{inst.}} = 2 \int d^4x \frac{1}{4} e^{-4\phi} \partial_\mu e^{2\phi} \partial^\mu e^{2\phi} = \frac{1}{2} \int d^4x e^{-4\phi} \partial_\mu \left( e^{2\phi} \partial^\mu e^{2\phi} \right) \ . \tag{9.31}$$

This is a total derivative, up to terms which vanish for $\Delta e^{2\phi} = 0$:

$$S[\phi, B]_{\text{inst.}} = -\frac{1}{2} \int d^4x \partial_\mu \left( e^{-2\phi} \partial^\mu e^{2\phi} \right) \ . \tag{9.32}$$

We then use Stoke's theorem to write this as an integral over the boundary of the integration region

$$S = -\frac{1}{2} \oint d^3\Sigma_\mu e^{-2\phi} \partial^\mu e^{2\phi} = - \oint d^3\Sigma_\mu \partial^\mu \phi \ . \tag{9.33}$$

We evaluate this expression on a single instanton solution (9.19). Since the solution is singular at $r = 0$, the integration region is $\mathbb{R}^4 - \{0\}$, and the boundaries can be taken to be asymptotic three-spheres $S^3_r$ with $r \to \infty$ and $r \to 0$. For $r \to \infty$ the solution approaches a ground states, because $\phi$ goes to the constant value $\phi_\infty$. Since $e^\phi$ is the (field-dependent) heterotic string coupling, $e^{\phi_\infty}$ is the (constant) value $g_S$ of the heterotic string coupling in this ground state. With the specified boundaries, the instanton action is

$$S[\phi, B]_{\text{inst.}} = -\frac{1}{2} \lim_{R \to \infty} \oint_{S^3_R} d^3\Omega \, r^3 e^{-2\phi} \partial_r e^{2\phi} + \frac{1}{2} \lim_{R' \to 0} \oint_{S^3_{R'}} d^3\Omega \, r^3 e^{-2\phi} \partial_r e^{2\phi} \ . \tag{9.34}$$

We compute

$$r^3 e^{-2\phi} \partial_r e^{2\phi} = \frac{-2C}{e^{2\phi_\infty} + \frac{C}{r^2}} = \frac{-2Cr^2}{e^{2\phi_\infty} r^2 + C} \ . \tag{9.35}$$

This approaches a constant value for $r \to \infty$, but vanishes for $r \to 0$. The resulting instanton action is

$$S[\phi, B]_{\text{inst.}} = -\frac{1}{2} \Omega_3 \lim_{r \to \infty} \frac{-2C}{e^{2\phi_\infty} + \frac{C}{r^2}} = \Omega_3 C e^{-2\phi_\infty} \ , \tag{9.36}$$

[21] More precisely, we only require this for $r > 0$ and admit a source term at $r = 0$. As we will see below the boundary at $r = 0$ does not contribute to the integral.

where $\Omega_3 = 2\pi^2$ is the volume of the unit three-sphere. Using the relation between $\phi$ and the heterotic string coupling, we see the typical dependence of an instanton action on the coupling:

$$S[\phi, B]_{\text{inst.}} \sim \frac{1}{g_S^2} \ . \tag{9.37}$$

In fact the factor of proportionality is proportional to the absolute value of the instanton charge. To define the instanton charge, remember that the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} = 0$ for the field strength $H_{\mu\nu\rho}$ is violated in the presence of magnetic sources. The magnetic current is

$$j = \epsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma}$$

and the associated conserved charge is obtained by integrating $j$ over the full Euclidean space. As usual for gauge theories, this charge can be rewritten as a surface charge, because the current $j$ is a total derivative:

$$j = \partial_\mu Q^\mu \ , \quad Q^\mu = \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \ . \tag{9.38}$$

We define the instanton charge to be proportional to the magnetic charge obtained by integrating the magnetic current, and include a conventional factor for later convenience:

$$Q_{\text{inst}} = \frac{1}{\sqrt{2}\,3!}\int d^4x j = \frac{1}{\sqrt{2}\,3!}\oint d^3\Sigma_\mu \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \ . \tag{9.39}$$

We used Stoke's theorem to rewrite the volume integral as a surface integral, where the surface encloses all the magnetic charges. For multicentered harmonic functions, $j$ is a linear combination of delta functions concentrated at the centers. Here we restrict ourselves to single-centered instanton solutions. Using (9.25) we obtain:

$$Q_{\text{inst}} = \pm\frac{1}{2}\oint d^3\Sigma_\mu \partial^\mu e^{2\phi} \ . \tag{9.40}$$

The sign depends on the choice of the constant $A = \pm\sqrt{2}$ in (9.25). Let us take a single-instanton solution (9.19) and choose the surface to be the three-sphere of radius $R > 0$, centered at the singularity of the harmonic function:

$$Q_{\text{inst}} = \pm\frac{1}{2}\oint_{S_R^3} d^3\Sigma_\mu \partial^\mu e^{2\phi} = \pm\frac{1}{2}\Omega_3 R^3 \left(\partial_r e^{2\phi}\right)_{r=R} = \mp\Omega_3 C \ . \tag{9.41}$$

Remember that the constant $C$ must be positive, because we require that instanton solutions are regular outside $r = 0$. The constant $C$ is thus proportional to the instanton charge. Instantons with $Q_{\text{inst}} > 0$ correspond to taking $A = -\sqrt{2}$, while anti-instantons, i.e. solutions with $Q_{\text{inst}} < 0$ correspond to taking $A = \sqrt{2}$.

As in the case of type-IIB D-instanton, there are also dual solutions which carry electric charge with respect to $H_{\mu\nu\rho}$. This electric charge is related to the Noether current associated with the abelian two-form gauge symmetry $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}A_{\nu]}$. By Dirac quantisation generalised to $p$-form gauge fields the allowed spectrum of charges is therefore discrete. The sources of magnetic $B$-charge have a zero-dimensional Euclidean world volume and are therefore $(-1)$-branes. Their electric duals have a two-dimensional Euclidean world volume. To keep terminology consistent with using the term $(-1)$-branes for zero-dimensional Euclidean world volume, they should be called 1-branes. The analogous objects in ten-dimensional type-IIB string theory are D7-branes.

Using the instanton charge, we can now express the instanton action as:

$$S[\phi, B]_{\text{inst.}} = \frac{|Q_{\text{inst.}}|}{g_S^2} \,. \tag{9.42}$$

Next, we show that instanton solutions have minimal action for given charge. This is done by deriving a Bogomol'nyi bound. The action (9.23) is bounded from below by zero, and it can be re-written as the sum of a perfect square and a remainder:

$$S^{(\text{def})}_{(0,4)}[\phi, B] = \int d^4x \left( \partial_\mu \phi \pm \frac{1}{\sqrt{2 \cdot 3!}} e^{-2\phi} \epsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma} \right)^2 \mp 2 \int d^4x \frac{1}{\sqrt{23!}} \partial_\mu \phi e^{-2\phi} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \,. \tag{9.43}$$

The perfect square vanishes, if and only if we impose the instanton ansatz (9.25).

$$S^{(\text{def})}_{(0,4)}[\phi, B] \geq \mp 2 \int d^4x \frac{1}{\sqrt{23!}} \partial_\mu \phi e^{-2\phi} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} = 2 \int d^4x \partial_\mu \phi \partial^\mu \phi \,. \tag{9.44}$$

This observation provides an alternative way of deriving the instanton ansatz, instead of requiring $T_{\mu\nu} = 0$ or Euclidean supersymmetry. As noticed above, the instanton ansatz implies, when combined with the Bianchi identity, already the equations of motion. Hence

$$S[\phi, B]_{\text{inst}} = \frac{|Q_{\text{inst}}|}{g_S^2} \geq 0 \,, \tag{9.45}$$

which shows explicitly that instantons solutions have minimal action for given charge.

Let us summarise the properties of the scalar-tensor instanton (9.25), (9.18) and of the underlying Euclidean action (9.23):

- The action is positive definite.
- The instanton is a solution of the field equations, with finite, minimal action $\frac{|Q_{\text{inst}}|}{g_S^2}$.

We close this section by relating our results to the literature. The ten-dimensional D-instanton can also be obtained from a scalar-tensor Lagrangian [30]. The main difference is that the instanton action is proportional to $g_S^{-1}$ rather than to $g_S^{-2}$. This is due to a different coupling of the axion to the dilaton and is related to the different wormhole geometries obtained in the string frame: finite neck instantons have an action proportional to $g_S^{-1}$, while semi-infinite wormholes have action proportional to $g_S^{-2}$. These remarks also apply to instantons in the hypermultiplet sector of four-dimensional $N = 2$ compactifications [7, 9].

We would also like to mention that the bosonic action (9.23) coincides with the bosonic part of the action of an $N = 1$ tensor multiplet. In other words our scalar-tensor instanton solution can be interpreted as a solution of $N = 1$ supergravity, which coincides with the solution found in [37].

### 9.4 Back to the scalar picture

Let us now dualise the scalar-tensor action (9.23) and show that this leads to the scalar action (9.12), plus a boundary term accounting for the correct instanton action. As a by-product we will see that the instanton solutions obtained from both actions are indeed identical.

The dualisation proceeds in the standard way. First we promote the Bianchi identity of $H_{\mu\nu\rho}$ to a field equation by introducing a Lagrange multiplier field $a$:

$$\hat{S}[\phi, H, a] = \int d^4x \left( \partial_\mu \phi \partial^\mu \phi + \frac{1}{2 \cdot 3!} e^{-4\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + \lambda a \epsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma} \right) . \tag{9.46}$$

Here $\lambda$ is a *real* constant, which we will fix later by imposing that the axion is normalised in the same way as in (9.12). The dualisation proceeds by eliminating the field $H_{\mu\nu\rho}$, which can now be treated as an independent tensor field, by its equation of motion. This entails that we have to integrate the third term in the above action by parts. Following the analogous analysis of the type-IIB D-instanton [30], we keep the resulting boundary term, despite that it does not contribute to the equations of motion.

We can now eliminate $H_{\mu\nu\rho}$ by its equation of motion

$$H_{\nu\rho\sigma} = 3! \lambda e^{4\phi} \epsilon_{\mu\nu\rho\sigma} \partial^\mu a \, . \tag{9.47}$$

Substituting this back into (9.46), and performing an integration by parts on the last term we obtain

$$\hat{S}[\phi, a] = \hat{S}_{\text{bulk}}[\phi, a] + \hat{S}_{\text{bound}}[\phi, a] \, . \tag{9.48}$$

The bulk term has the form

$$\hat{S}[\phi, a]_{\text{bulk}} = \int d^4x \left( \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} (3! \lambda)^2 e^{4\phi} \partial_\mu a \partial^\mu a \right) . \tag{9.49}$$

If we choose $\lambda^2 = \frac{1}{2} \cdot \frac{1}{(3!)^2}$, this agrees with (9.12):

$$\hat{S}[\phi, a]_{\text{bulk}} = S[\phi, a]^{(\text{def})}_{(0,4)} \, . \tag{9.50}$$

By combining (9.47) with (9.25) we obtain (9.16). Since we already saw that the condition (9.18) on $\phi$ is the same for both solutions, it follows that the two instanton solutions are the same.

The boundary term of $\hat{S}[\phi, a]$ is

$$\hat{S}_{\text{bound}}[\phi, a] = (3!\lambda)^2 \oint d^3\Sigma_\mu a \partial^\mu a e^{4\phi} . \tag{9.51}$$

If we set $(3!\lambda)^2 = \frac{1}{2}$, and evaluate the boundary term on the instanton solution (9.18), (9.20) we obtain

$$\begin{aligned} \hat{S}_{\text{bound}}[\phi, a] &= \frac{1}{2} \oint d^3\Sigma_\mu e^{2\phi} \partial^\mu e^{-2\phi} \mp \frac{D}{2} \oint d^3\Sigma_\mu e^{4\phi} \partial^\mu e^{-2\phi} \\ &= \hat{S}_{\text{inst.}} \pm \frac{\Omega_3 D}{2} , \end{aligned} \tag{9.52}$$

where $\hat{S}_{\text{inst.}}$ is the instanton action, $\Omega_3$ is the volume of the unit three-sphere, and $D$ is the integration constant in the solution (9.20) for the axion. When comparing to (9.32), (9.33), it is useful to note that

$$\frac{1}{2} \oint d^3\Sigma^\mu e^{2\phi} \partial_\mu e^{-2\phi} = - \oint d^3\Sigma^\mu \partial_\mu \phi = -\frac{1}{2} \oint d^3\Sigma^\mu e^{-2\phi} \partial_\mu e^{2\phi} .$$

Thus the boundary action gives precisely the instanton action, provided we set the integration constant $D = 0$. We have no other way of fixing this integration constant, since the axion only enters into the bulk action and into the equations of motion through its first derivatives. Thus there is no obvious contradiction in setting $D = 0$. When we add the boundary term to the bulk action (or, in other words, if we keep it after dualisation), then the improved action

$$\hat{S}_{(0,4)}[\phi, a] = S[\phi, a]^{(\text{indef})}_{(0,4)} + \hat{S}_{\text{bound}}[\phi, a]$$

agrees with the scalar-tensor action $S[\phi, B]^{(\text{def})}_{(0,4)}$ when evaluated on instanton solutions. However, the improved action also has one feature which is different from the scalar-tensor action. Since the boundary term contains the axion field $a$ explicitly, the axionic shift symmetry is broken, in contrast to the manifest gauge invariance of the $B$-field in the scalar-tensor action. At the classical level the breaking of axionic shift invariance by the boundary term is not an issue, because this term does not contribute to the equations of motion. The implications on the quantum theory need to be investigate in a different set-up, e.g., by the investigation of instanton corrections to quantum transition amplitudes. This will be left to future work. Also note that there are other boundary terms which evaluate to the correct instanton action but do not break axionic shift symmetry. Explicit examples will be given when we consider the dimensional lifting of instanton solutions to black holes.

We should also provide an interpretation for the instanton charge in the scalar picture. Since the tensor field $B_{\mu\nu}$ and the axion $a$ are related by Hodge duality, magnetic (electric) $B$-charge corresponds to electric (magnetic) charge for the $a$-field. A non-vanishing 'electric' charge density with respect to the axionic shift symmetry $a \to a + \text{const.}$ corresponds to adding a source term to the equation of motion for $a$:

$$j = \partial^\mu \left( e^{4\phi} \partial_\mu a \right) . \tag{9.53}$$

For instanton solutions a delta-function type charge density is located at the centers of the harmonic functions. This density is indeed proportional to the 'magnetic' density associated with the tensor field $B_{\mu\nu}$, as expected. The associated charge is obtained by integration over four-dimensional space. Since the density is a total derivative, it can be rewritten as a surface charge, which we can normalise such that it is equal to the instanton charge (9.39):

$$Q_{\text{inst.}} = \frac{1}{2} \lim_{r\to\infty} \oint_{S_r^3} d^3\Sigma_\mu e^{4\phi} \partial^\mu a . \tag{9.54}$$

### 9.5 Discussion of instantons, Euclidean actions and boundary terms

One particular feature of the Euclidean action (9.12) is that it is indefinite. While this is necessary for the existence of instanton solutions, it prevents us from using the expontial of the action $\exp\left(-S[\phi,a]^{(\text{indef})}_{(0,4)}\right)$ to define a functional measure. Here the natural candidate is the definite action (9.22), which leads to a damped measure factor $\exp\left(-S[\phi,a]^{(\text{def})}_{(0,4)}\right)$, but does not have instanton solutions. Thus regarding instanton corrections at the quantum level, we seem to be stuck with two actions which both are deficient. This problem is not unique to our class of models, but occurs generally if one wants to construct non-trivial Euclidean finite action solutions involving axionic scalars. Examples which have been discussed previously in the literature include scalar field wormholes [38], the D-instanton solution of type-IIB supergravity [30], and instanton solutions involving hypermultiplets [7, 9, 10].

Since the scalar-tensor action (9.23) is both positive definite and has instanton solutions, one option is to base the quantum theory exclusively on it. There are several potential problems with this. One is that the complete theory involves vector multiplets or vector-tensor multiplets, and, as already mentioned there are problems and subtleties with the Hodge dualisation of the full supermultiplets. Another, more general point is the question whether and how precisely the duality between axions and antisymmetric tensors works at the quantum level. This cannot be answered by just looking at actions, but requires the investigation of instanton contributions to quantum amplitudes. Studies performed on similar models in the literature show that boundary conditions play an important role [38, 39, 10]. A central question is the fate of the axionic shift symmetry, which corresponds to the gauge symmetry of the $B$-field under duality. Here we encounter an asymmetry between the scalar picture and the scalar-tensor picture. The boundary term generated in the dualisation, which is needed to obtain the correct instanton action, contains the axion explicitly and breaks the axionic shift symmetry. The corresponding measure factor $\exp\left(-S[\phi,a]^{(\text{indef})}_{(0,4)} - \hat{S}_{\text{bound}}[\phi,a]\right)$ is still invariant under discrete imaginary shifts. In contrast, the corresponding gauge symmetry in the scalar-tensor picture cannot be broken. The general expectation is that instanton effects break the continuous axionic shift symmetry to a discrete subset, and it is not obvious how this can be expressed in the scalar-tensor picture. Therefore, a better understanding of the scalar picture and of its relation to the scalar-tensor picture is required. Note that it is not completely clear to us whether the boundary term found by dualisation is responsible for the expected breaking of axionic shift symmetries in the quantum theory. As we will see later, one can motivate other boundary terms, which provide the correct instanton action, but do not break the axionic shift symmetry. Within the classical realm, we are not aware of a criterion which could allow us to single out one of these candidate boundary terms as the correct one.

While the investigation of quantum amplitudes is left to future work, we can already make a few observations. The two scalar actions are related by analytic continuation of the axion, but so far we have only related the indefinite scalar action directly to the scalar-tensor action. Let us now display the dualised action, including the boundary term, without fixing the parameter $\lambda$:

$$\begin{aligned}\hat{S}[\phi,a] = &\int d^4x \left(\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{4\phi}(3!\lambda)^2\partial_\mu a\partial^\mu a\right) \\ &+ (3!\lambda)^2 \oint d^3\Sigma_\mu \left(a\partial^\mu a e^{4\phi}\right) \, . \end{aligned} \tag{9.55}$$

Here it is manifest that for real $\lambda$ with $(3!\lambda)^2 = \frac{1}{2}$ the bulk terms equals the indefinite scalar action (9.12), while for imaginary $\lambda$ with $(3!\lambda)^2 = -\frac{1}{2}$ we obtain the definite scalar action (9.22). Thus we can either preserve the saddle points of the scalar-tensor action by choosing $\lambda$ real, or preserve its definiteness by choosing $\lambda$ imaginary, but not both. The choice of an imaginary Lagrange multiplier is unconventional from the classical point of view, because it does not preserve the equations of motion, but natural within the context of Euclidean functional integrals, because it corresponds to implementing the Bianchi identity for the $B$-field through a functional delta function. It is a particular feature of Euclidean signature that definiteness and saddle points cannot be preserved simultanously. In Minkowski signature a real Lagrange multiplier preserves both properties, and corresponds to implementing the Bianchi identity through a functional delta function.

Thus the definite scalar action seems to be correct choice for defining the quantum theory dual to the scalar-tensor theory. While the instanton is not a saddle point in the strict sense, it can be regarded as a complex saddle point, and there are several examples of path integrals and functional integrals in quantum mechanics and quantum field theory which are dominated by complex saddle points [38, 39]. In this interpretation both scalar actions play a role: the definite action defines the measure, the indefinite action identifies the saddle point. In fact, it is convenient to regard $a$ as a complex field, and to view the two real scalar actions as arising from a single complex scalar action. Note that not only the bulk actions but also the boundary actions obtained for real and imaginary $\lambda$ respectively, are related by the analytic continuation $a \to ia$. The boundary term is needed to obtain the correct instanton action, irrespective of whether we work with the definite or the indefinite real action.

Since $a$ and $\phi$ are in the same supermultiplet, we could also promote $\phi$ (and all the other fields which have been truncated out) to complex fields, and view different real Euclidean actions as different real forms of one underlying complex 'master action'. In the scalar sector this corresponds to the complexification of the (pseudo-)Riemannian target space, resulting in a complex-Riemannian space. From such a 'complex point of view' it is natural to work with complex saddle points. A necessary and sufficient condition for a pseudo-Riemannian manifold to admit a complexification is that the manifold and the metric are *real analytic*. Contrary to complex manifolds, which are complex analytic and a fortiori real analytic, para-complex manifolds are not always analytic. Therefore it is not possible to obtain every para-Kähler manifold as a real section of a complex-Riemannian manifold. This implies that a para-Kähler manifold cannot be in general Wick rotated into a (pseudo-)Kähler manifold. Viewing target space geometries, which are related by dimensional reduction over either space or time, or by analytic continuation of axions, as different real sections of one underlying complex space should lead to a more unified picture of instantons, solitons and other solutions of supergravity theories, since these are often related by analytical continuation in either time or target space. We also expect that the relation between Minkowskian and Euclidean supersymmetry, and their relation to the concepts of pseudo- or fake-supersymmetry can be understood systematically in such a framework. A similar point of view has been taken recently with regards to ten-dimensional supergravity in [41].

### 9.6 Summary of the relation between actions

In this subsection we summarise the properties and mutual relations between the various actions occuring in this paper. For concreteness we refer to the truncated version of the actions, which contains one scalar together with one axion, or tensor field or (five-dimensional) gauge field. However, the same properties and relations hold between the complete supersymmetric actions (modulo subtleties with regards to the off-shell dualisation of vector multiplets into vector-tensor multiplets).

Figure 1 only involves the actions which we actually encountered in previous sections. For each action the fields are specified. All actions contain a scalar $\phi$ while the second field is either a five-dimensional gauge field $A = A_{\hat{\mu}}$, or an axion $a$ or an antisymmetric tensor field $B = B_{mn}$. For each action the space-time dimension and signature is specified as a lower label. For Euclidean actions an additional upper label provides the information whether the action positive definite or indefinite.

The basic operations relating the actions are: dimensional reduction/lifting with respect to space or time, denoted $D_S$, $D_T$, respectively, Wick rotation between Minkowksi space and Euclidean space, denoted $W$, and Hodge dualisation between an axion and an antisymmetric tensor, denoted $H$. As apparent from the diagram, all actions can be obtained by composing these basic operation. There are two Euclidean actions involving $\phi$ and $a$, for two related reasons: (i) in Euclidean signature, Wick rotation and Hodge dualisation do not commute, and (ii) dimensional reduction over space followed by Wick rotation gives a result different from reduction over space. We have also displayed further maps between the actions, which are equivalent to compositions of the basic operations $D_S, D_T, H, W$. These are the analytic continuation of scalars $a \to ia$, denoted $A$, the modified Wick rotation $W'$ (which combines analytic continuation of time with analytic contiunation of axions), and the modified Hodge dualisation $H'$, which uses an imaginary Lagrange multiplier and thus combines Hodge dualisation with analytic continuation of axions. The first diagram is not complete, in the sense that further actions can be obtained by composing the basic transformations in different order. For completeness we present a second diagram 2 which contains all the eight four-dimensional actions which can be obtained this way. In this extended diagram the Minkowski space actions also carry a label def/indef, which specifies whether the kinetic terms (the terms quadratic in the first time derivatives) are definite or indefinite. For actions involving an axion this label specifies whether the target space metric is definite or indefinite. Lorentzian signature actions with indefinite target space geometries occur in string theory when performing T-duality transformations along a time-like direction [42]. A particular example is provided by the II* string theories. The existence of precisely eight different four-dimensional actions reflects a three-fold binary alternative: the action can either contain an axion or an antisymmetric tensor, space-time signature can be Euclidean or Minkowskian, the action (for Minkowski signature, its kinetic terms) can be definite or indefinite. From the diagram it is clear that

$$
\begin{tikzcd}
& S_{(1,4)}[\phi, A] \arrow[dl, leftrightarrow, "D_S"'] \arrow[dr, leftrightarrow, "D_T"] & \\
S_{(1,3)}[\phi, a] \arrow[rr, leftrightarrow, "W'"] \arrow[dr, leftrightarrow, "W"] \arrow[dd, leftrightarrow, "H"] & & S^{(\mathrm{indef})}_{(0,4)}[\phi, a] \arrow[dl, leftrightarrow, "A"] \arrow[dd, leftrightarrow, "H"] \\
& S^{(\mathrm{def})}_{(0,4)}[\phi, a] \arrow[dr, leftrightarrow, dashed, "H'"] & \\
S_{(1,3)}[\phi, B] \arrow[rr, leftrightarrow, "W"] & & S^{(\mathrm{def})}_{(0,4)}[\phi, B]
\end{tikzcd}
$$

**Figure 1:** This diagram summarises the relations between the actions occuring in section 9. Further explanations are given in the text.

$$
\begin{tikzcd}
& & S^{(\mathrm{def})}_{(1,4)}[\phi, A] \arrow[dl, leftrightarrow, "D_S"'] \arrow[drr, leftrightarrow, "D_T"] & & \\
& S^{(\mathrm{def})}_{(1,3)}[\phi, a] \arrow[rrr, leftrightarrow, "W'"] \arrow[dl, leftrightarrow, "W"'] \arrow[dd, leftrightarrow, "H"] & & & S^{(\mathrm{indef})}_{(0,4)}[\phi, a] \arrow[dl, leftrightarrow, "W"] \arrow[dd, leftrightarrow, "H"] \\
S^{(\mathrm{def})}_{(0,4)}[\phi, a] \arrow[rrr, leftrightarrow, "W'"] \arrow[dd, leftrightarrow, "H"] & & & S^{(\mathrm{indef})}_{(1,3)}[\phi, a] \arrow[dd, leftrightarrow, "H"] & \\
& S^{(\mathrm{def})}_{(1,3)}[\phi, B] \arrow[rrr, leftrightarrow, "W"] \arrow[dl, leftrightarrow, "W'"'] & & & S^{(\mathrm{def})}_{(0,4)}[\phi, B] \arrow[dl, leftrightarrow, "W'"] \\
S^{(\mathrm{indef})}_{(0,4)}[\phi, B] \arrow[rrr, leftrightarrow, "W"] & & & S^{(\mathrm{indef})}_{(1,3)}[\phi, B] &
\end{tikzcd}
$$

**Figure 2:** This extended diagram contains all four-dimensional actions which can be generated from a given action containing one normal scalar and one axion by applying Wick rotations and Hodge dualisations. We have also included the relation to a five-dimensional scalar-gauge field action via dimensional reduction/lifting. Further explanations are given in the text.

all eight theories can be related by using Wick roation $W$ and Hodge dualisation $H$. We have also included the modified Wick rotations $W'$, but not the analytic continuations $A$ and modified Hodge dualisations $H'$ in order to keep the diagram transparent. The relation to the five-dimensional Minkowksi space action has been included. While the diagram is complete with respect to four-dimensional actions, further five- and three-dimensional actions could be obtained by applying dimensional reduction to three dimension and Wick rotations and Hodge dualisations in five and three dimensions.

## 10. Dimensional lifting of four-dimensional instantons

### 10.1 Five-dimensional black holes

Instantons can be used as generating solutions for a variety of higher-dimensional solitons. In this section the one-charge instanton solution obtained previously will be lifted to five dimensions. We will show that we obtain an extremal black hole, and that the ADM mass of the black hole equals the instanton action. Both the ADM mass and the instanton action are boundary terms, which agree on black hole/instanton solutions, and we observe that such a boundary term can be generated by transforming the four-dimensional Einstein-Hilbert term from the Einstein frame into another conformal frame, which we call the Kaluza-Klein frame. In this frame, the metric of the instanton solution agrees with the metric of the black hole, restricted to a space-like hypersurface.

Since we know the explicit relation between the five-dimensional action (6.2) and the four-dimensional action (7.9), it is straightforward to lift four-dimensional instantons to five-dimensional space-times. Let us apply this to the one-charge instanton solution (9.18), (9.20) of the Euclidean STU-model. This model lifts to five-dimensional supergravity coupled to two vector multiplets, which is a subsector of the effective field theory of the heterotic string theory compactified on $K3 \times S^1$.

The only field excited in the four-dimensional one-charge instanton is the four-dimensional heterotic dilaton

$$S = \epsilon i_\epsilon z^1 = \epsilon i_\epsilon (x^1 + i_\epsilon y^1) \ .$$

According to (6.11), the relation between the $y^i$ and the five-dimensional scalars $h^i$ is $y^i = 6^{1/3} e^\sigma h^i$, while the $x^i$ lift to the temporal components of the five-dimensional gauge potentials. We can compute the Kaluza-Klein scalar using the constraint $c_{ijk} h^i h^j h^k = 1$:

$$y^1 y^2 y^3 = \frac{1}{6} c_{ijk} y^i y^j y^k = e^{3\sigma} \ .$$

In the one-charge solution $y_2, y_3$ are constant, $y_2 y_3 = B$, and therefore

$$e^{3\sigma} = e^{-2\phi} B \ .$$

The Kaluza-Klein vector is trivial, and therefore the four-dimensional Einstein frame metric $ds^2_{\text{Einstein}} = \delta_{\mu\nu} dx^\mu dx^\nu$ lifts to the five-dimensional static metric

$$ds^2_{(5)} = -e^{2\sigma} dt^2 + e^{-\sigma} \delta_{\mu\nu} dx^\mu dx^\nu \ . \tag{10.1}$$

Since this metric is asymtotically flat, we impose that it approaches the canonically normalised Minkowski metric $\eta_{\hat{\mu}\hat{\nu}}$ at infinity. This implies that the constant $B$ is related to the value of the four-dimensional dilaton at infinity by

$$B = e^{2\phi_\infty} .$$

We can now express the five-dimensional metric in terms of the four-dimensional dilaton:

$$ds^2_{(5)} = -e^{-4/3(\phi-\phi_\infty)}dt^2 + e^{2/3(\phi-\phi_\infty)}\delta_{\mu\nu}dx^\mu dx^\nu \tag{10.2}$$

By comparing to [32], and using that $e^{2\phi}$ is harmonic, we immediately recognize this solution as a supersymmetric extremal black hole, which is charged under a single $U(1)$. In the single center case we have

$$e^{-3\sigma} = e^{2(\phi-\phi_\infty)} = 1 + \frac{e^{-2\phi_\infty}C}{r^2} ,$$

and

$$ds^2_{(5)} = -\left(1 + \frac{e^{-2\phi_\infty}C}{r^2}\right)^{-2/3} dt^2 + \left(1 + \frac{e^{-2\phi_\infty}C}{r^2}\right)^{1/3} \delta_{\mu\nu}dx^\mu dx^\nu . \tag{10.3}$$

If we fix a space-like hypersurface by setting $t =$ const., we obtain the four-dimensional positive definite metric

$$ds^2_{t=const.} = \left(1 + \frac{e^{-2\phi_\infty}C}{r^2}\right)^{1/3} \delta_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{e^{-2\phi_\infty}C}{r^2}\right)^{1/3} \left(dr^2 + r^2 d\Omega^2_{(3)}\right) . \tag{10.4}$$

This is a semi-infinite wormhole akin to (9.21). However, due to the different power of the harmonic function in front, the volume of the three-sphere transverse to the throat goes to zero in the limit $r \to 0$. This is as expected, because a supersymmetric five-dimensional black hole needs to carry at least three charges in order to have a non-vanishing horizon area. Since the semi-infinite wormhole (10.4) describes the spatial geometry of a degenerate black hole, we call it a degenerate semi-infinite wormhole.

From the four-dimensional point of view the conformal frame where we obtain the spatial geometry of the five-dimensional black hole is neither the Einstein frame where four-dimensional geometry is flat, nor the string frame (9.21). We call the conformal frame defined by (10.4) the Kaluza-Klein frame. Its relation to the other two frames is given by

$$ds^2_{\mathrm{KK}} = e^{-\sigma}ds^2_{\mathrm{Einstein}} = e^{-2\phi-\sigma}ds^2_{\mathrm{String}} . \tag{10.5}$$

So far we have seen that the horizon area of the black hole is given by the size of the asymptotic three-sphere of the instanton in the Kaluza-Klein frame. To extend our instanton–black hole dictionary, we will compare the ADM mass of the black hole to the instanton action. The ADM mass measures the flow generated by asymptotic time translations through an asymptotic sphere at spatial infinity [43]. The relevant formulae for higher-dimensional black holes can be found in [44]. Let

$$ds^2 = -h_{tt}dt^2 + 2h_{t\mu}dtdx^\mu + h_{\mu\nu}dx^\mu dx^\nu$$

be the line element of an $(n+1)$-dimensional space-time. We have chosen a parametrisation where $t = \text{const}$ defines a foliation by spacelike hypersurfaces, and where the spatial part $h_{\mu\nu}$ of the metric approaches the flat Euclidean $n$-dimensional metric $ds^2_{\text{flat}} = \delta_{\mu\nu}dx^\mu dx^\nu = dr^2 + r^2 d\Omega^2_{n-1}$ at infinity, where $d\Omega^2_{n-1}$ is the line element of the unit $(n-1)$-sphere. We choose one of the spatial hypersurfaces and denote its asymptotic boundary by $S^{n-1}_\infty$. Then the ADM mass is given by

$$16\pi G_N M_{\text{ADM}} = \oint_{S^{n-1}_\infty} d\Sigma^\mu \left(\partial^\nu h_{\mu\nu} - \partial_\mu(\delta^{\rho\sigma}h_{\rho\sigma})\right) := \lim_{r\to\infty} \oint_{S^{n-1}_r} d\Sigma^\mu \left(\partial^\nu h_{\mu\nu} - \partial_\mu(\delta^{\rho\sigma}h_{\rho\sigma})\right) \ , \tag{10.6}$$

where $G_N$ is Newton's constant, $d\Sigma^\mu$ is the vectorial volume element of the sphere $S^{n-1}_r$.[22] It is known that (10.6) is independent of the choice of the asymptotically flat coordinate system if the scalar curvature of the metric $h_{\mu\nu}$ is norm-integrable [45].

For the solutions obtained above, the spatial metric is conformally flat and takes the form

$$h_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{m}{r^{n-2}} + \cdots\right)(dr^2 + r^2 d\Omega^2_{n-1}) \ ,$$

where $m$ is a constant. The evaluation of the integral (10.6) gives

$$16\pi G_N M_{\text{ADM}} = (n-1)(n-2)\Omega_{n-1} m \ ,$$

where $\Omega_{n-1}$ is the area of the unit $(n-1)$ sphere, which reproduces the result of [44].

Using that $n = 4$ and that $h_{\mu\nu}dx^\mu dx^\nu = e^{-\sigma}\delta_{\mu\nu}dx^\mu dx^\nu$, we find

$$16\pi G_N M_{\text{ADM}} = -3 \oint_{S^3_\infty} d\Sigma^\mu \partial_\mu e^{-\sigma} = \lim_{r\to\infty} \oint_{S^3_r} \partial_r \left(1 + \frac{e^{-2\phi_\infty}C}{r^2}\right)^{\frac{1}{3}} r^3 d\Omega^3_{S^3_1} = \frac{2A\Omega_3 C}{e^{2\phi_\infty}} \ . \tag{10.7}$$

In the previous sections of this paper we have used units where $8\pi G_N = 1$. Since the instanton charge satisfies $|Q_{\text{inst}}| = \Omega_3 C$, we see that the ADM mass of the five-dimensional black hole equals the action of the four-dimensional instanton:

$$M_{\text{ADM}} = \frac{|Q_{\text{inst}}|}{e^{2\phi_\infty}} = S_{\text{inst}} \ .$$

[22] This means that $d\Sigma^\mu = n^\mu d\text{vol}$, where $n^\mu$ is the Euclidean unit normal of $S^{n-1}_r$, and where $d$vol is the canonical volume element of $S^{n-1}_r$.

As we discussed previously, the bulk action (9.12) vanishes when evaluated on the instanton solution. To find the instanton action, we either need to work in the scalar-tensor picture, or to add a boundary term. One way to obtain a boundary term which gives the same instanton action as the scalar-tensor formulation of the theory is to apply Hodge dualisation. However, the ADM mass of the lifted solution is an alternative candidate for the boundary term. Besides the above observation, there is a general reason to expect a relation between the ADM mass of a soliton and the action of the instanton obtained by dimensional reduction. As is well known, $p$-brane solitons can be obtained from $(p+1)$-branes by double dimensional reduction, and in this case the respective brane tensions are related by the volume of the internal dimension. One should expect that this extends to 0-branes (solitons) and $(-1)$ branes (instantons), where the brane tension is the mass and the action, respectively.[23]

In order to see that the relation between ADM mass and instanton action is general rather than accidental, we take the formula which expresses the ADM mass as a boundary term and re-write it in terms of the four-dimensional dilaton instead of the Kaluza-Klein scalar:

$$M_{ADM} = -\frac{3}{2} \oint d^3\Sigma^\mu \partial_\mu e^{-\sigma} = -\frac{3}{2} \oint d^3\Sigma^\mu \partial_\mu e^{\frac{2}{3}(\phi-\phi_\infty)} = - \oint d^3\Sigma^\mu e^{\frac{2}{3}(\phi-\phi_\infty)} \partial_\mu \phi \; . \tag{10.8}$$

This can now be compared to the boundary term obtained by Hodge-dualisation of the scalar-tensor action:

$$S_{\rm bd} = - \oint d^3\Sigma^\mu \partial_\mu \phi \; .$$

Both boundary terms are different, but give the same result whenever the additional factor $e^{\frac{2}{3}(\phi-\phi_\infty)}$ in the ADM boundary term approaches its constant limit value fast enough. This is in particular the case when $e^{2\phi}$ is harmonic. If one considers more complicated instanton solutions, which involve several scalar fields, the role of the four-dimensional dilation is played by a particular combination of all scalar fields, but the fall-off properties of the boundary terms remain the same, and the relation between ADM mass and instanton action is seen to hold generally [33].

When relating actions by dimensional reduction one usually neglects boundary terms. This raises the question whether the boundary term which accounts for the instanton action can be obtained by keeping the boundary terms occuring in the dimensional reduction of the action. In section 6 we have performed the reduction such that we went from the

[23] When reducing the five-dimensional action in Section 6, we did not include an explicit parameter for the volume of the internal circle. This volume factor, which controlls the ratio between the higher- and lower-dimensional Newton constant, and, hence, sets the ratio between soliton mass and instanton action, could of course be easily reinstated.

five-dimensional Einstein frame to the four-dimensional Einstein frame. More generally, we can use the following family of parametrisations:

$$ds^2 = -e^{2\beta\sigma}(dt + A_\mu dx^\mu)^2 + e^{2\alpha\sigma} g_{\mu\nu} dx^\mu dx^\nu \ .$$

While the choice $\alpha = \frac{1}{2}$, $\beta = 1$ brings us to the four-dimensional Einstein frame, the alternative choice $\alpha = 0$, $\beta = 1$ brings us to the four-dimensional Kaluza-Klein frame introduced above. The Ricci scalars corresponding to the two frames are related by (see [46], Appendix D):

$$R_{KK} = e^\sigma \left( R_E + 3\nabla^\mu \partial_\mu \sigma - \frac{3}{2} \partial_\mu \sigma \partial^\mu \sigma \right) \ . \tag{10.9}$$

For $\alpha = 0$, $\beta = 1$, the temporal reduction of the five-dimensional action

$$S_{(1,4)} = \frac{1}{2} \int d^5 x \sqrt{g_{(5)}} (R_{(5)} + \cdots)$$

gives[24]

$$\begin{aligned} S^{KK}_{(0,4)} &= -\frac{1}{2} \int d^4 x \sqrt{g_{KK}} (e^\sigma R_{KK} + \cdots) \\ &= -\frac{1}{2} \int d^4 x \sqrt{g_E} (R_E + 3\nabla^\mu \partial_\mu \sigma - \frac{3}{2} \partial_\mu \sigma \partial^\mu \sigma + \cdots) \\ &= S^E_{(0,4)} - \frac{3}{2} \oint d^3 \Sigma^\mu \partial_\mu \sigma = S^E_{(0,4)} + \oint d^3 \Sigma^\mu \partial_\mu \phi \ . \end{aligned} \tag{10.10}$$

Thus the boundary term obtained by transforming from the Kaluza-Klein frame to the Einstein frame is precisely the instanton action:

$$M_{ADM} = S_{\text{inst}} = S^E_{(0,4)} - S^{KK}_{(0,4)} \ . \tag{10.11}$$

As already noted, the two metrics entering into the ADM formula can be identified with the four-dimensional Kaluza-Klein frame and Einstein frame metrics. This observation is interesting, as it relates the ADM mass formula to an action. Notice that the equation (10.9) shows that the boundary term satisfies

$$0 < \frac{3}{2} \int d^4 x \sqrt{g_E} \nabla^\mu \partial_\mu \sigma = - \oint d^3 \Sigma^\mu \partial_\mu \phi = M_{ADM}$$

if the scalar curvature satisfies $R_{KK} > 0$, in accordance with the relation between positivity of scalar curvature and positivity of the mass, familiar from the positive mass theorem.

[24]Since we define Euclidean actions with an explicit minus sign, the temporal reduction gives minus the Euclidean action.

## 10.2 Ten-dimensional Five-branes

In the context of string compactifications, five-dimensional supersymmetric black holes can be interpreted in terms of ten-dimensional components, which are wrapped $p$-branes or other stringy solitons.[25] The particular black hole we have obtained by lifting the four-dimensional one-charge instanton can be further lifted to a ten-dimensional five-brane.

To see this, remember that the string-frame metric of a solitonic five-brane in ten dimensions is:

$$\begin{aligned} ds^2_{\text{String}} &= -dt^2 + (dy^1)^2 + \cdots + (dy^5)^2 + H(x)\left((dx^1)^2 + \cdots (dx^4)^2\right) , \\ e^{2(\Phi-\Phi_\infty)} &= H(x) , \quad dB = \star_4 dH(x) , \\ \Delta_4 H &= 0 . \end{aligned} \tag{10.12}$$

Here $\Phi$ is the ten-dimensional dilaton, $B$ the universal two-form gauge field, $\star_4$ is the Hodge operator with respect to four transverse directions. All fields are given in terms of a function $H(x)$, which is harmonic in the four transverse coordinates $x^1, \ldots, x^4$. This solution only excites fields in the universal sector common to any theory of closed oriented strings and exists for both heterotic and type-II string theories. Dimensional reduction along five spatial world volume directions results in the following five-dimensional string frame metric:

$$ds^2_{\text{String}(5)} = -dt^2 + H(x)\left((dx^1)^2 + \cdots (dx^4)^2\right) .$$

The five-dimensional dilaton equals the ten-dimensional one, while the two-form reduces to a gauge potential, under which the solution is charged. The relation between the the five-dimensional string and Einstein frames is[26] $ds^2_{\text{String}} = e^{\frac{4}{3}\Phi} ds^2_{\text{Einstein}}$. Using that $e^{2\Phi} = H(x)$ is harmonic, we obtain

$$ds^2_{\text{Einstein}(5)} = -H^{-\frac{2}{3}} dt^2 + H^{\frac{1}{3}}\left((dx^1)^2 + \cdots (dx^4)^2\right) .$$

For the single-center case this is precisely the five-dimensional black hole (10.3), which can therefore be lifted to a wrapped five-brane. Further reduction along a time-like circle gives

[25] Here 'wrapping' refers to embeddings of the $(p+1)$-dimensional brane world volume $\Sigma$ into a total space-time of the form $N \times K$, where $N$ is not compact and interpreted as the 'dimensionally reduced space-time', where $K$ is compact and interpreted as internal space, and where the image of the world volume is of the form $\Sigma_1 \times \Sigma_2 \subset N \times K$. A totally wrapped brane corresponds to an embedding of $\Sigma$ into $K$.

[26] In general, the relation between string frame and Einstein frame metric is $ds^2_{\text{String}} = e^{m\Phi} ds^2_{\text{Einstein}}$, where $m$ is chosen such that $\sqrt{|h_{\text{String}}|} e^{-2\Phi} R_{h_{\text{String}}} = \sqrt{|h_{\text{Einstein}}|} R_{h_{\text{Einstein}}} + \cdots$, where the omitted terms do not involve the space-time curvature. Using the transformation properties of the metric under Weyl transformations (see for example [46], Appendix D), one finds that $m = \frac{4}{d-2}$, where $d > 2$ is the dimension of space-time.

the four-dimensional instanton which can thus be interpreted as a five-brane where all six world volume directions have been wrapped.

Further details depend on the string theory into which one embedds the solution. Since we constructed instanton solutions in the vector multiplet sector of an $N = 2$ compactification, we need to pick a string compactification which preserves $N = 2$ supersymmetry, and where the dilaton sits in a vector multiplet. This happens for the heterotic string, compactified on $K3 \times S^1$ to five dimensions. Therefore the microscopic description of the four-dimensional instanton (9.18), (9.20) is a completely wrapped heterotic five-brane. This observation strongly suggests that the instanton solutions considered in this paper are the supergravity approximations of string instantons. One difference compared to the string instanton calculus is that we reduce over time instead of considering Euclidean wrappings (which implicitly assumes that the world volume time has been Wick rotated).

Other instanton solutions of Euclidean vector multiplets will have different microscopic interpretations. Consider for example the Euclidean STU-model. Since this has, at the classical level, a permutation symmetry between the fields $S$, $T$ and $U$, we can immediately replace $S$ by any of the other two fields. From the supergravity point of view this appears to be rather trivial, but the microscopic interpretation of these new solutions is completely different. Whereas $S$ is the dilaton, $T$ and $U$ are geometric moduli, and the solutions do not involve the string coupling. Therefore they cannot be space-time instantons, but must be world sheet instantons (or more precisely the effective supergravity description thereof). The detailed investigation of the microscopic, stringy aspects of vector multiplet instantons is left to future work.

## 11. Outlook

In this paper we have defined projective special para-Kähler manifolds and shown that they arise as target manifolds for the scalars of Euclidean $N = 2$ vector multiplets coupled to gravity. A subset of these theories can be obtained by dimensional reduction of five-dimensional vector multiplets over time, which defines a temporal version of the $r$-map. To understand the geometry of the scalar sector it was sufficient to focus on the bosonic sector of the theory. For rigid vector multiplets the fermionic terms and supersymmetry transformations rules were found in [1], and it is desirable to extend this to the local case in the future. To complete the programme of characterising the special geometries of Euclidean $N = 2$ supersymmetry, and relating the various special geometries by geometric constructions, which was started in [1, 2] and continued in this paper, we finally need to explore para-quaternion-Kähler geometry of Euclidean hypermultiplets and its relation to projective special $\epsilon$-Kähler geometry through the $c$-map.

Potential applications of our work include the systematic construction of instanton solutions and the generation of solitonic solutions through dimensional lifting, which we have illustrated with a detailed example. General solutions involving an arbitrary number of scalar fields will be discussed in [33]. We have also seen that Euclidean actions and instanton solutions involving axionic scalars involve ambiguities and subtleties which deserve further study. The geometric framework provided by [1, 2] and this paper should be useful in this respect. Another question, which we only touched upon briefly, is the microscopic, 'stringy' interpretation of Euclidean supergravity solutions.

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