Infinite matroid theory exercise sheet 11

1. (a) Prove that any finite family $(M_k|k \in K)$ of finitary matroids on a common ground set E has a covering if and only if for every finite set $X \subseteq E$ the following holds.

$$\sum_{k\in K} r_{M_k}(X) \geq |X|$$

   (b) Is this also true if we leave out the assumption that the family is finite?

2.** Is the covering conjecture true for arbitrary families of finitary matroids?

3. Let $\overline{P}$ and $\overline{Q}$ be disjoint subsets of the ground set E of a matroid M with $\kappa_M(\overline{P}, \overline{Q}) = k \in \mathbb{N}$. Let $((P_i, Q_I)|i \in I)$ be a family of subsets of E with $\overline{P} \subseteq P_i$, $\overline{Q} \subseteq Q_i$ and $\kappa_M(P_i) = k$ for each $i \in I$. Show that $\kappa_M(\bigcup_{i\in I} P_i) = k$.

4. (a) Let M be a matroid with ground set E and let B be a base of M. Let G_B be the bipartite graph on $(B, E \setminus B)$ with an edge from $e \in B$ to $f \notin B$ if and only if $e \in C_B^f$. Prove that G_B is connected if and only if M is connected.

   (b) Let M be a connected matroid in which all circuits and cocircuits are countable. Prove that the ground set of M is countable.

Reminder:

Conjecture 0.1 (Covering Conjecture)**.** *A family of matroids $(M_k|k \in K)$ on the same ground set E has a covering if and only if the following is true for every $X \subseteq E$.*

If $(M_k\restriction_X|k \in K)$ has a packing, then it also has a covering. (1)