# Infinite matroid theory exercise sheet 1

1. Determine for which values for $m$ and $n$, the uniform matroid $U_{m,n}$ is graphic.
2. Let $\mathcal{C} \subseteq \mathcal{P}(E)$ satisfy the following.

   (C3) (Circuit elimination) Let $C, C' \in \mathcal{C}$ and $x \in C \cap C'$ and $z \in C \setminus C'$. Then there is some $C'' \in \mathcal{C}$ with $z \in C'' \subseteq (C \cup C') - x$.

   Show that the set of minimal nonempty elements of $\mathcal{C}$ is the set of circuits of a matroid.
3. A function $r : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ *is the rank function of a matroid* $M$ if for every $A \subseteq E$ the value $r(A)$ is the size of the largest independent subset of $A$.

   Show that a function $r : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid $M$ if and only if it satisfies the following.

   (R1) $\forall A \subseteq E, r(A) \leq |A|$

   (R2) $\forall A \subseteq E, x \in E, r(A) \leq r(A + x) \leq r(A) + 1$

   (R3) (Submodularity) $\forall A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

   In these circumstances, show that the independent sets of $M$ are precisely those sets $A \subseteq E$ satsifying $r(A) = |A|$.

**Reminder: Independence axioms**

A subset $\mathcal{I}$ of $\mathcal{P}(E)$ is called the set of *independence sets of a finite matroid* if and only if it satisfies the following.

(I1) $\varnothing \in \mathcal{I}(M)$.

(I2) $\mathcal{I}(M)$ is closed under taking subsets.

(I3) Given $I_1, I_2 \in \mathcal{I}(M)$ and $x \in I_1 \setminus I_2$ such that $I_2 + x \notin \mathcal{I}(M)$, there exists a $y \in I_2 \setminus I_1$ such that $I_1 - x + y \in \mathcal{I}(M)$.

**Circuit axioms**

(C1) $\emptyset \notin \mathcal{C}$

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) (Circuit elimination) Let $C, C' \in \mathcal{C}$ and $x \in C \cap C'$ and $z \in C \setminus C'$. Then there is some $C'' \in \mathcal{C}$ with $z \in C'' \subseteq (C \cup C') - x$.

## Hints

Concerning question 3:

First show that if $r(A) + 1 = r(A + x)$ for some $x \notin A$, then $r(B) + 1 = r(B + x)$ for every $B \subseteq A$.