Infinite matroid theory exercise sheet 8

1. For two matroids M_1 and M_2 that share only one edge e, the *2-sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 is the matroid whose edge set is the symmetric difference of those of M_1 and M_2 and whose scrawls are those symmetric differences of scrawls of M_1 and M_2 that do not contain e.

   Prove that $M_1 \oplus_2 M_2$ is a matroid.

2. If $L = M_1 \oplus_2 M_2$, which values are possible for $\kappa_L(E(M_1) \cap E(L), E(M_2) \cap E(L))$?

3. Let $L = M_1 \oplus_2 M_2$. Describe the bases, circuits and cocircuits of L in terms of the bases, circuits and cocircuits of M_1 and M_2, respectively. Prove that $(M_1^* \oplus_2 M_2^*)^* = M_1 \oplus_2 M_2$.

4. Prove that the 2-sum is associative, in the sense that if M_1 and M_2 meet in only one edge and M_2 and M_3 meet in only one edge, and M_1 has no edge in common with M_3, then $(M_1 \oplus_2 M_2) \oplus_2 M_3 = M_1 \oplus_2 (M_2 \oplus_2 M_3)$.

5.* Let M be a connected matroid, and e be one of its edges. Prove that either $M/\{e\}$ or $M\backslash\{e\}$ is connected.

   Is it true for every $F \subseteq E(M)$ that there is a partition of F into sets A and B such that $M/A\backslash B$ is connected?

6.* Let M be a connected matroid such that all its circuits and cocircuits are finite. Prove that M is finite. Deduce that every matroid that is finitary and cofinitary is a direct sum of finite matroids.