Problem sheet 5

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=36004
until 20:00, January 12.

1. Define on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a new probability measure

$$\mathbb{P}^a\{A\} = \mathbb{E}\,\mathbb{I}_A e^{a\eta - \frac{a^2}{2}}, \quad A \in \mathcal{F},$$

where $\eta \sim N(0,1)$. Show that $\mathbb{P}^a$ is a probability measure on Ω.

2. Let $N(t)$, $t \geq 0$, be a Poisson process. Define $N_n(t) = \frac{1}{n}N(nt)$, $t \geq 0$, for all $n \geq 1$.

   (a) Show that for every $t > 0$ the family $(N_n(t))_{n\geq 1}$ satisfies the LDP in $\mathbb{R}$ (with $a_n = \frac{1}{n}$) and find the corresponding rate function.

   HW1 **[4 points]** Show that for every $t_1 < t_2 < \dots < t_d$ the family $((N_n(t_1), \dots, N_n(t_d)))_{n\geq 1}$ satisfies the LDP in $\mathbb{R}^d$ (with $a_n = \frac{1}{n}$) and find the corresponding rate function.

   HW2 **[2 points]** Which form should have the rate function in the LDP for the family of processes $\{N_n(t),\ t \in [0,T]\}_{n\geq 1}$ in the space $\mathrm{C}_0[0,T]$?

HW3 **[5 points]** Show that for any $f \in H_0^2[0,T]$ there exists a sequence $(f_n)_{n\geq 1}$ from $\mathrm{C}_0^2[0,T]$ such that $f_n \to f$ in $\mathrm{C}_0[0,T]$ and $I(f_n) \to I(f)$ as $n \to \infty$, where

$$I(f) = \begin{cases} \frac{1}{2}\int_0^T \dot{f}^2(t)dt & \text{if } f \in H_0^2[0,T], \\ +\infty & \text{otherwise.} \end{cases}$$

(*Hint:* Use first the fact that $\mathrm{C}^1[0,T]$ is dense in $L_2[0,T]$. Then show that if $h_n \to h$ in $L_2[0,T]$, then $\int_0^\cdot h_n(s)ds$ tends to $\int_0^\cdot h(s)ds$ in $\mathrm{C}_0[0,T]$, using Hölder's inequality)

3. Let $h \in \mathrm{C}^1[0,T]$ and $w(t)$, $t \in [0,T]$, be a Brownian motion. Show that

$$\int_0^T h(t)dw(t) = h(T)w(T) - h(0)w(0) - \int_0^T h'(t)w(t)dt.$$

(*Hint:* Take a partition $0 = t_0 < t_1 < \dots < t_n = T$ and check first that functions $h_n = \sum_{k=1}^n h(t_k)\mathbb{I}_{[t_{k-1},t_k)}$ converge to h in $L_2[0,T]$ as the mesh of partition goes to 0, using e.g. the uniform continuity of h on $[0,T]$. Next show that

$$\sum_{k=1}^n h(t_{k-1})(w(t_k) - w(t_{k-1})) = h(t_{n-1})w(T) - h(0)w(0) - \sum_{k=1}^{n-1} w(t_k)(h(t_k) - h(t_{k-1}))$$

Then prove that the first partial sum converges to the integral $\int_0^T h(t)dw(t)$ in L_2 and the second partial sum converges to $\int_0^T w(t)dh(t)$ a.s. as the mesh of partition goes to 0)

HW4 **[3 bonus points]** Show that for every $a \in \mathbb{R}$ and $\delta > 0$

$$\mathbb{P}\{w_{\sigma^2}(t) + at < \delta, \quad t \in [0,T]\} > 0,$$

where $w_{\sigma^2}(t)$, $t \geq 0$, is a Brownian motion with diffusion rate σ^2.

(*Hint:* Use the Cameron-Marting formula and the fact that $\sup_{t\in[0,T]} w_{\sigma^2}(t)$ and $|w_{\sigma^2}(T)|$ have the same distribution[1])

[1]see Proposition 13.13 [Kallenberg]