Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Problem sheet 4

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=34225
until 20:00, December 17.

HW 1 [6 points] Let E be a metric space and $f : E \to [-\infty, +\infty]$. Define

$$f_{\mathrm{lsc}}(x) = \sup \left\{ \inf_{y \in G} f(y) : \ G \ni x \text{ and } G \text{ is open} \right\}. \tag{1}$$

(a) Show that if $x_n \to x$, then $f_{\mathrm{lsc}}(x) \le \underline{\lim}_{n\to\infty} f(x_n)$.

(*Hint:* Use Lemma 5.2, namely that the function f_{lsc} is lower semi-continuous and $f_{\mathrm{lsc}} \le f$)

(b) Show that for each the supremum in (1) can only be taken over all ball with center x, namely

$$f_{\mathrm{lsc}}(x) = \sup_{r>0} \inf_{y \in B_r(x)} f(y) \tag{2}$$

(*Hint:* Use the fact that any open set G containing x also contains a ball $B_r(x)$ for some $r > 0$. It will allow to prove the inequality $f_{\mathrm{lsc}}(x) \le \sup_{r>0} \inf_{y\in B_r(x)} f(y)$. The inverse inequality just follows from the observation that supremum in the right hand side of (2) is taken over smaller family of open sets)

(c) Prove that for each $x \in E$ there is a sequence $x_n \to x$ such that $f(x_n) \to f_{\mathrm{lsc}}(x)$ (the constant sequence $x_n = x$ is allowed here). This gives the alternate definition

$$f_{\mathrm{lsc}}(x) = \min \left\{ f(x), \underline{\lim}_{y \to x} f(y) \right\}.$$

(*Hint:* Use part b) of the exercise to construct the corresponding sequence x_n, $n \ge 1$)

1. Let $I : \mathrm{C}_0[0, T] \to [0, +\infty]$ be defined by

$$I(f) = \begin{cases} \frac{1}{2}\int_0^T \dot f^2(x)dx, & \text{if } f \in H_0^2[0,T], \\ +\infty, & \text{otherwise .} \end{cases}$$

Show that the set $\{f \in \mathrm{C}_0[0,T] : \ I(f) \le \alpha\}$ is equicontinuous and bounded in $\mathrm{C}_0[0,T]$ for all $\alpha \ge 0$. Conclude that I is good.

(*Hint:* Using Hölder's inequality, show that $|f(t) - f(s)|^2 \le |t-s| \int_0^T \dot f^2(x)dx$ for all $t, s \in [0,T]$ and each $f \in H_0^2[0,T]$)

2. Prove that a family $(\xi_\varepsilon)_{\varepsilon>0}$ is exponentially tight in E if and only if for any $b > 0$ there exists a compact $K \subset E$ and $\varepsilon_0 > 0$ such that

$$\mathbb{P}\left\{\xi_\varepsilon \notin K\right\} \le e^{-\frac{1}{\varepsilon}b}, \quad \varepsilon \in (0, \varepsilon_0).$$

3. Let E be a complete and separable metric space.

a) Show that exponential tightness implies tightness for a countable family of random variables.

(*Hint:* Prove a similar inequality to one in the previous exercise and then use the fact that any random element on a complete and separable metric space is tight

b) Show that tightness does not imply exponential tightness.

HW 2 **[3 points]** Let $(\xi_\varepsilon)_{\varepsilon>0}$ be a family of random variables in $\mathbb{R}$ such that there exist $\lambda > 0$ and $\kappa > 0$ such that $\mathbb{E}\, e^{\frac{\lambda}{\varepsilon}|\xi_\varepsilon|} \leq \kappa^{\frac{1}{\varepsilon}}$ for all $\varepsilon > 0$. Show that this family is exponentially tight.

(*Hint:* Use Chebyshev's inequality)

HW3 **[3 bonus points]** Find a simpler proof of Proposition 6.3 in the case $E = \mathbb{R}^d$.

(*Hint:* Cover a level set $\{x \in R^d : \ I(x) \leq \beta\}$ by an open ball and use the upper bound)