Problem sheet 3

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=32488
until 20:00, December 8.

HW1 **[3 points]** Let $a_n > b_n$, $n \geq 1$, be positive real numbers such that there exist limits (probably infinite)

$$a := \lim_{n\to\infty} \frac{1}{n} \ln a_n \quad \text{and} \quad b := \lim_{n\to\infty} \frac{1}{n} \ln b_n$$

and $a > b$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \ln(a_n - b_n) = a.$$

(*Hint:* Show that $\frac{b_n}{a_n} \to 0$, $n \to \infty$)

1. For any random vector $\xi \in \mathbb{R}^d$ and non-singular $d \times d$ matrix A, show that $\varphi_{A\xi}(\lambda) = \varphi_\xi(\lambda A)$ and $\varphi^*_{A\xi}(x) = \varphi^*_\xi(A^{-1}x)$.

2. For any pair of independent random vectors ξ and η show that $\varphi_{\xi,\eta}(\lambda, \mu) = \varphi_\xi(\lambda) + \varphi_\eta(\mu)$ and $\varphi^*_{\xi,\eta}(x, y) = \varphi^*_\xi(x) + \varphi^*_\eta(y)$.

   (*Hint:* To prove the second equality, use the equality $\sup\limits_{\lambda,\mu} f(\lambda, \mu) = \sup\limits_{\lambda} \sup\limits_{\mu} f(\lambda, \mu)$)

3. Let $\xi_1, \xi_2, \dots$ be independent random vectors in $\mathbb{R}^d$ whose coordinates are independent exponentially distributed random variables with rate γ. Show that the empirical means $\left(\frac{1}{n} S_n\right)_{n\geq 1}$ satisfies the LDP in $\mathbb{R}^d$ and find the corresponding rate function I.

HW2 **[5 points]** Let $\xi_1, \xi_2, \dots$ be independent normal distributed random vectors in $\mathbb{R}^d$ with mean 0 and positively defined covariance matrix C. Show that the empirical means $\left(\frac{1}{n} S_n\right)_{n\geq 1}$ satisfies the LDP in $\mathbb{R}^d$ and find the corresponding rate function I.

4. Show that the function $f(x) = 1 - |x - 1|$, $x \in [0, 2]$, belongs to $H^2_0[0, 2]$ but is not continuously differentiable.

HW3 **[2 points]** Let $f_\lambda : E \to \mathbb{R}$, $\lambda \in \mathbb{R}$, be a family of continuous functions, where E is a metric space. Show that the function $f(x) = \sup\limits_{\lambda \in \mathbb{R}} f_\lambda(x)$, $x \in E$, is lower semi-continuous.