Classification and Stability of Low-Dimensional Heteroclinic Structures

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Hiermit versichere ich, dass ich die wissenschaftlichen Arbeiten in dieser Habilitationsschrift bzw. meinen Anteil daran selbstständig und ohne fremde Hilfe verfasst habe.

Außerdem versichere ich, dass ich mich bisher nicht andernorts einem Habilitationsverfahren unterzogen habe.

Hamburg, November 2021

Alexander Lohse

For the mountains may move and the hills disappear, but even then my faithful love for you will remain. My covenant of blessing will never be broken.

Isaiah 54:10

Contents

Introduction

This habilitation thesis encompasses my mathematical research activities since 2015, i.e. exclusively after completion of my PhD. It is made up of eight published papers with several co-authors and one preprint that at the time of writing is under review for publication. The central theme of these works is low-dimensional heteroclinic attractors: my contribution to the scientific literature on this subject can be best understood by dividing them into papers that (i) focus on the classification and systematic study of certain classes of heteroclinic objects between equilibria in $\mathbb{R}^4$, and (ii) investigate different phenomena in the context of heteroclinic dynamics that are not necessarily restricted by the dimension of the system and might involve more complex nodes than just equilibria. This thesis is thus divided into two respective parts. After a short introduction to heteroclinic dynamics, we proceed give an overview of both.

Heteroclinic structures have been an intriguing area of research within dynamical systems for several decades. They are invariant objects in phase space, consisting of finitely many different states — often hyperbolic equilibria, but possibly more complex sets like periodic orbits or chaotic attractors — that are connected by non-empty intersections of (some of) their stable and unstable manifolds. Such connections may form a heteroclinic cycle and several cycles can constitute a heteroclinic network. These objects appear as organizing centers of bifurcation scenarios, but also exist robustly within certain classes of dynamical systems. In order for a heteroclinic object to persist under perturbations of a system, the intersections of the invariant manifolds must be transverse, hence their robustness is usually due to the existence of invariant subspaces containing the connections. There are different mechanisms that naturally lead to such invariant subspaces, e.g. fixed-point spaces in equivariant systems, synchrony subspaces in coupled cell systems or extinction hyperplanes in Lotka-Volterra systems. An early overview of what is mentioned in this paragraph is provided in [8].

Heteroclinic objects display an intricate variety of attraction properties, ranging from *fragmentary asymptotic stability* („attracting a set of positive measure“) [15] through *essential asymptotic stability* („attracting almost everything in a neighbourhood“) [1, 14] to the classic notion of *asymptotic stability*. The former two are often referred to as non-asymptotic stability properties. A particularly useful tool for investigating these is the *(local) stability index* which was introduced in 2011 by [16] and has since been used by various authors to characterize long-term behaviour in different types of systems. Calculating stability indices for heteroclinic objects involves the standard return map technique, i.e. composing local and global maps to approximate trajectories near the object to quantify the local extent of its basin of attraction.

A trajectory converging to a heteroclinic attractor displays what is sometimes called stop-and-go dynamics: it spends increasingly long periods of time close to the nodes of the attractor that are interrupted by fast transitions along the connections towards different states. This sort of behavior has been observed in various applications making heteroclinic dynamics an interesting subject from a modeling perspective as well, see e.g. the introduction in [12].

Part I. A Systematic Study in $\mathbb{R}^4$

The lowest-dimensional space in which heteroclinic attractors can display the full array of intermediate stability properties is $\mathbb{R}^4$. Efforts have been made in the past to understand and classify heteroclinic structures with certain properties here: for equivariant dynamics the work by Krupa and Melbourne [9, 10, 11] is widely considered a milestone. They introduce the notion of *robust simple cycles* between hyperbolic equilibria, which are roughly speaking cycles with one-dimensional saddle-sink connections in two-dimensional fixed-point spaces of an equivariant system. These are divided into types A, B and C depending on properties of a certain isotypic decomposition of $\mathbb{R}^4$. The definition of simple cycle is refined by Podvigina and Chossat in [17], leading to the new class of *pseudo-simple cycles* which differ in a subtle way from simple cycles that had been seemingly overlooked until then. All simple cycles of types B and C are listed in [11] and their stability is discussed exhaustively in [16] and [13]. A complete classification of homoclinic cycles (where the only connections are from an equilibrium to itself) can be found in [18, 19].

The papers making up the first part of this thesis primarily extend the efforts to fully classify simple and pseudo-simple heteroclinic cycles and networks in $\mathbb{R}^4$. In [6] there appears yet another, more general type of cycles introduced called *quasi-simple*. These encompass all simple cycles in the sense of Krupa and Melbourne, but not the pseudo-simple cycles of Podvigina and Chossat. Quasi-simple cycles were introduced by the authors of [6] to label a class of cycles for which a certain stability analysis can be carried out successfully. Since this leads to a very large class of quasi-simple cycles, there is no hope to list all of them (even in $\mathbb{R}^4$), thus classification attempts are focussed on the more restrictive notions of simple and pseudo-simple.

1. Construction of Heteroclinic Networks in $\mathbb{R}^4$

This work is one of the first steps in the systematic study of heteroclinic networks which can be constructed by joining two or more simple cycles in $\mathbb{R}^4$. Examples of such networks had been known for a long time already and studied extensively, e.g. by Kirk and Silber in [7], and a classification of simple type B/C networks in $\mathbb{R}^4$ can be found in my PhD thesis [12], see also [3]. The central results of this paper are (i) a complete list of simple heteroclinic networks of type A in $\mathbb{R}^4$ that satisfy a certain geometric assumption and (ii) a detailed stability analysis for the type A networks that we discover by applying the stability index approach of [16]. This extends the classification of simple networks in $\mathbb{R}^4$ that was begun in my PhD thesis [12].

2. Simple Heteroclinic Networks in $\mathbb{R}^4$
Here we take a different approach at classifying all simple networks in $\mathbb{R}^4$: by associating a graph with a given group $\Gamma \subset O(4)$ we determine all such Γ that admit simple networks in a Γ-equivariant system in $\mathbb{R}^4$. This leads to a full classification of simple networks in $\mathbb{R}^4$, i.e. not limited by the geometric assumption from the previous chapter. Furthermore, we derive general necessary and sufficient conditions for fragmentary and essential asymptotic stability of type A networks, thus completing the picture of simple networks in $\mathbb{R}^4$.

3. Pseudo-simple Heteroclinic Cycles in $\mathbb{R}^4$
This paper builds on [17] where pseudo-simple cycles are first mentioned. We provide an initial step towards the full classification of pseudo-simple cycles in $\mathbb{R}^4$ by determining all finite subgroups $\Gamma \subset O(4)$ admitting pseudo-simple cycles in a Γ-equivariant system. This is done by explicitly constructing equivariant vector fields possessing robust cycles. Our exposition makes use of the relationship between quaternions and subgroups of SO(4). Analysing the isotropy subgroups for a given Γ allows us to derive geometric conditions (necessary and sufficient) for the existence of cycles. We do not explicitly characterize all cycles that may exist for a given group – however, with our results this can individually be determined from the respective isotropy subgroup structure. Moreover, we provide an example of a system with $\Gamma \not\subset SO(4)$ and a pseudo-simple cycle which we prove to be fragmentarily asymptotically stable – it had previously been shown in [20] that pseudo-simple cycles with $\Gamma \subset SO(4)$ are generically completely unstable.

4. A Hybrid Heteroclinic Cycle
In this work we establish the existence of a heteroclinic structure in $\mathbb{R}^4$ that possesses properties of both simple and pseudo-simple cycles, but does not fall into any of the established categories. Our cycle has a two-dimensional connection and is therefore not simple. The same is also true, however, for all its subcycles with one-dimensional connections due to a two-dimensional component in the Γ-isotypic decomposition of $\mathbb{R}^4$. Though this is reminiscent of a pseudo-simple cycle, there are no such cycles in our system since the relevant subspaces – while dynamically invariant – are not fixed-point spaces. To our knowledge, this provides the first explicit example of a cycle with transitions between equilibria of types both A and B. Even though two subcycles are quasi-simple, none of the known stability results apply here and reduction of the return maps to the level of transition matrices is not possible due to the type A connection. This shows that even once the task of classifying simple and pseudo-simple cycles has been completed, there are still intermediate structures in $\mathbb{R}^4$ of similar complexity that do not fall into these categories.

Part II. Heteroclinic Phenomena Beyond $\mathbb{R}^4$

The papers forming the second part of this thesis address a variety of phenomena associated with heteroclinic dynamics that go beyond the classification of cycles or networks between hyperbolic equilibria. This includes cycles with one ore more periodic orbits as nodes, which we study as the organizing center of a two-parameter bifurcation scenario (chapter 5) and for which we derive a complete set of dynamic invariants under conjugacy (chapter 6). Moreover, in chapter 7 we deal with switching — a term referring to the question which (finite or infinite) paths of nodes are followed by trajectories converging to a given heteroclinic network. In chapter 8 we carry out a novel stability analysis for a heteroclinic network occurring in a system of coupled oscillators. Finally, in chapter 9 we propose a method to design vector fields which realize a given graph as a heteroclinic network with certain desirable properties, where the dimension of the resulting system is equal to the number of nodes in the graph.

5. Boundary Crisis for Degenerate Singular Cycles
Heteroclinic trajectories do not only appear between equilibria, but can also connect more general invariant sets. In this work we investigate a *singular EP1t-cycle*: a heteroclinic cycle connecting a hyperbolic equilibrium E and a hyperbolic periodic solution P through a quadratic tangency between the unstable manifold of P and the stable manifold of E, making this object a codimension two phenomenon on a compact three-dimensional manifold without boundary. Extending the work by others in this context, most notably Champneys et al. [4], we investigate how such a cycle acts as an organizing center for bifurcations associated with E and P. In particular, we establish the existence of chaotic dynamics by showing there are two different kinds of horseshoes in an open and dense set of vector fields satisfying some hypotheses. Moreover, we describe the shape of curves associated with (i) the occurrence of multipulse-homoclinic solutions to E, i.e. trajectories converging to E in forward and backward time while taking a prescribed number of turns around the periodic solution P in between, and (ii) homoclinic tangencies to P. Our results yield a division of the two-dimensional parameter space into regions with different dynamical behavior.

6. Moduli of Stability for Heteroclinic Cycles of Periodic Solutions
A set of quantities for a dynamical system that are invariant under topological conjugacy and fully characterize the conjugacy classes is called a *complete set of invariants*. In this paper we derive such a set of eight invariants for a system with a heteroclinic cycle between two hyperbolic periodic solutions with two-dimensional connections in both directions, and show that it is minimal in the sense that there is no complete set of invariants with fewer elements. Previous work, see [2] and references therein, established similar results for heteroclinic connections/cycles involving at most one periodic solution (and hyperbolic equilibria otherwise). We find the invariants by analyzing sequences of hitting times associated with trajectories that are attracted to the cycle. Completeness is proved through adapting an argument by Takens [21], which allows us to define an injective and continuous way of recovering orbits from hitting time sequences, which eventually establishes the desired conjugacy between two systems for which our invariants are identical.

7. Switching in Heteroclinic Networks
The term *switching* refers to the question which sequences of connections in a given heteroclinic network X are followed by trajectories converging to X. There are different levels of switching, ranging from the weakest (switching along single connection or node that belongs to more than one cycle) to the strongest (infinite switching). We show here that switching

along a connection – and thus all stronger forms of switching – is prevented by a geometric obstruction in the global map for the transition along the common connection. This result is valid in $\mathbb{R}^n$ for any $n \geq 4$, but the obstruction is more likely to occur in low dimensions. We investigate two examples in $\mathbb{R}^5$ where it does not hold and show that these may indeed possess some form of switching – a correction to one of our results was later made in Corollary 4.2.7 of [5], adding an extra condition to the statement of our Lemma 3.7.

8. Heteroclinic Dynamics of Localized Frequency Synchrony: Stability of Heteroclinic Cycles and Networks

In systems of coupled oscillators heteroclinic cycles and networks may appear robustly due to the invariance of synchrony subspaces. In this paper we analyze a system of four populations with two oscillators each where we establish the existence of a heteroclinic network involving nine (relative) equilibria connected by one-dimensional intersections of their respective invariant manifolds. We show that the cycles in the network are quasi-simple and thus apply results from [6] to obtain stability indices for all connections in the network. This yields an unprecedented degree of detail in the description of possible stability configurations and attraction properties in such a setting, giving insight on how the coupling structure of oscillator systems can influence its potential attractors.

9. Almost Complete and Equable Heteroclinic Networks

It has become increasingly clear over the last decade that higher-dimensional heteroclinic connections are pivotal for understanding attraction of and the dynamics near objects made up of lower-dimensional connections. This leads to the question of *completeness*: for a given heteroclinic structure, what can one say about the smallest possible set containing all unstable manifolds of its nodes? In this work we take the opposite approach: given a transitive directed graph we develop a method for realizing it as a heteroclinic network with some desirable properties. In particular, we show that even though completeness can usually not be expected, our realizations can be made *almost complete*, i.e. all unstable manifolds are contained up to a set of measure zero. We further define a new property called *equability* which relates to the dimensions of different outgoing connections of the same node. This can be loosely interpreted as a fairness condition, where none of the connections prescribed by the initial graph is favored over another in the realization. We also discuss how almost complete equable networks appear naturally in a one-step discrete-time Markov switching process defined on a network.

The following chapters contain all published papers in the publisher's pdf layout and the preprint in its most recent form. Page numbers in the table of contents refer to the actual pages of this document and they are not displayed on pages belonging to a published paper or preprint. For each publication there is a header page explaining the origin of the idea and giving an overview of the work process as well as the contributions of the different authors (if applicable). Literature references on these header pages refer to the numbering/labeling in the bibliography of the respective paper.

Two projects are mentioned in several of the header paragraphs, these are:

1. *Dynamics and Stability in Heteroclinic Networks*, Incentivo/MAT/UI0144/2014, funded by the Fundação para a Ciência e a Tecnologia, Portugal, 04/2015–09/2015

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Part I.

A Systematic Study in $\mathbb{R}^4$

1. Construction of Heteroclinic Networks in $\mathbb{R}^4$

This version of my habilitation thesis does not contain the full published papers, but rather links to where they can be found:

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2. Simple Heteroclinic Networks in $\mathbb{R}^4$

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3. Pseudo-simple Heteroclinic Cycles in $\mathbb{R}^4$

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Examples and Counterexamples 2, 2022

Part II.

Heteroclinic Phenomena Beyond $\mathbb{R}^4$

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8. Heteroclinic Dynamics of Localized Frequency Synchrony: Stability of Heteroclinic Cycles and Networks

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List of the Publications making up this Thesis

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Construction of heteroclinic networks in $\mathbb{R}^4$
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[I-2] O. Podvigina and A. Lohse (2019)
Simple heteroclinic networks in $\mathbb{R}^4$
Nonlinearity 32, 3269–3293

[I-3] P. Chossat, A. Lohse, O. Podvigina (2018)
Pseudo-simple heteroclinic cycles in $\mathbb{R}^4$
Physica D 372, 1–21

[I-4] S.B.S.D. Castro and A. Lohse (2022)
A hybrid heteroclinic cycle
Examples and Counterexamples 2

[II-5] A. Lohse and A.P. Rodrigues (2017)
Boundary crisis for degenerate singular cycles
Nonlinearity 30, 2211–2245

[II-6] M. Carvalho, A. Lohse, A.P. Rodrigues (2019)
Moduli of stability for heteroclinic cycles of periodic solutions
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[II-7] S.B.S.D. Castro and A. Lohse (2016)
Switching in heteroclinic networks
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[II-8] C. Bick and A. Lohse (2019)
Heteroclinic Dynamics of Localized Frequency Synchrony: Stability of Heteroclinic Cycles and Networks
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