Provability Logics of Constructive Theories

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Core Logic,
Wednesday, November 14, 2007

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Overview

Provability Logic

Friedman's Classical Problem

Friedman's Problem: the Constructive Variant

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The First Incompleteness Theorem

Let T be a theory that interprets a reasonable weak theory of arithmetic like Buss' S^1_2. In this talk we will also consider the possibility that such a theory is constructive.

We write $\Box_T A$ for $\mathrm{Prov}_T(\underline{\ulcorner A \urcorner})$.

The Gödel sentence for T:

- $T \vdash G \leftrightarrow \neg \Box_T G$.

We have:

$$\begin{array}{rcl} T \vdash G & \Rightarrow & T \vdash \Box_T G \\ & \Rightarrow & T \vdash \neg G \\ & \Rightarrow & T \vdash \bot \end{array}$$

The Second Incompleteness Theorem

We formalize the above reasoning in T.

$$\begin{array}{rcl} T \vdash \Box_T G & \rightarrow & \Box_T \Box_T G \\ & \rightarrow & \Box_T \neg G \\ & \rightarrow & \Box_T \bot \end{array}$$

We find $T \vdash G \leftrightarrow \neg \Box_T \bot$.

So the second incompleteness theorem follows from the first.

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Arithmetical Interpretations

We interpret the language of modal propositional logic into T via interpretations $(\cdot)^*$ that send the propositional atoms to arbitrary sentences, commute with the propositional connectives and satisfy:

- $(\Box\phi)^* := \Box_T \phi^*$.

We say that ϕ is (an) arithmetically valid (scheme) for T iff, for all $(\cdot)^*$, we have $T \vdash \phi^*$.

Löb's Logic

Löb's Logic aka GL is the modal propositional theory axiomatized by classical propositional logic plus the following axioms and rules.

L1. $\vdash (\Box\phi \wedge \Box(\phi \to \psi)) \to \Box\psi$,

L2. $\vdash \Box\phi \to \Box\Box\phi$,

L3. $\vdash \Box(\Box\phi \to \phi) \to \Box\phi$,

L4. $\vdash \phi \quad \Rightarrow \quad \vdash \Box\phi$.

Löb's Logic is arithmetically sound for all classical theories that interpret Buss' S^1_2. It is arithmetically complete for all classical Σ^0_1-sound theories that interpret EA (Elementary Arithmetic) aka $I\Delta_0 + \mathrm{Exp}$. (Solovay 1976)

Some Theorems

GL is complete for finite transitive irreflexive Kripke models.

A variable p is *modalized* in ϕ iff all its occurrences are in the scope of a box. We write $\boxplus\phi$ for $\phi \wedge \Box\phi$.

Bernardi, de Jongh, Sambin: Suppose p is modalized in ϕp.

- $\vdash (\boxplus(p \leftrightarrow \phi p) \wedge \boxplus(q \leftrightarrow \phi q)) \rightarrow (p \leftrightarrow q)$.

Sambin, de Jongh: Suppose p is modalized in $\phi p\vec{q}$. Then, there is a $\psi\vec{q}$, such that:

- $\vdash \psi\vec{q} \leftrightarrow \phi(\psi\vec{q})\vec{q}$.

E.g. if ϕp is $\neg\Box p$, then ψ is $\neg\Box\bot$.

Shavrukov: GL has uniform interpolation.

The Constructive Case

Provability Logics of theories are not mononotonic in these theories!

iGL is sound for extensions of iS^1_2.

Principles for Heyting Arithmetic aka HA.

Leivant's Principle $\vdash \Box(\phi \vee \psi) \to \Box(\phi \vee \Box\psi)$.

Markov's Rule $\vdash \Box\neg\neg\Box\phi \to \Box\Box\phi$.

Anti-Markov's Rule $\vdash \Box(\neg\neg\Box\phi \to \Box\phi) \to \Box\Box\phi$.

In *classical* GL plus Leivant's Principle we have:

$$\begin{aligned} \vdash \Box(\Box\bot \vee \neg\Box\bot) &\to \Box(\Box\bot \vee \Box\neg\Box\bot) \\ &\to \Box\Box\bot \end{aligned}$$

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The Problem

The closed fragment of provability logic is simply the logic for zero propositional variables.

Friedman's 35th problem was to give a decision procedure for the closed fragment of the provability logic of Peano Arithmetic, PA. (Friedman 1975) It was indepently solved by van Benthem, Boolos and Bernardi & Montagna.

The van Benthem-Boolos-Bernardi-Montagna result holds for Σ^0_1-sound theories that interpret S^1_2.

Degrees of Falsity

Let $\omega^+ := \omega \cup \{\infty\}$. We equip ω^+ with the usual ordering and define $\infty + 1 := \infty$. Note that the successor function remains injective under this extension.

We define *the modal degrees of falsity* as follows.

- $\Box^0 \bot := \bot$,
- $\Box^{n+1} \bot := \Box\Box^n \bot$,
- $\Box^\infty \bot := \top$.

We have:

1. $\vdash (\Box^\alpha \bot \wedge \Box^\beta \bot) \leftrightarrow \Box^{\min(\alpha,\beta)} \bot$.
2. $\vdash (\Box^\alpha \bot \vee \Box^\beta \bot) \leftrightarrow \Box^{\max(\alpha,\beta)} \bot$.
3. $\vdash \Box(\Box^\alpha \bot \to \Box^\beta \bot) \leftrightarrow \Box^\infty \bot$, if $\alpha \leq \beta$.
4. $\vdash \Box(\Box^\alpha \bot \to \Box^{\beta+1} \bot) \leftrightarrow \Box^\beta \bot$, if $\alpha < \beta$.

The Basic Idea

Suppose ϕ is a Boolean combination of degrees of falsity.

$$\begin{aligned}
\vdash \Box\phi \quad & \leftrightarrow \quad \Box \bigwedge \bigvee \pm\Box^{\alpha}\bot \\
& \leftrightarrow \quad \Box \bigwedge (\bigvee \Box^{\beta}\bot \vee \neg \bigwedge \Box^{\gamma}\bot) \\
& \leftrightarrow \quad \Box \bigwedge (\Box^{\delta}\bot \rightarrow \Box^{\varepsilon}\bot) \\
& \leftrightarrow \quad \bigwedge \Box(\Box^{\delta}\bot \rightarrow \Box^{\varepsilon}\bot) \\
& \leftrightarrow \quad \Box^{\eta}\bot
\end{aligned}$$

We now prove, by induction on ψ, that any ψ in the closed fragment is a equivalent to a Boolean combination of degrees of falsity.

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Target Theories

We can characterize the closed fragments for HA, $\mathrm{HA} + \mathrm{MP}$, $\mathrm{HA}^\star$ and PA.

Markov's Principle MP:

- $\vdash (\forall x\,(Ax \vee \neg Ax) \wedge \neg\neg\,\exists x\; Ax) \to \exists x\; Ax$.

Open: $\mathrm{HA} + \mathrm{ECT}_0$ and $\mathrm{MA} = \mathrm{HA} + \mathrm{ECT}_0 + \mathrm{MP}$.

Visser (1985, 1994, 2002): solution for HA using translation methods and a computation of semi-normal forms modulo a suitable equivalence relation..

Theories of Degrees of Falsity

We write α for $\Box^{\alpha}\bot$. We consider theories in the propositional language where the degrees of falsity are treated as propositional constants.

We work in a propositional language with the constants α without variables. The theory Basic is axiomatized by Intuitionistic Propositional Logic plus $\vdash \alpha \to \beta$, for $\alpha \leq \beta$.

We consider extensions Γ of Basic.

- Γ is *p-sound* if $\Gamma \vdash \alpha \to \beta$ implies $\alpha \leq \beta$.
- Γ is *decent* if, for every ϕ and for every *n* larger than all *m* occurring in ϕ, we have $\Gamma \vdash n \to \phi$ implies $\Gamma \vdash \phi$.
- $\alpha_\Gamma(\phi) := \max\{\alpha \mid \Gamma \vdash \alpha \to \phi\}$.

Salient Theories of Degrees

- Stronglöb := Basic + $\{((\alpha \to \beta) \to \beta) \mid \beta < \alpha\}$,
- Stable := Basic + $\{\neg\neg\,\alpha \to \alpha \mid \alpha \in \omega^+\}$,
- Classical := Basic + $\{\alpha \vee \neg\,\alpha \mid \alpha \in \omega^+\}$.

1. Basic corresponds to HA.
2. Stronglöb corresponds to HA*.
3. Stable corresponds to HA + MP.
4. Classical corresponds to PA.

From Theories of Degrees to Closed Fragments

Suppose Γ is a decent theory of degrees. We define the closed fragment AL_Γ by introducing a modal operator setting $\Box\phi :\leftrightarrow \alpha_\Gamma(\phi) + 1$. We find that AL_Γ is a closed fragment and that its theory of degrees of falsity is Γ.

Intuition: the box of AL_Γ is the strongest or most informative box for closed modal theories compatible with Γ.

We prove $\mathsf{AL}_\Gamma \vdash \Box(\Box\phi \to \phi) \to \Box\phi$. In case $\alpha_\Gamma(\phi) = \infty$, we are easily done. Let $n := \alpha_\Gamma(\phi)$. We have:

1. $\vdash n \to ((n+1) \to \phi)$, since $\vdash n \to \phi$.
2. $\nvdash (n+1) \to ((n+1) \to \phi)$, since $\nvdash (n+1) \to \phi$.

So $\alpha_\Gamma(\Box\phi \to \phi) = n$.

From Theories of Degrees to Closed Fragments

Theorem
The closed fragments of HA, $\mathrm{HA}^\star$, $\mathrm{HA}+\mathrm{MP}$ and PA are respectively $\mathrm{AL}_{\mathrm{Basic}}$, $\mathrm{AL}_{\mathrm{Stronglöb}}$, $\mathrm{AL}_{\mathrm{Stable}}$, $\mathrm{AL}_{\mathrm{Classical}}$.

I.o.w., we have $\mathrm{CF}_T = \mathrm{AL}_{\mathrm{TDF}_T}$ for these theories. We might say: we have 'box-elimination' for these fragments.