Foundations of Mathematics.

- **Two weeks ago:** First-order logic as a foundations of mathematics, completeness theorem.
- **Last week:** Frege's logicism: can we derive the basic mathematical concepts from logic alone? No! So, we need axioms and show that they form a consistent foundations: Hilbert's Programme.

Basic four areas of mathematical logic:

- Set Theory
- Proof Theory
- Recursion Theory
- Model Theory

The Continuum Hypothesis (1).

If AC holds, then the real numbers $\mathbb{R}$ are wellorderable. That means there is an ordinal α such that $\mathbb{R}$ and α are equinumerous. Let $\mathfrak{c}$ be the least such ordinal. We know by Cantor's theorem that this cannot be a countable ordinal. There is an ordinal that is not equinumerous to the natural numbers. We call it ω_1.

Question. What is the relationship between $\mathfrak{c}$ and ω_1?

CH. $\omega_1 = \mathfrak{c}$. The least ordinal that is not equinumerous to the natural numbers is the least ordinal that is equinumerous to the real numbers.

The Continuum Hypothesis (2).

Hilbert (1900). ICM in Paris: Mathematical Problems for the XXth century.

> *"Es erhebt sich nun die Frage, ob das Continuum auch als wohlgeordnete Menge aufgefaßt werden kann, was Cantor bejahen zu müssen glaubt."*

In other words: CH implies "there is a wellordering of the real numbers".

- **Question 1.** Does $\mathsf{ZF} \vdash \mathsf{AC}$?
- **Question 2.** Does $\mathsf{ZF} \vdash \mathsf{CH}$?
- **Question 2*.** Does $\mathsf{ZFC} \vdash \mathsf{CH}$?

All of these questions were wide open in 1930.

Hilbert's Programme (1).

- 1900: *Hilbert's 2nd problem.* "Is there a finitistic proof of the consistency of the arithmetical axioms?"
- 1917-1921: Hilbert develops a predecessor of modern first-order logic.
- Paul Bernays (1888-1977)

  - Assistant of Zermelo in Zürich (1912-1916).
  - Assistant of Hilbert in Göttingen (1917-1922).
  - Completeness of propositional logic.
  - "Hilbert-Bernays" (1934-1939).
- Hilbert-Ackermann (1928).
- **Goal.** Axiomatize mathematics and find a finitary consistency proof.

Hilbert's Programme (2).

- 1922: Development of ε-calculus (Hilbert & Bernays). General technique for consistency proofs: "ε-substitution method".
- 1924: Ackermann presents a (false) proof of the consistency of analysis.

Richard **Zach**, The practice of finitism: epsilon calculus and consistency proofs in Hilbert's program, **Synthese** 137 (2003), p. 211-259

Richard **Zach**, Hilbert's 'Verunglückter Beweis', the first epsilon theorem, and consistency proofs, **History and Philosophy of Logic** 25 (2004), p. 79-94

Hilbert's Programme (2).

- 1922: Development of ε-calculus (Hilbert & Bernays). General technique for consistency proofs: "ε-substitution method".
- 1924: Ackermann presents a (false) proof of the consistency of analysis.
-  1925: John von Neumann (1903-1957) corrects some errors and proves the consistency of an ε-calculus without the induction scheme.
- 1928: At the ICM in Bologna, Hilbert claims that the work of Ackermann and von Neumann constitutes a proof of the consistency of arithmetic.

Brouwer (1).

L. E. J. (Luitzen Egbertus Jan) Brouwer
(1881-1966)

- Student of Korteweg at the UvA.
- 1909-1913: Development of topology. Brouwer's Fixed Point Theorem.
- 1913: Succeeds Korteweg as full professor at the UvA.
- 1918: *"Begründung der Mengenlehre unabhängig vom Satz des ausgeschlossenen Dritten"*.

Brouwer (2).

- 1920: *"Besitzt jede reelle Zahl eine Dezimalbruch-Entwickelung?"*. Start of the *Grundlagenstreit*.
-  1921: Hermann Weyl (1885-1955), *"Über die neue Grundlagenkrise der Mathematik"*
- 1922: Hilbert, *"Neubegründung der Mathematik"*.
- 1928-1929: ICM in Bologna; *Annalenstreit*. Einstein and Carathéodory support Brouwer against Hilbert.

Intuitionism.

- Constructive interpretation of existential quantifiers.
- As a consequence, rejection of the *tertium non datur*.
- The big three schools of philosophy of mathematics: logicism, formalism, and intuitionism.
- Nowadays, different positions in the philosophy of mathematics are distinguished according to their view on ontology and epistemology. Main positions are: (various brands of) Platonism, Social Constructivism, Structuralism, Formalism.

Gödel (1).

Kurt Gödel (1906-1978)

- Studied at the University of Vienna; PhD supervisor Hans Hahn (1879-1934).
- Thesis (1929): Gödel Completeness Theorem.
- 1931: "*Über formal unentscheidbare Sätze der* Principia Mathematica *und verwandter Systeme I*". Gödel's First Incompleteness Theorem and a proof sketch of the Second Incompleteness Theorem.

Gödel (2).

- 1935-1940: Gödel proves the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the axioms of set theory (solving one half of Hilbert's 1st Problem).
- 1940: Emigration to the USA: Princeton.
- Close friendship to Einstein, Morgenstern and von Neumann.
- Suffered from severe hypochondria and paranoia.
- Strong views on the philosophy of mathematics.

Gödel's Incompleteness Theorem (1).

1928: At the ICM in Bologna, Hilbert claims that the work of Ackermann and von Neumann constitutes a proof of the consistency of arithmetic.

- 1930: Gödel announces his result (G1) in Königsberg in von Neumann's presence.
- Von Neumann independently derives the Second Incompleteness Theorem (G2) as a corollary.
- Letter by Bernays to Gödel (January 1931): There may be finitary methods not formalizable in $\mathbf{PA}$.
- 1931: Hilbert suggests new rules to avoid Gödel's result. Finitary versions of the ω-rule.
- By 1934, Hilbert's programme in the original formulation has been declared dead.

Gödel's Incompleteness Theorem (2).

Theorem (Gödel's Second Incompleteness Theorem). If T is a consistent axiomatizable theory containing $\mathbf{PA}$, then $T \nvdash \mathrm{Cons}(T)$.

- "consistent": $T \nvdash \bot$.
- "axiomatizable": T can be listed by a computer ("computably enumerable", "recursively enumerable").
- "containing $\mathbf{PA}$": $T \vdash \mathbf{PA}$.
- "$\mathrm{Cons}(T)$": The formalized version (in the language of arithmetic) of the statement 'for all T-proofs P, $\bot$ doesn't occur in P'.

Gödel's Incompleteness Theorem (3).

- Thus: Either $\mathbf{PA}$ is inconsistent or the deductive closure of $\mathbf{PA}$ is not a complete theory.
- All three conditions are necessary:
  - **Theorem** (Presburger, 1929). There is a weak system of arithmetic that proves its own consistency ("Presburger arithmetic").
  - If T is inconsistent, then $T \vdash \varphi$ for all φ.
  - If $\mathbb{N}$ is the standard model of the natural numbers, then $\mathrm{Th}(\mathbb{N})$ is a complete extension of $\mathbf{PA}$ (but not axiomatizable).

Gentzen.

Gerhard Gentzen (1909-1945)

- Student of Hermann Weyl (1933).
- 1934: Hilbert's assistant in Göttingen.
- 1934: Introduction of the Sequent Calculus.
- 1936: Proof of the consistency of $\mathbf{PA}$ from a transfinite wellfoundedness principle.

Theorem (Gentzen). Let $T \supseteq \mathbf{PA}$ such that T proves the existence and wellfoundedness of (a code for) the ordinal ε_0. Then $T \vdash \mathrm{Cons}(\mathbf{PA})$.

Arithmetic and orderings (1).

What is ε_0?
The first transfinite closure ordinal of the ordinal operations "addition", "multiplication", and "exponentiation".

But: Ordinals are not objects of arithmetic (neither first-order not second-order). So what should it mean that an arithmetical theory proves that "ε_0 is well-ordered"?

Arithmetic and orderings (2).

What should it mean that an arithmetical theory proves that "ε_0 is well-ordered"?

Let α be a countable ordinal. By definition, there is some bijection $f : \mathbb{N} \to \alpha$. Define

$$n <_f m :\leftrightarrow f(n) < f(m).$$

Clearly, f is an isomorphism between $\langle \mathbb{N}, <_f \rangle$ and α.

If $g : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ is an arbitrary function, we can interpret it as a binary relation on $\mathbb{N}$:

$$n <_g m :\leftrightarrow g(n, m) = 1.$$

Arithmetic and orderings (3).

Let us work in second-order arithmetic

$$\langle \mathbb{N}, \mathbb{N}^{\mathbb{N}}, 2^{\mathbb{N}\times\mathbb{N}}, +, \times, 0, 1, \mathrm{app} \rangle$$

$g : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ codes a wellfounded relation if and only if

$$\neg \exists F \in \mathbb{N}^{\mathbb{N}} \forall n \in \mathbb{N}(g(F(n+1), F(n)) = 1).$$

“Being a code for an ordinal $< \varepsilon_0$” is definable in the language of second-order arithmetic (ordinal notation systems).
$\mathrm{TI}(\varepsilon_0)$ is defined to be the formalization of “every code g for an ordinal $< \varepsilon_0$ codes a wellfounded relation”.

More proof theory (1).

$\mathrm{TI}(\varepsilon_0)$: “every code g for an ordinal $< \varepsilon_0$ codes a wellfounded relation”

Generalization: If “being a code for an ordinal $< \alpha$” can be defined in second-order arithmetic, then let $\mathrm{TI}(\alpha)$ mean “every code g for an ordinal $< \alpha$ codes a wellfounded relation”.

The proof-theoretic ordinal of a theory T.

$$|T| := \sup\{\alpha\,;\, T \vdash \mathrm{TI}(\alpha)\}$$

Rephrasing Gentzen. $|\mathsf{PA}| = \varepsilon_0$.

More proof theory (2).

Results from Proof Theory.

- The proof-theoretic ordinal of primitive recursive arithmetic is ω^ω.
- (Jäger-Simpson) The proof-theoretic ordinal of arithmetic with arithmetical transfinite recursion is Γ_0 (the limit of the Veblen functions).

These ordinals are all smaller than ω_1^{CK}, the least noncomputable ordinal.