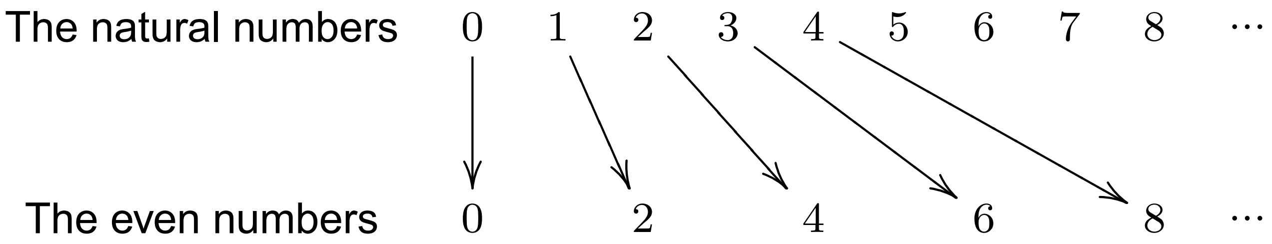
# Cantor.

Georg Cantor
(1845-1918)
studied in Zürich, Berlin, Göttingen
Professor in Halle

- Work in analysis leads to the notion of cardinality (1874): most real numbers are transcendental.
- Correspondence with Dedekind (1831-1916): bijection between the line and the plane.
- Perfect sets and iterations of operations lead to a notion of ordinal number (1880).

# Cardinality (1).




- There is a 1-1 correspondence (bijection) between $\mathbb{N}$ and the even numbers.
- There is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.
- There is a bijection between $\mathbb{Q}$ and $\mathbb{N}$.
- There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.
- There is no bijection between $\mathbb{R}$ and $\mathbb{N}$.

# Cardinality (2).

**Theorem** (Cantor). There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.

**Proof.** Suppose that $F$ were such a bijection, *i.e.*, for every $n$, $F_n$ is an infinite 0-1 sequence and for every such sequence $x$ there is some $n$ such that $F_n = x$. Define $d(n) := 1 - F_n(n)$. Then $d$ is an infinite 0-1 sequence. So there is some $m$ such that $d = F_m$. But $F_m(m) = d(m) = 1 - F_m(m)$. Contradiction! q.e.d.

**Theorem** (Cantor). There is a bijection between the real line and the real plane.

**Proof.** Let's just do it for the set of infinite 0-1 sequences and the set of pairs of infinite 0-1 sequences:
If $x$ is an infinite 0-1 sequence, then let $x_0(n) := x(2n)$, and $x_1(n) := x(2n+1)$. Let $F(x) := \langle x_0, x_1 \rangle$. $F$ is a bijection. q.e.d.

Cantor to Dedekind (1877): "*Ich sehe es, aber ich glaube es nicht!*"

# Transfiniteness (1).

If $X \subseteq \mathbb{R}$ is a set of reals, we call $x \in X$ isolated in $X$ if no sequence of elements of $X$ converges to $x$.

**Cantor's goal:** Given any set $X$, give a construction of a nonempty subset that doesn't contain any isolated points.

**Idea:** Let $X^{\text{isol}}$ be the set of all points isolated in $X$, and define $X' := X \backslash X^{\text{isol}}$.

**Problem:** It could happen that $x \in X'$ was the limit of a sequence of points isolated in $X$. So it wasn't isolated in $X$, but is now isolated in $X'$.

**Solution:** Iterate the procedure: $X_0 := X$ and $X_{n+1} := (X_n)'$.

# Transfiniteness (2).

$X' := X \backslash X^{\rm isol}$; $X_0 := X$ and $X_{n+1} := (X_n)'$.

**Question:** Is $\bigcap_{n\in\mathbb{N}} X_n$ a set without isolated points?

**Answer:** In general, no!

So, you could set $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$, and then $X_{\infty+1} := (X_\infty)'$; in general, $X_{\infty+n+1} := (X_{\infty+n})'$.

The indices used in transfinite iterations like this are called ordinals.

# Sets (Preview).

The notion of cardinality needs a general notion of function as a special relation between sets. In order to make the notion of an ordinal precise, we also need sets.

## What is a set?

*Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen.* (Cantor 1895)

# Syllogistics versus Propositional Logic.

*Deficiencies of Syllogistics:*

**Not expressible:**

Every $X$ is a $Y$ and a $Z$. *Ergo...* Every $X$ is a $Y$.

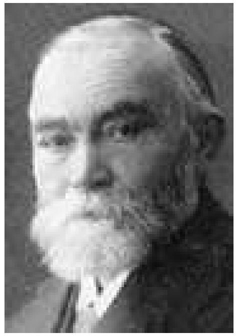
*Deficiencies of Propositional Logic:*

- $X\mathrm{a}Y$ can be represented as $Y \rightarrow X$.
- $X\mathrm{e}Y$ can be represented as $Y \rightarrow \neg X$.

**Not expressible:**

$X\mathrm{i}Y$ and $X\mathrm{o}Y$.

# Frege.

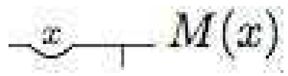


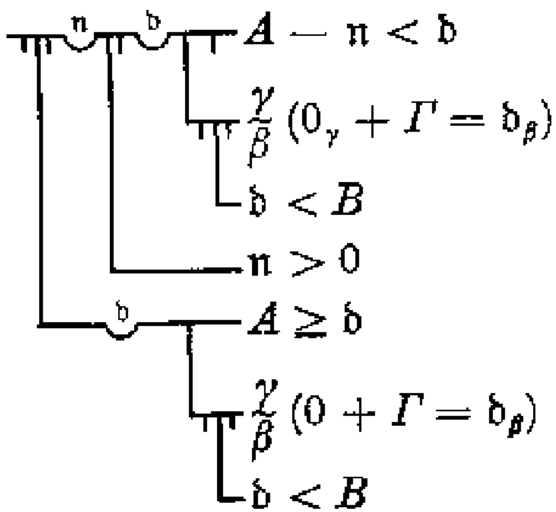
**Gottlob Frege**
1848 - 1925

- Studied in Jena and Göttingen.
- Professor in Jena.
- *Begriffsschrift* (1879).
- *Grundgesetze der Arithmetik* (1893/1903).

"Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician. (G. Frege)"

# Frege's logical framework.

| | | |
|---|---|---|
| "Everything is $M$" |  | $\forall x\, M(x)$ |
| "Something is $M$" |  | $\exists x\, M(x) \equiv \neg\forall x\, \neg M(x)$ |
| "Nothing is $M$" |  | $\forall x\, \neg M(x)$ |
| "Some $P$ is an $M$" |  | $\exists x\, (P(x) \wedge M(x))$ $\equiv \neg\forall x\, (P(x) \rightarrow \neg M(x))$ |



Second order logic allowing for quantification over properties.

# Frege's importance.

- Notion of a formal system.
- Formal notion of proof in a formal system.
- Analysis of number-theoretic properties in terms of second-order properties.
  $\rightsquigarrow$ Russell's Paradox
  (*Grundlagekrise der Mathematik*)

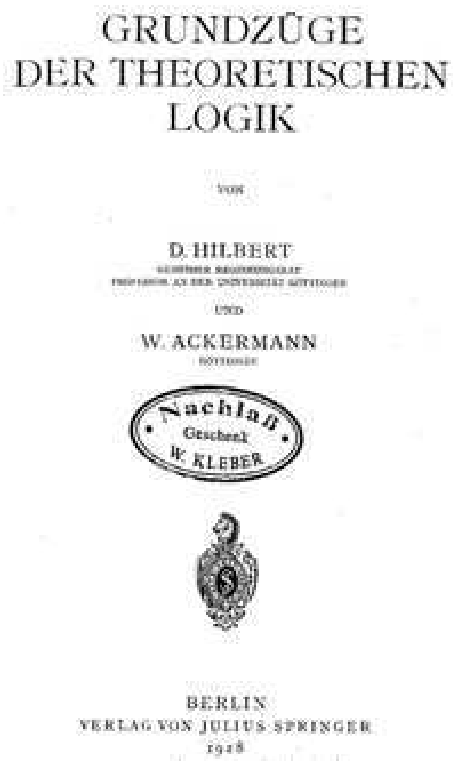
# Hilbert (1).

David Hilbert (1862-1943)
Student of Lindemann
1886-1895 Königsberg
1895-1930 Göttingen

1899: *Grundlagen der Geometrie*

"*Man muss jederzeit an Stelle von 'Punkten', 'Geraden', 'Ebenen' 'Tische', 'Stühle', 'Bierseidel' sagen können.*"
"It has to be possible to say 'tables', 'chairs' and 'beer mugs' instead of 'points', 'lines' and 'planes' at any time."

# Hilbert (2).



GRUNDZÜGE
DER THEORETISCHEN
LOGIK

VON

D. HILBERT

UND

W. ACKERMANN



BERLIN
VERLAG VON JULIUS SPRINGER
1928

1928: **Hilbert-Ackermann**
*Grundzüge der Theoretischen Logik*

Wilhelm Ackermann (1896-1962)

# First order logic (1).

A first-order language $\mathcal{L}$ is a set $\{\dot{\mathrm{f}}_i\,;\, i \in I\} \cup \{\dot{\mathrm{R}}_j\,;\, j \in J\}$ of function symbols and relation symbols together with a signature $\sigma : I \cup J \to \mathbb{N}$.

- $\sigma(\dot{\mathrm{f}}_i) = n$ is interpreted as "$\dot{\mathrm{f}}_i$ represents an $n$-ary function".
- $\sigma(\dot{\mathrm{R}}_i) = n$ is interpreted as "$\dot{\mathrm{R}}_i$ represents an $n$-ary relation".

In addition to the symbols from $\mathcal{L}$, we shall be using the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, equality $=$, and a set of variables $\mathrm{Var}$.

# First order logic (2).

We fix a first-order language $\mathcal{L} = \{\dot{\mathrm{f}}_i\ ;\ i \in I\} \cup \{\dot{\mathrm{R}}_j\ ;\ j \in J\}$ and a signature $\sigma : I \cup J \to \mathbb{N}$.

**Definition of an $\mathcal{L}$-term.**

- Every variable is an $\mathcal{L}$-term.
- If $\sigma(\dot{\mathrm{f}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{f}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-term.
- Nothing else is an $\mathcal{L}$-term.

**Example.** Let $\mathcal{L} = \{\dot{\times}\}$ be a first order language with a binary function symbol.

- $\dot{\times}(x, x)$ is an $\mathcal{L}$-term (normally written as $x \dot{\times} x$, or $x^2$).
- $\dot{\times}(\dot{\times}(x, x), x)$ is an $\mathcal{L}$-term (normally written as $(x \dot{\times} x) \dot{\times} x$, or $x^3$).

# First order logic (3).

**Definition of an $\mathcal{L}$-formula.**

- If $t$ and $t^*$ are $\mathcal{L}$-terms, then $t = t^*$ is an $\mathcal{L}$-formula.
- If $\sigma(\dot{\mathrm{R}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{R}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-formula.
- If $\varphi$ and $\psi$ are $\mathcal{L}$-formulae and $x$ is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $\forall x\,(\varphi)$ and $\exists x\,(\varphi)$ are $\mathcal{L}$-formulae.
- Nothing else is an $\mathcal{L}$-formula.

An $\mathcal{L}$-formula without free variables is called an $\mathcal{L}$-sentence.

# Semantics (1).

We fix a first-order language $\mathcal{L} = \{\dot{\mathrm{f}}_i\, ;\, i \in I\} \cup \{\dot{\mathrm{R}}_j\, ;\, j \in J\}$ and a signature $\sigma : I \cup J \to \mathbb{N}$.

A tuple $\mathbf{X} = \langle X, \langle f_i\, ;\, i \in I\rangle, \langle R_j\, ;\, j \in J\rangle\rangle$ is called an $\mathcal{L}$-structure if $f_i$ is an $\sigma(\dot{\mathrm{f}}_i)$-ary function on $X$ and $R_i$ is an $\sigma(\dot{\mathrm{R}}_i)$-ary relation on $X$.

An $X$-interpretation is a function $\iota : \mathrm{Var} \to X$.

If $\iota$ is an $X$-interpretation and $\mathbf{X}$ is an $\mathcal{L}$ then $\iota$ extends to a function $\hat{\iota}$ on the set of all $\mathcal{L}$-terms.

If $\mathbf{X}$ is an $\mathcal{L}$-structure and $\iota$ is an $X$-interpretation, we define a semantics for all $\mathcal{L}$-formulae by recursion.

# Semantics (2).

If $\mathbf{X}$ is an $\mathcal{L}$-structure and $\iota$ is an $X$-interpretation, we define a semantics for all $\mathcal{L}$-formulae by recursion.

- $\mathbf{X}, \iota \models t = t^*$ if and only if $\hat{\iota}(t) = \hat{\iota}(t^*)$.
- $\mathbf{X}, \iota \models \dot{\mathrm{R}}_j(t_1, ..., t_n)$ if and only if $R(\hat{\iota}(t_1), ..., \hat{\iota}(t_n))$.
- $\mathbf{X}, \iota \models \varphi \wedge \psi$ if and only if $\mathbf{X}, \iota \models \varphi$ and $\mathbf{X}, \iota \models \psi$.
- $\mathbf{X}, \iota \models \neg\varphi$ if and only if it is not the case that $\mathbf{X}, \iota \models \varphi$.
- $\mathbf{X}, \iota \models \forall x\,(\varphi)$ if and only if for all $X$-interpretations $\iota^*$ with $\iota \sim_x \iota^*$, we have $\mathbf{X}, \iota^* \models \varphi$.
- $\mathbf{X} \models \varphi$ if and only if for all $X$-interpretations $\iota$, we have $\mathbf{X}, \iota \models \varphi$.

Object Language $\leftrightarrow$ Metalanguage.

# Semantics (3).

Object Language $\leftrightarrow$ Metalanguage.

Let $\mathbf{X}$ be an $\mathcal{L}$-structure. The theory of $\mathbf{X}$, $\mathrm{Th}(\mathbf{X})$, is the set of all $\mathcal{L}$-sentences $\varphi$ such that $\mathbf{X} \models \varphi$.

Under the assumption that the *tertium non datur* holds for the metalanguage, the theory of $\mathbf{X}$ is always complete:

For every sentence $\varphi$, we either have $\varphi \in \mathrm{Th}(\mathbf{X})$ or $\neg\varphi \in \mathrm{Th}(\mathbf{X})$.

# Deduction (1).

Let $\Phi$ be a set of $\mathcal{L}$-sentences. A $\Phi$-proof is a finite sequence $\langle \varphi_1, ..., \varphi_n \rangle$ of $\mathcal{L}$-formulae such that for all $i$, one of the following holds:

- $\varphi_i \equiv t = t$ for some $\mathcal{L}$-term $t$,
- $\varphi_i \in \Phi$, or
- there are $j, k < i$ such that $\varphi_j$ and $\varphi_k$ are the premisses and $\varphi_i$ is the conclusion in one of the rows of the following table.

| Premisses | | Conclusion |
|---|---|---|
| $\varphi \wedge \psi$ | | $\varphi$ |
| $\varphi \wedge \psi$ | | $\psi$ |
| $\varphi$ | $\psi$ | $\varphi \wedge \psi$ |
| $\varphi$ | $\neg\varphi$ | $\psi$ |
| $\varphi \rightarrow \psi$ | $\neg\varphi \rightarrow \psi$ | $\psi$ |
| $\forall x\,(\varphi)$ | | $\varphi\frac{s}{x}$ |
| $\varphi\frac{y}{x}$ | | $\forall x\,(\varphi))$ |
| $t = t^*$ | $\varphi\frac{t}{x}$ | $\varphi\frac{t^*}{x}$ |

# Deduction (2).

If $\Phi$ is a set of $\mathcal{L}$-sentences and $\varphi$ is an $\mathcal{L}$-formula, we write $\Phi \vdash \varphi$ if there is a $\Phi$-proof in which $\varphi$ occurs.

We call a set $\Phi$ of sentences a theory if whenever $\Phi \vdash \varphi$, then $\varphi \in \Phi$ ("$\Phi$ is deductively closed").

**Example.** Let $\mathcal{L} = \{\leq\}$ be the language of partial orders. Let $\Phi_{\text{p.o.}}$ be the axioms of partial orders, and let $\Phi$ be the deductive closure of $\Phi_{\text{p.o.}}$. $\Phi$ is not a complete theory, as the sentence $\forall x \forall y(x \leq y \vee y \leq x)$ is not an element of $\Phi$, but neither is its negation.

# Completeness.

Kurt Gödel (1906-1978)

*Semantic entailment.* We write $\Phi \models \varphi$ for "whenever $\mathbf{X} \models \Phi$, then $\mathbf{X} \models \varphi$".

**Gödel Completeness Theorem** (1929).

$\Phi \vdash \varphi$ if and only if $\Phi \models \varphi$.

"there is a $\Phi$-proof of $\varphi$" "for all $\mathbf{X} \models \Phi$, we have $\mathbf{X} \models \varphi$"

$\Phi \nvdash \varphi$ if and only if $\Phi \not\models \varphi$.

"no $\Phi$-proof contains $\varphi$" "there is some $\mathbf{X} \models \Phi \wedge \neg\varphi$"

# Applications (1).

**The Model Existence Theorem.**

If $\Phi$ is consistent (*i.e.*, $\Phi \nvdash \bot$), then there is a model $\mathbf{X} \models \Phi$.

**The Compactness Theorem.**

Let $\Phi$ be a set of sentences. If every finite subset of $\Phi$ has a model, then $\Phi$ has a model.

*Proof.* If $\Phi$ doesn't have a model, then it is inconsistent by the **Model Existence Theorem**.
So, $\Phi \vdash \bot$, *i.e.*, there is a $\Phi$-proof $P$ of $\bot$.
But $P$ is a finite object, so it contains only finitely many elements of $\Phi$. Let $\Phi_0$ be the set of elements occurring in $P$. Clearly, $P$ is a $\Phi_0$-proof of $\bot$, so $\Phi_0$ is inconsistent. Therefore $\Phi_0$ cannot have a model. q.e.d.

# Applications (2).

**The Compactness Theorem.** Let $\Phi$ be a set of sentences. If every finite subset of $\Phi$ has a model, then $\Phi$ has a model.

**Corollary 1.** Let $\Phi$ be a set of sentences that has arbitrary large finite models. Then $\Phi$ has an infinite model.

*Proof.* Let $\psi_{\geq n}$ be the formula stating "there are at least $n$ different objects". Let $\Psi := \{\psi_{\geq n} \, ; \, n \in \mathbb{N}\}$. The premiss of the theorem says that every finite subset of $\Phi \cup \Psi$ has a model. By compactness, $\Phi \cup \Psi$ has a model. But this must be infinite. q.e.d.

Let $\mathcal{L} := \{\leq\}$ be the first order language with one binary relation symbol. Let $\Phi_{\mathrm{p.o.}}$ be the axioms of partial orders.

**Corollary 2.** There is no sentence $\sigma$ such that for all partial orders $\mathbf{P}$, we have

$$\mathbf{P} \text{ is finite if and only if } \mathbf{P} \models \sigma.$$

[If $\sigma$ is like this, then **Corollary 1** can be applied to $\Phi_{\mathrm{p.o.}} \cup \{\sigma\}$.]