Coalgebras and Coalgebra Automata

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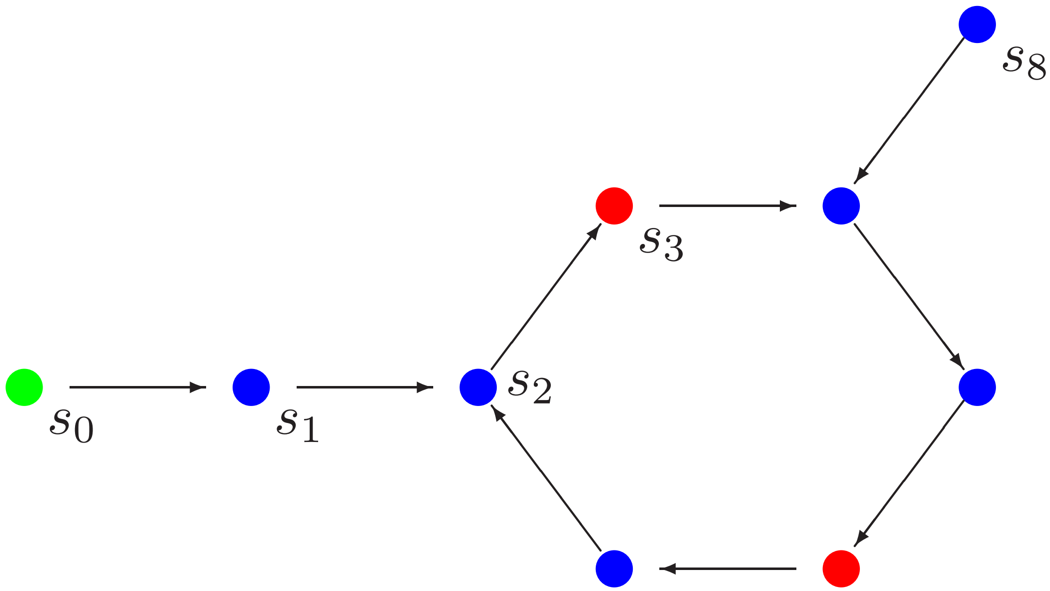
Japan Advanced Institute of Science and Technology

Overview of talk

- Examples
- Coalgebra
- Automata for coalgebras
- Finally, . . .

Streams



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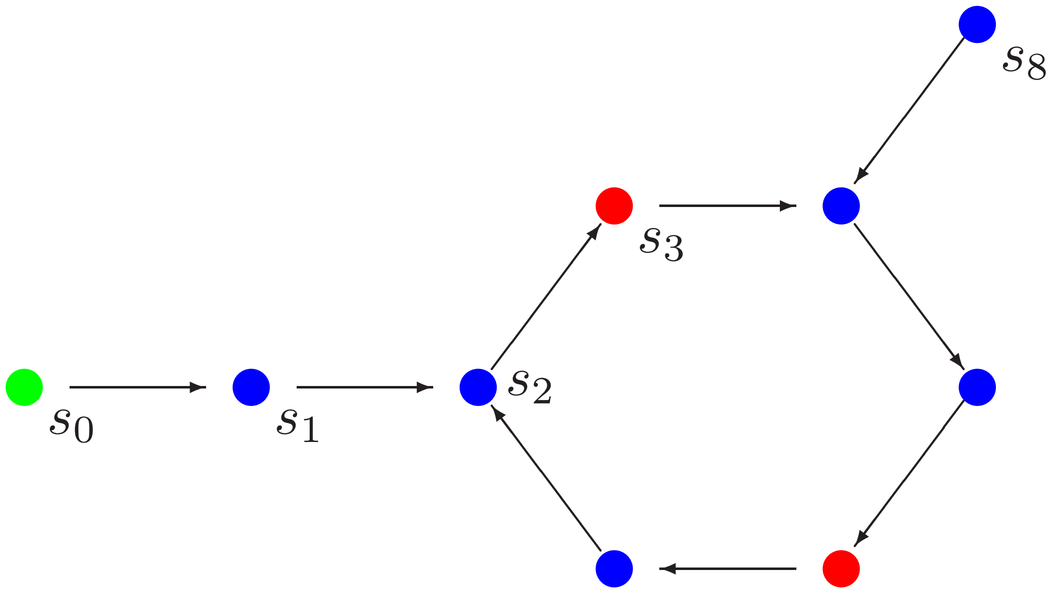

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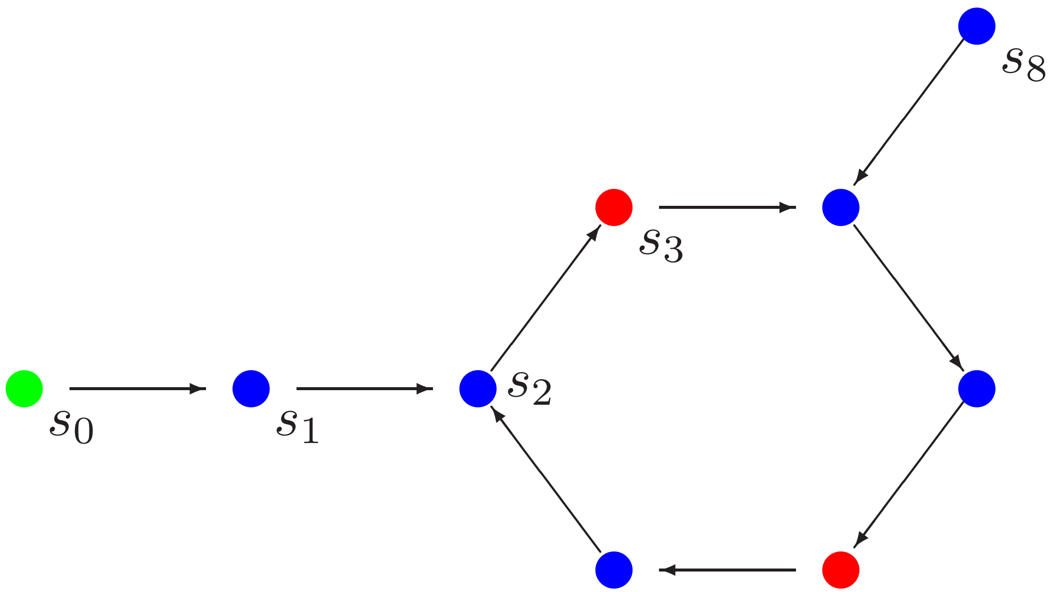

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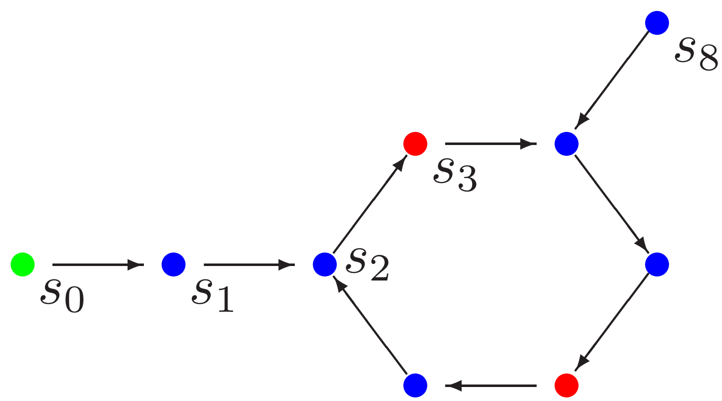
Streams as Coalgebras



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E.g. model an infinite word $c_0 c_1 c_2 \ldots$ as $\langle \omega, \lambda n.(c_n, n+1) \rangle$.

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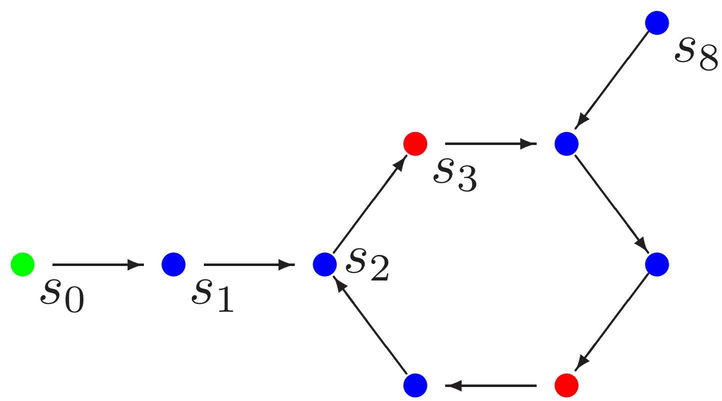

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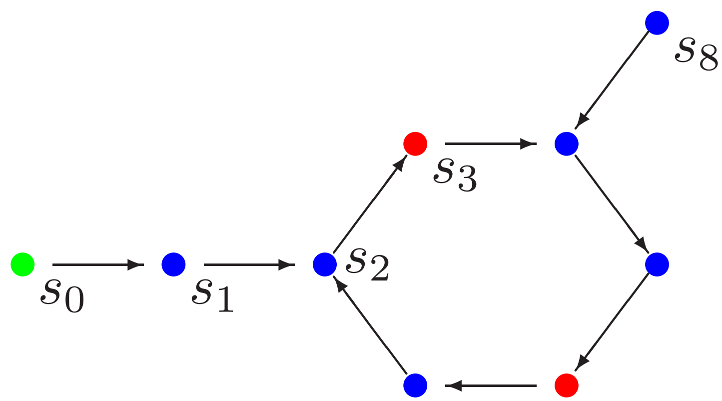

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Behaviour: $\mathsf{Beh}(s) = \mathsf{col}(s)\mathsf{col}(\mathsf{nxt}(s))\mathsf{col}(\mathsf{nxt}^2(s)) \ldots$

Behavioral equivalence

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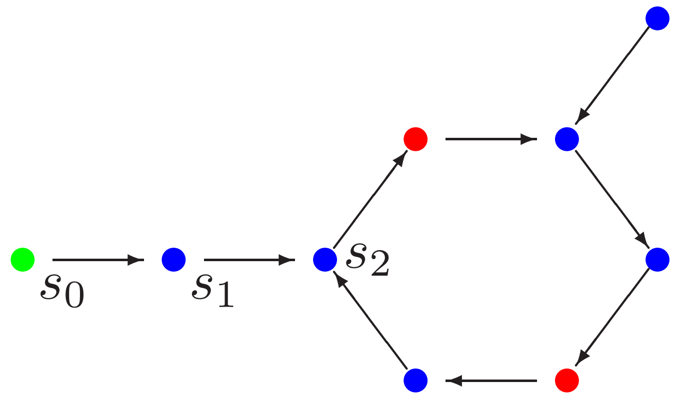
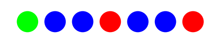
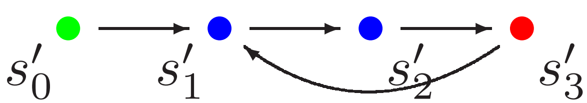


Beh(s_0) =  $\cdots$



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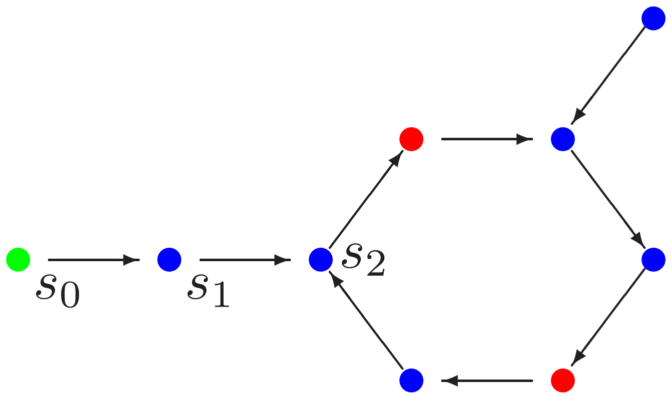
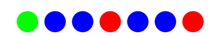
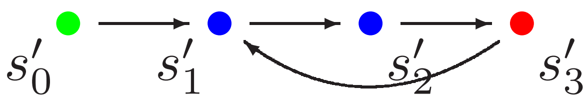


Beh(s_0) =  $\cdots$



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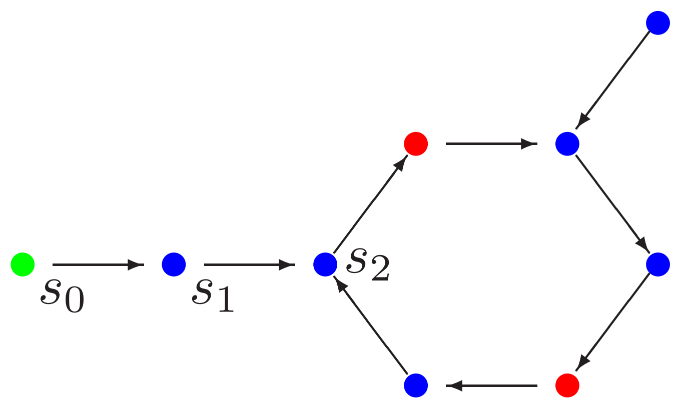
Definition: s in $\mathbb{S}$ is behaviorally equivalent to s' in $\mathbb{S}'$ if Beh(s) = Beh(s').

Bisimilarity



$Z \subseteq S \times S'$ is a bisimulation if for all $(s, s') \in Z$:

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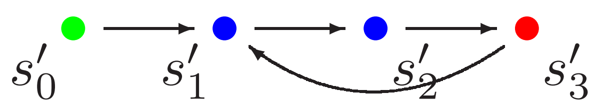
2. $(\mathsf{nxt}(s), \mathsf{nxt}(s') \in Z$.



s in $\mathbb{S}$ and s' in $\mathbb{S}'$ are bisimilar if linked by some bisimulation Z.

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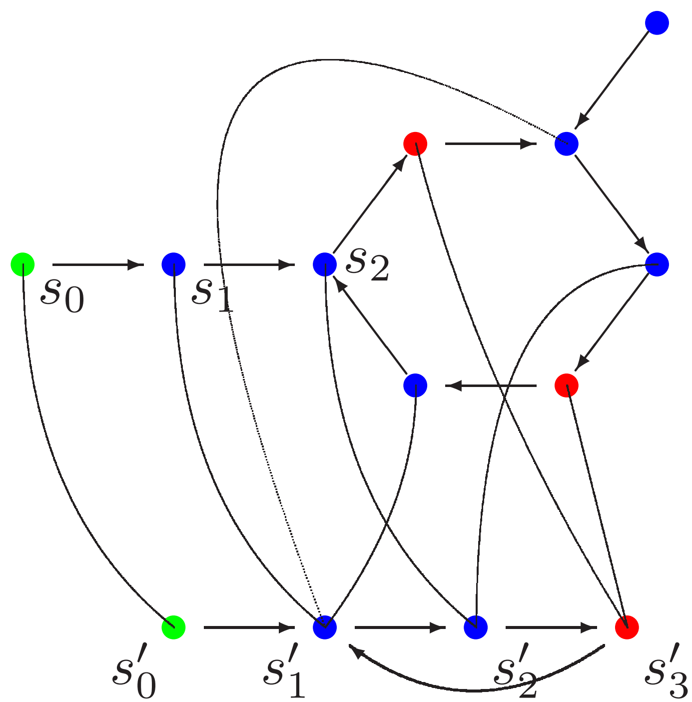

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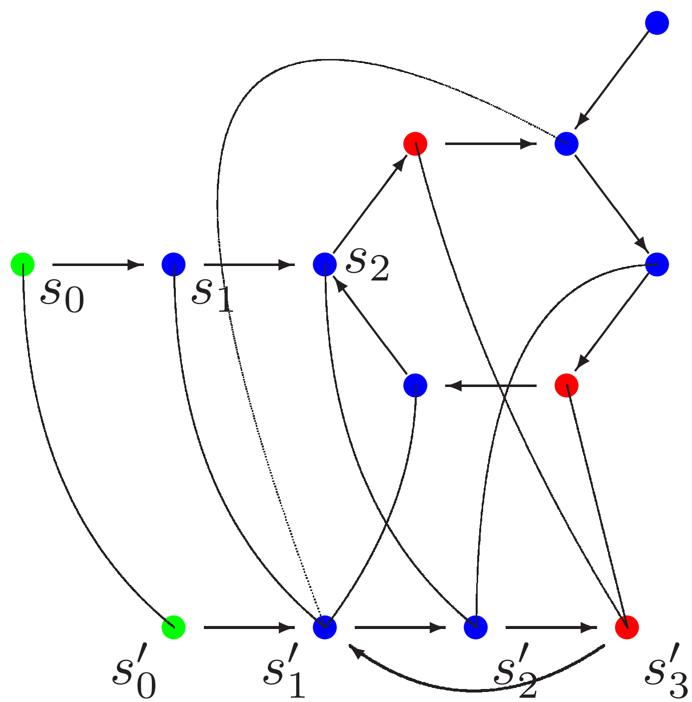

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Theorem: bisimilarity = behavioral equivalence

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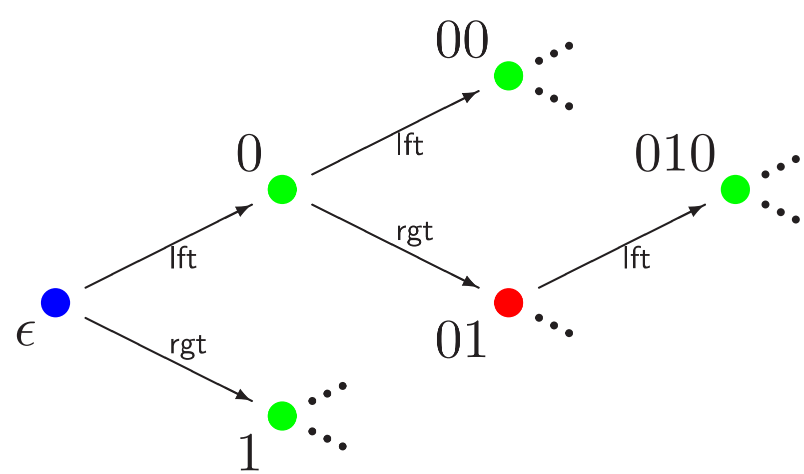
E.g. model an infinite C-labeled binary tree

$\gamma : \{0,1\}^* \to C$ as

$\langle \{0,1\}^*, \lambda s.(\gamma(s), s0, s1) \rangle$.



Bistreams bisimulations

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$Z \subseteq S \times S'$ is a bisimulation if for all $(s, s') \in Z$:

1. $\mathsf{col}(s) = \mathsf{col}(s')$,
2. $(\mathsf{lft}(s), \mathsf{lft}(s')) \in Z$ and $(\mathsf{rgt}(s), \mathsf{rgt}(s')) \in Z$,

Definition: An infinite C-labeled binary tree is regular iff it is bisimilar to a finite bistream

Kripke Models

Kripke structure: pair $\mathbb{S} = \langle S, \sigma : S \to \wp\mathsf{Prop} \times \wp S \rangle$, with

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Abbreviate $\mathsf{K}S := \wp\text{Prop} \times \wp S$.

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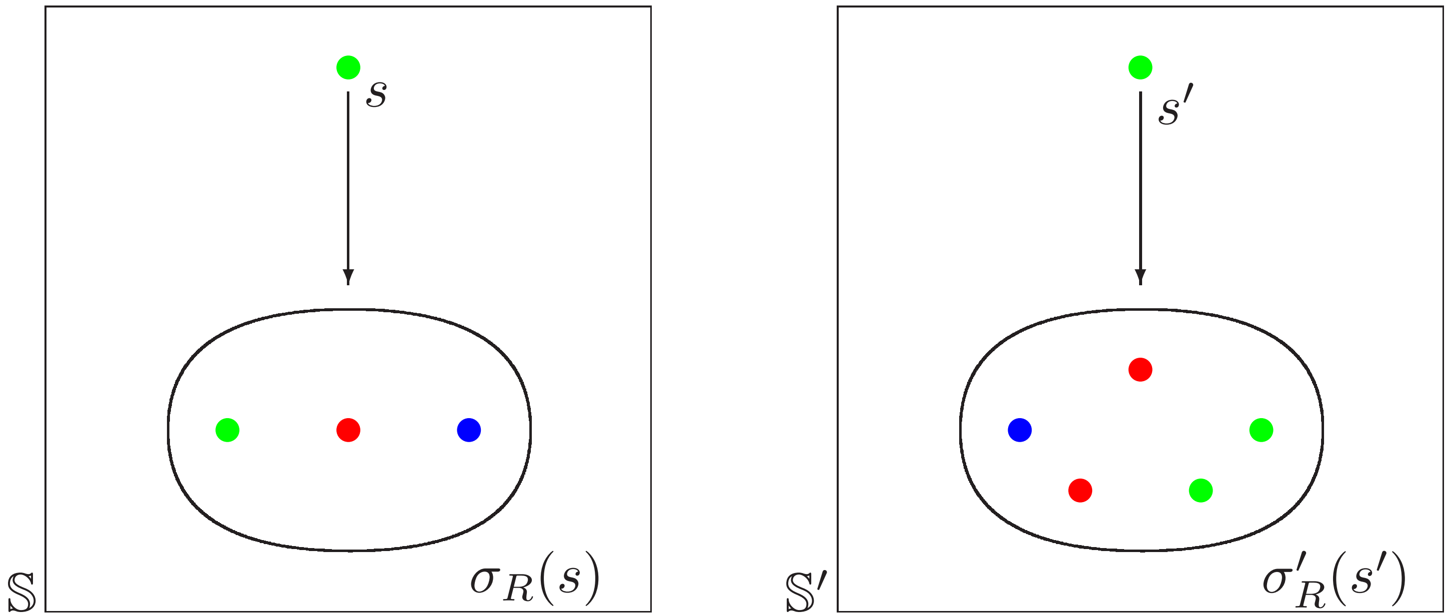

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Proposition: *Let $Z \subseteq S \times S'$ for two Kripke structures $\mathbb{S}$ and $\mathbb{S}'$.*
Z is a bisimulation iff it is a local bisimulation for $(\sigma(s), \sigma'(s'))$ whenever $(s, s') \in Z$.

Bisimilarity game

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Theorem: For all s, s': $(s, s') \in Win_{\exists}(\mathcal{B})$ iff $\mathbb{S}, s \leftrightarrows \mathbb{S}', s'$.

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- Automata for coalgebras
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  input, output, non-determinism, interaction, probability, . . .
- A pointed F-coalgebra is a pair $(\mathbb{S}, s_0)$ where $\mathbb{S}$ is a coalgebra, and s_0 is a designated point in $\mathbb{S}$.

Examples

- streams: $\mathsf{F}S = C \times S$
- bi-streams: $\mathsf{F}S = C \times S \times S$
- Kripke frames: $\mathsf{F}S = \wp(S)$
- Kripke models: $\mathsf{F}S = \wp(\mathsf{Prop}) \times \wp(S)$

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  $\mathbb{A} = \langle A, \alpha : A \leftarrow \mathsf{F} A \rangle$
  $\mathbb{C} = \langle C, \gamma : C \rightarrow \mathsf{F} C \rangle$
- congruence vs bisimulation
- induction vs coinduction
- varieties vs covarieties
- . . .

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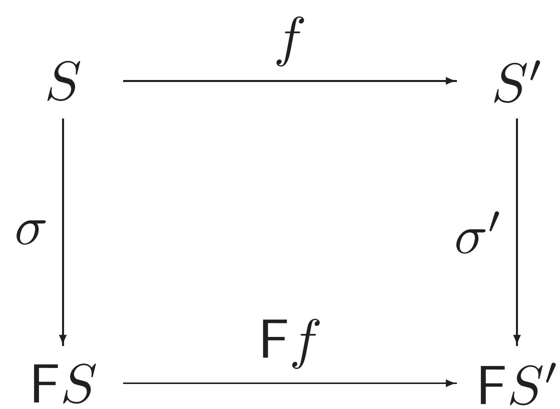
- A coalgebra homomorphism between two coalgebras $\mathbb{S}$ and $\mathbb{S}'$ is a map $f : S \to S'$ such that $\sigma' \circ f = \mathsf{F}f \circ \sigma$:



Overview of talk

- Examples
- Coalgebra
- Automata for coalgebras
- Finally, . . .

Automata Theory

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Claim: Coalgebra is a natural level of generality for studying automata

(Local) Bisimulation revisited

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Fix a coalgebra type F.

Fact: Any relation $Z \subseteq S \times S'$ can be lifted to a relation $\overline{\mathsf{F}}(Z) \subseteq \mathsf{F}S \times \mathsf{F}S'$.

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Kripke models $(\mathsf{F}S = \wp(\mathsf{Prop}) \times \wp(S))$
$((\pi,T),(\pi',T')) \in \overline{\mathsf{F}}(Z)$ iff $\pi = \pi'$ & $(T,T') \in \overline{\wp}(Z)$.

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$Z \subseteq A \times S$ is a bisimulation iff $(\alpha(a), \sigma(s)) \in \overline{\mathsf{F}}(Z)$ whenever $(a, s) \in Z$.

Bisimilarity game

With F-coalgebras $\mathbb{A} = \langle A, \alpha \rangle$ and $\mathbb{S} = \langle S, \sigma \rangle$,

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Theorem: For all a, s: $(a, s) \in Win_{\exists}(\mathcal{B})$ iff $\mathbb{A}, a \leftrightarroweq \mathbb{S}, s$.

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Definition A coalgebra automaton of type F is a triple $\mathbb{A} = \langle A, \Delta, Acc \rangle$.

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Definition: A pointed F-automaton $(\mathbb{A}, a)$ accepts a pointed F-coalgebra $(\mathbb{S}, s)$ if $(a, s) \in Win_{\exists}(\mathcal{B}(\mathbb{A}, \mathbb{S}))$.

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Existing automata on words, trees, graphs, etc, are all special instances of coalgebra automata.

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F-automaton	$\mathbb{A}$	$=$	$\langle A, \Delta : A \to \wp(\mathsf{F}A), Acc \rangle$

- Acceptance generalizes bisimilarity.
- Separate the combinatorics (Acc) from the dynamics (Δ).

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- Determinization There is a construction transforming an alternating F-automaton into an equivalent nondeterministic one.
- Logic The above results have various corollaries in fixpoint logics.

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