Attempts to resolve the paradoxes.

- **Theory of Types.**
- **Axiomatization of Set Theory.**
- **Foundations of Mathematics.**

The Axiomatization of Set Theory (1).

-  **Ernst Zermelo** (1871-1953).

  Zermelo Set Theory (1908) Z^-. Union Axiom, Pairing Axiom, *Aussonderungsaxiom* (Separation), Power Set Axiom, Axiom of Infinity.

  Zermelo Set Theory with Choice ZC^-. Axiom of Choice.

- **Hausdorff** (1908/1914). *Are there any regular limit cardinals?* "weakly inaccessible cardinals".

  > "The least among them has such an exorbitant magnitude that it will hardly be ever come into consideration for the usual purposes of set theory."

The Axiomatization of Set Theory (2).

- **1911-1913.** Paul Mahlo generalizes Hausdorff's questions in terms of fixed point phenomena ($\rightsquigarrow$ Mahlo cardinals).

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Thoralf Skolem	Abraham Fraenkel
(1887-1963)	(1891-1965)

  1922: *Ersetzungsaxiom* (Replacement) $\rightsquigarrow$ ZF^- and ZFC^-.

- **von Neumann** (1929): Axiom of Foundation $\rightsquigarrow$ Z, ZF and ZFC.

The Axiomatization of Set Theory (3).

- **Zermelo** (1930): ZFC doesn't solve Hausdorff's question (independently proved by Sierpiński and Tarski).
- **Question.** Does ZF prove AC?

Cardinals & Ordinals (1).

Cardinality. Two sets A and B are called equinumerous if there is a bijection $\pi : A \to B$. Equinumerosity is an equivalence relation. The cardinality of A is its equinumerosity equivalence class.

Ordinals. A linear order $\langle X, \leq \rangle$ is called a well-order if there is no infinite strictly descending chain, *i.e.*, a sequence

$$x_0 > x_1 > x_2 > \ldots$$

Examples. Finite linear orders, $\langle \mathbb{N}, \leq \rangle$.

Nonexamples. $\langle \mathbb{Z}, \leq \rangle$, $\langle \mathbb{Q}, \leq \rangle$, $\langle \mathbb{R}, \leq \rangle$.

Cardinals and Ordinals (2).

Important: If $\langle X, \leq \rangle$ is not a wellorder, that does not mean that the set X cannot be wellordered.

$\ldots$ -4 -3 -2 -1 0 1 2 3 4 $\ldots$

Cardinals and Ordinals (2).

Important: If $\langle X, \leq \rangle$ is not a wellorder, that does not mean that the set X cannot be wellordered.

-1	-2	-3	-4	-5	...
0	1	2	3	4	...

Cardinals and Ordinals (2).

Important: If $\langle X, \leq \rangle$ is not a wellorder, that does not mean that the set X cannot be wellordered.

	-1	-2	-3	-4	-5	...
	0	1	2	3	4	...
$\rightsquigarrow$	0	-1	1	-2	2	...

$$z \sqsubseteq z^* :\leftrightarrow |z| < |z^*| \ \vee \ (|z| = |z^*| \ \& \ z \leq z^*)$$

There is an isomorphism between $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{Z}, \sqsubseteq \rangle$. The order $\langle \mathbb{Z}, \sqsubseteq \rangle$ is a wellorder, thus $\mathbb{Z}$ is wellorderable.
If $\mathbf{L}$ and $\mathbf{L}^*$ are wellorders then either $\mathbf{L}$ is orderisomorphic to an initial segment of $\mathbf{L}^*$ or vice versa.

Cardinals and Ordinals (3).

If $\mathbf{L}$ and $\mathbf{L}^*$ are wellorders then either $\mathbf{L}$ is orderisomorphic to an initial segment of $\mathbf{L}^*$ or vice versa.

The class of wellorders is wellordered by

$$\mathbf{L} \preccurlyeq \mathbf{L}^* \leftrightarrow \mathbf{L} \text{ is orderisomorphic to an initial segment of } \mathbf{L}^*.$$

Ordinals are the equivalence classes of orderisomorphism. We let Ord be the class of all ordinals.

Operations on ordinals (1).

If $\mathbf{L} = \langle L, \leq \rangle$ and $\mathbf{M} = \langle M, \sqsubseteq \rangle$ are linear orders, we can define their sum and product:

$\mathbf{L} \oplus \mathbf{M} := \langle L \dot{\cup} M, \preceq \rangle$ where $x \preceq y$ if

- $x \in L$ and $y \in M$, or
- $x, y \in L$ and $x \leq y$, or
- $x, y \in M$ and $x \sqsubseteq y$.

$\mathbf{L} \otimes \mathbf{M} := \langle L \times M, \preceq \rangle$ where $\langle x, y \rangle \preceq \langle x^*, y^* \rangle$ if

- $y \sqsubset y^*$, or
- $y = y^*$ and $x \leq x^*$.

Operations on ordinals (2).

Fact. $\mathbb{N} \oplus \mathbb{N}$ is isomorphic to $\mathbb{N} \otimes \mathbf{2}$.

Exercise. These operations are not commutative: there are linear orders such that $\mathbf{L} \oplus \mathbf{M}$ is not isomorphic to $\mathbf{M} \oplus \mathbf{L}$ and similarly for $\otimes$. (Exercise 36.)

Observation. If $\mathbf{L}$ and $\mathbf{M}$ are wellorders, then so are $\mathbf{L} \oplus \mathbf{M}$ and $\mathbf{L} \otimes \mathbf{M}$.

The Axiom of Choice (1).

The Axiom of Choice (AC**).** For every function f defined on some set X with the property that $f(x) \neq \varnothing$ for all x, there is a choice function F defined on X, such that

for all $x \in X$, we have $F(x) \in f(x)$.

- Implicitly used in Cantor's work.
- Isolated by Peano (1890) in Peano's Theorem on the existence of solutions of ordinary differential equations.
- **1904.** Zermelo's wellordering theorem.

The Axiom of Choice (2).

Question. Are all sets wellorderable?

Theorem (Zermelo's Wellordering Theorem). If AC holds, then all sets are wellorderable.

The Continuum Hypothesis (1).

If AC holds, then the real numbers $\mathbb{R}$ are wellorderable. That means there is an ordinal α such that $\mathbb{R}$ and α are equinumerous. Let $\mathfrak{c}$ be the least such ordinal. We know by Cantor's theorem that this cannot be a countable ordinal. There is an ordinal that is not equinumerous to the natural numbers. We call it ω_1.

Question. What is the relationship between $\mathfrak{c}$ and ω_1?

CH. $\omega_1 = \mathfrak{c}$. The least ordinal that is not equinumerous to the natural numbers is the least ordinal that is equinumerous to the real numbers.

The Continuum Hypothesis (2).

Hilbert (1900). ICM in Paris: Mathematical Problems for the XXth century.

> *“Es erhebt sich nun die Frage, ob das Continuum auch als wohlgeordnete Menge aufgefaßt werden kann, was Cantor bejahen zu müssen glaubt.”*

In other words: CH implies “there is a wellordering of the real numbers”.

- **Question 1.** Does $\mathsf{ZF} \vdash \mathsf{AC}$?
- **Question 2.** Does $\mathsf{ZF} \vdash \mathsf{CH}$?
- **Question 2*.** Does $\mathsf{ZFC} \vdash \mathsf{CH}$ or does $\mathsf{ZFC} \vdash \neg\mathsf{CH}$?

All of these questions were wide open in 1930.

The Continuum Hypothesis (3).

Question 2*. Does ZFC $\vdash$ CH or does ZFC $\vdash \neg$CH?

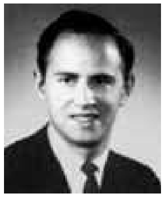
Gödel's *constructible universe*: $\mathbf{L}$.
Theorem (Gödel; 1938). $\mathbf{L} \models \mathsf{ZFC} + \mathsf{CH}$.

Corollary. If ZF is consistent, then ZFC + CH is consistent.

Consequences. The second disjunct of **Question 2*** cannot be true.

The Continuum Hypothesis (4).

Question 2*. Does ZFC $\vdash$ CH or does ZFC $\vdash \neg$CH?

 Paul Cohen (b. 1934)

Technique of Forcing (1963). Take a model M of ZFC and a partial order $\mathbb{P} \in M$. Then there is a model construction of a new model $M^{\mathbb{P}}$, the forcing extension. By choosing $\mathbb{P}$ carefully, we can control properties of $M^{\mathbb{P}}$.

Let κ be an uncountable cardinal not in bijection with ω_1. If $\mathbb{P}$ is the set of fi nite partial functions from $\kappa \times \omega$ into 2, then $M^{\mathbb{P}} \models \neg$CH.

Theorem (Cohen). ZFC $\nvdash$ CH.

Consequences. The first disjunct of **Question 2*** cannot be true, so the answer to **Question 2*** must be **No!**

Hilbert's Programme (1).

- 1900: *Hilbert's 2nd problem.* "Is there a finitistic proof of the consistency of the arithmetical axioms?"
- 1917-1921: Hilbert develops a predecessor of modern first-order logic.
- Paul Bernays (1888-1977)

  - Assistant of Zermelo in Zürich (1912-1916).
  - Assistant of Hilbert in Göttingen (1917-1922).
  - Completeness of propositional logic.
  - "Hilbert-Bernays" (1934-1939).
- Hilbert-Ackermann (1928).
- **Goal.** Axiomatize mathematics and find a finitary consistency proof.

Hilbert's Programme (2).

- 1922: Development of ε-calculus (Hilbert & Bernays). General technique for consistency proofs: "ε-substitution method".
- 1924: Ackermann presents a (false) proof of the consistency of analysis.
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  1925: John von Neumann (1903-1957) corrects some errors and proves the consistency of an ε-calculus without the induction scheme.
- 1928: At the ICM in Bologna, Hilbert claims that the work of Ackermann and von Neumann constitutes a proof of the consistency of arithmetic.

Brouwer (1).

L. E. J. (Luitzen Egbertus Jan) Brouwer
(1881-1966)

- Student of Korteweg at the UvA.
- 1909-1913: Development of topology. Brouwer's Fixed Point Theorem.
- 1913: Succeeds Korteweg as full professor at the UvA.
- 1918: *"Begründung der Mengenlehre unabhängig vom Satz des ausgeschlossenen Dritten"*.

Brouwer (2).

- 1920: *"Besitzt jede reelle Zahl eine Dezimalbruch-Entwickelung?"*. Start of the *Grundlagenstreit*.
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  1921: Hermann Weyl (1885-1955), *"Über die neue Grundlagenkrise der Mathematik"*
- 1922: Hilbert, *"Neubegründung der Mathematik"*.
- 1928-1929: ICM in Bologna; *Annalenstreit*. Einstein and Carathéodory support Brouwer against Hilbert.

Intuitionism.

- Constructive interpretation of existential quantifiers.
- As a consequence, rejection of the *tertium non datur*.
- The big three schools of philosophy of mathematics: logicism, formalism, and intuitionism.
- Nowadays, different positions in the philosophy of mathematics are distinguished according to their view on ontology and epistemology. Main positions are: (various brands of) Platonism, Social Constructivism, Structuralism, Formalism.

Gödel (1).

Kurt Gödel (1906-1978)

- Studied at the University of Vienna; PhD supervisor Hans Hahn (1879-1934).
- Thesis (1929): Gödel Completeness Theorem.
- 1931: "*Über formal unentscheidbare Sätze der* Principia Mathematica *und verwandter Systeme I*". Gödel's First Incompleteness Theorem and a proof sketch of the Second Incompleteness Theorem.

Gödel (2).

- 1935-1940: Gödel proves the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the axioms of set theory (solving one half of Hilbert's 1st Problem).
- 1940: Emigration to the USA: Princeton.
- Close friendship to Einstein, Morgenstern and von Neumann.
- Suffered from severe hypochondria and paranoia.
- Strong views on the philosophy of mathematics.

Gödel's Incompleteness Theorem (1).

1928: At the ICM in Bologna, Hilbert claims that the work of Ackermann and von Neumann constitutes a proof of the consistency of arithmetic.

- 1930: Gödel announces his result (G1) in Königsberg in von Neumann's presence.
- Von Neumann independently derives the Second Incompleteness Theorem (G2) as a corollary.
- Letter by Bernays to Gödel (January 1931): There may be finitary methods not formalizable in $\mathbf{PA}$.
- 1931: Hilbert suggests new rules to avoid Gödel's result. Finitary versions of the ω-rule.
- By 1934, Hilbert's programme in the original formulation has been declared dead.

Gödel's Incompleteness Theorem (2).

Theorem (Gödel's Second Incompleteness Theorem). If T is a consistent axiomatizable theory containing $\mathbf{PA}$, then $T \nvdash \mathrm{Cons}(T)$.

- "consistent": $T \nvdash \bot$.
- "axiomatizable": T can be listed by a computer ("computably enumerable", "recursively enumerable").
- "containing $\mathbf{PA}$": $T \vdash \mathbf{PA}$.
- "$\mathrm{Cons}(T)$": The formalized version (in the language of arithmetic) of the statement 'for all T-proofs P, $\bot$ doesn't occur in P'.

Gödel's Incompleteness Theorem (3).

- Thus: Either $\mathbf{PA}$ is inconsistent or the deductive closure of $\mathbf{PA}$ is not a complete theory.
- All three conditions are necessary:
  - **Theorem** (Presburger, 1929). There is a weak system of arithmetic that proves its own consistency ("Presburger arithmetic").
  - If T is inconsistent, then $T \vdash \varphi$ for all φ.
  - If $\mathbb{N}$ is the standard model of the natural numbers, then $\mathrm{Th}(\mathbb{N})$ is a complete extension of $\mathbf{PA}$ (but not axiomatizable).