Universiteit van Amsterdam
Institute for Logic, Language and Computation

Recursion Theory

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Homework Set # 3. Deadline: October 9th, 2003

Exercise # 1.
Prove the following theorem:

> For each $n \in \mathbb{N}$, there is a recursive function $\mathsf{concat}^n : \mathbb{N}^n \to \mathbb{N}$ such that for all $e_1, \ldots, e_n \in \mathbb{N}$, we have
> $$\varphi_{\mathsf{concat}(e_1,\ldots,e_n)}(x) = \varphi_{e_1} \circ \cdots \circ \varphi_{e_n}(x).$$

Exercise # 2.
Prove the following theorem:

> Let p be a recursive binary function and e and f indices. Then there is a recursive function $\mathsf{plugin} : \mathbb{N}^2 \to \mathbb{N}$ such that
> $$\varphi_{\mathsf{plugin}(e,f)}(x) = p(\varphi_e(x), \varphi_f(x)).$$

Exercise # 3.
The following "recursion formula" defines the **Ackermann function**:

$\mathsf{ACK}(0, 0, y) = y$,
$\mathsf{ACK}(0, x + 1, y) = \mathsf{ACK}(0, x, y) + 1$,
$\mathsf{ACK}(1, 0, y) = 0$,
$\mathsf{ACK}(z + 2, 0, y) = 1$,
$\mathsf{ACK}(z + 1, x + 1, y) = \mathsf{ACK}(z, \mathsf{ACK}(z + 1, x, y), y)$.

This is not a primitive recursive derivation, and in fact there is none for the Ackermann function (notice the nested recursion in the last line of the definition). Compute $\mathsf{ACK}(4, 3, 2)$. Let $\mathcal{C}$ be the smallest class of functions which is closed under (I) to (V) and contains the Ackermann function. Is $\mathcal{C}$ the class of all recursive functions?