Universiteit van Amsterdam
Institute for Logic, Language and Computation

Advanced Topics in Set Theory

2003/2004; 1st Semester
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Homework Set # 2. Deadline: September 30th, 2003

Exercise 4 (Normal ultrafilters)
Let κ be a measurable cardinal and U a κ-complete ultrafilter on κ. Show that U is normal if and only if κ is represented by the identity function in the ultrapower $\mathrm{Ult}(\mathbf{V}, U)$.

Exercise 5 (Reflection at measurables)
Let κ be a measurable cardinal and Φ a first-order sentence in the language of set theory. Show the following **Reflection Theorem**: If $\kappa \models \Phi$, then there is a set $A \subseteq \kappa$ of cardinality κ such that for all $\alpha \in A$, we have $\alpha \models \Phi$.

Extend the **Reflection Theorem** to second-order sentences: formulate and prove it.
Hint. Keep in mind that $\mathbf{V}_{\kappa+1} \subseteq \mathrm{Ult}(\mathbf{V}, U)$ for the normal ultrafilter U witnessing measurability.

Exercise 6 (Strong cardinals)
Let $j : \mathbf{V} \to M$ be a nontrivial elementary embedding with $\mathrm{crit}(j) = \kappa$ and $\mathbf{V}_{\kappa+2} \subseteq M$. Show that κ cannot be the least measurable cardinal.
Remark. Such a κ is called a **$\kappa + 2$-strong cardinal**.

Exercise 7 (Inaccessible cardinals and $\mathbf{L}$)
Assume $\mathbf{V}$=$\mathbf{L}$ + (IC) and let κ be inaccessible. Show that $\mathbf{L}_\kappa = \mathbf{V}_\kappa$. Use this to get a countable ordinal α such that $\mathbf{L}_\alpha \models \mathsf{ZFC}$.
Hint. Familiarize yourself with the *Skolem Hull Argument* and Theorem 0.5. Use Gödel's Condensation Lemma.

Exercise 8 (Relative constructibility)
Let x be a real number (pick your favourite set-theoretic concept of "real number": Dedekind cuts, Cauchy sequences, subsets of ω, *etc.*). Show that $\mathbf{L}(x) = \mathbf{L}[x]$. Furthermore, argue why this result doesn't depend on the choice of the concept of "real number" you chose.