UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

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Homework Set # 2 Deadline: Monday, May 5th, 2003

Exercise 2.1 (On Limit Ordinals)

In the following assume EST, but please be very careful about the use of other axioms of set theory. In each part of the exercise, list the axioms that you needed to prove the claims.

(1) Show: If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal. Show that $\bigcap C$ is the least element of C.
(2) Show: If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal.
(3) Show that α is a limit ordinal if and only if α is closed under the successor operation, *i.e.*, if $\gamma \in \alpha$ implies $\gamma + 1 \in \alpha$.
(4) Call a set X of ordinals **unbounded** if for each $\alpha \in X$ there is some $\beta \in X$ with $\alpha \in \beta$. Prove that if X is unbounded, then $\bigcup X$ is a limit ordinal.
(5) Look at the set of ordinals $X := \{\omega + n\,;\, n \in \mathbb{N}\}$ and its union $\bigcup X$. Draw a dot diagram of $\bigcup X$ in the style of the dot diagrams of ω and $\omega + 1$ in the lecture. Show that $\bigcup X$ is countable by presenting a bijection between it and ω.
(6) Discuss the existence of the ordinal $\bigcup X$ of the last item: What axioms do you need to prove its existence?

Exercise 2.2 (Normal Sequences)

Let F be a functional class term on the class of pairs of ordinals (informally: $F : \mathrm{Ord} \to \mathrm{Ord}$). We call F a **normal sequence** if it is increasing (*i.e.*, if $\alpha < \beta$ implies $F(\alpha) < F(\beta)$) and continuous (*i.e.* for every limit ordinal λ we have $F(\lambda) = \bigcup\{F(\alpha)\,;\, \alpha < \lambda\}$).

Show that every normal sequence has unboundedly many fixed points, *i.e.* for each α there is $\beta > \alpha$ such that $F(\beta) = \beta$.

(**Hint.** Let $\alpha_{n+1} := F(\alpha_n)$ recursively.)

Exercise 2.3 (Rules of Calculation for Ordinals)
Select two of the following five equations and prove them.

(1) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
(2) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
(3) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
(4) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
(5) $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$

Exercise 2.4 (Epsilon Numbers)
An ordinal α is called an **epsilon number** (sometimes called **Principal Numbers of Exponentiation**) if it is a fixed point of the operation $x \mapsto \omega^x$. Give abstract reasons for the existence of epsilon numbers, in fact for the existence of a proper class of epsilon numbers. (*cf.* **Exercise 2.2**).
More concretely, look at the sequence recursively defined by $\alpha_0 := \omega$, $\alpha_{n+1} := \omega^{\alpha_n}$. Let $\varepsilon_0 := \bigcup\{\alpha_n\,;\,n \in \omega\}$. Show that ε_0 is the least epsilon number. What axioms do you need to show that ε_0 exists? What is the Cantor Normal Form of ε_0?