Cardinal Characteristics, Regularity Properties, Definability and the Structure of the Real Line and the Generalised Real Line.

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Contents

1 Introduction

1.1 History and motivations

This *Habilitationsschrift* deals with the interplay between *combinatorial, topological and structural properties* and *definability aspects* of the real number continuum; here, by *definability*, we refer to the descriptive complexity of sets of real numbers in terms of formal logic. In addition to this main theme, we explore smaller side avenues, such as: technical aspects of *forcing* (the central tool in set theory for obtaining independence proofs), infinite combinatorics, and *generalised Baire spaces*—a sub-field of set theory that has caught the attention of set theorists in recent years, where one studies analogues of the real numbers at higher cardinalities.

Real numbers are ubiquitous in nearly all areas of science and mathematics. Although known since antiquity, the exact nature of real numbers was not clearly understood and the use of real numbers other than the rationals use was limited to concrete cases, such as the number π in geometry. The first use of the concept of a real number in its full and unbridled form was the development of calculus in the late 17th century, which required an abstract treatment of convergent sequences and limit processes. Even then, the concept of a *real number* was not clearly defined, and it was not until the late 19th century that the *real number continuum* was given a proper mathematical definition.

As 19th century mathematics progressed and its methods became more abstract, increasingly complex and unexpected aspects of the real number continuum were being discovered. One of these was Cantor's famous theorem about the uncountability of real numbers, employing the diagonalisation method for the first time. Having discovered the surprising result that there were "different sizes of infinity", Cantor went on to conjecture the famous *Continuum Hypothesis* (CH)—the statement that the cardinality of the continuum is the smallest cardinality above the countable one. Cantor himself, and many of his successors, devoted considerable effort to proving or refuting this conjecture, but were all met with failure.

Another unexpected aspect of the real number continuum was the existence of non-measurable sets, leading to such bizarre results as the famous Banach-Tarski paradox. But proofs of the existence of such non-measurable sets were *non-constructive*, in the sense that they did not provide concrete definitions of the non-measurable set. Rather, this existence was established indirectly, using an evocation of the Axiom of Choice, a fundamental principle of mathematics. This realisation made some mathematicians ask the following question: what if we restrict attention to *definable* sets of reals? How counter-intuitive can they be, and can "paradoxes" be resolved by restricting attention to sets of reals of a given complexity? In 1917, Mikhail Suslin was led to study analytic sets of reals (a set is *analytic* if it is the image of a Borel set under a continuous function; or, equivalently, if it can be defined in second-order number theory by a Σ^1_1-sentence with real parameters). Suslin proved that all analytic sets are measurable, in addition to having a number of other desirable properties (in particular, having the *Baire property* and the *perfect set property*). This provided the first clear example of a relationship between *mathematical* properties of sets on one hand, and their *logical* definability on the other.

Proceeding higher through the *projective hierarchy*, one can define $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$ and $\boldsymbol{\Delta}^1_n$ sets of reals as those definable in second-order number theory by a sentence of the corresponding complexity (a purely topological definition can also be given). This hierarchy typically serves as a natural measure of the "complexity" of a given set of reals. A natural question then arises: at which level do counter-intuitive sets (e.g., non-measurable sets) first occur? Efforts to resolve this fundamental question grounded immediately on the first level beyond Suslin's result: it was impossible to determine whether all $\mathbf{\Sigma}^1_2$ sets were Lebesgue measurable. The obstacles encountered here were so severe that some mathematicians were prompted to speculate on the potential "unsolvability" of such problems. Nikolai Luzin described the state of affairs in 1925 thus:

> "The theory of analytic sets presents a perfect harmony: any analytic set is either countable or of the cardinality of the continuum; an analytic set is never a set of the third category [satisfies the Baire property] . . . finally, an analytic set is always measurable.
>
> There remains but one significant gap: one does not know whether every complementary analytic [$\mathbf{\Pi}^1_1$] uncountable set has the cardinality of the continuum.
>
> The efforts that I exerted on the resolution of this question led me to the following totally unexpected discovery: there exists a family . . . consisting of effective [definable] sets, such that one does not know *and one will never know* whether every set from this family, if uncountable, has the cardinality of the continuum, nor whether it is of the third category, nor whether it is measurable. . . . This is the family of the *projective sets* of Mr. H. Lebesgue. It remains but to recognise the nature of this new development." [33, p 1572]

This often quoted passage is remarkable for its prophetic quality, despite the fact that in 1925 mathematicians were not aware of the incompleteness phenomenon in mathematics, and it is unclear what Luzin might have meant by the phrase "one will never know".

However, Luzin was right. Both the Continuum Hypothesis and the question described above turned out to be *independent*, in the sense that neither the statement nor its negation could be deduced just from ZFC (the standard axiomatisation of set theory), although it took until the 1960s for this to be conclusively settled by work of Gödel, Cohen and Solovay [22, 10, 11, 39].

The methods employed in both cases—inner models and forcing—have now become standard staples in the establishments of such independence proofs. Logic and set theory have thus come out as the right tools for settling foundational questions about the nature of the real numbers.

1.2 Modern research area

Although set theory has evolved beyond recognition since its early beginnings described above, two fundamental research areas—*cardinal characteristics* and *descriptive set theory*—can be traced back to the independence of CH and the problems regarding definability of sets of reals.

The size of the continuum is perhaps the most fundamental problem in the foundations of the reals, but one may also ask more subtle questions, for example: what is the smallest number of Lebesgue-null sets needed to cover all the reals? What is the smallest cardinality of a measure-one set of reals? If CH is true then the answer is trivial ($\aleph_1$), but if CH is false, these numbers could have different values, and even be different from one another. Such questions are usually formulated in terms of a *cardinal characteristic*: a cardinal number motivated by combinatorial, topological or analytical questions whose exact value is independent of set theory. The study of cardinal characteristics, and the models constructed to prove independence results about them, have given rise to sophisticated forcing techniques, particularly with the work of Shelah and Goldstern [23, 38]. Moreover, statements about cardinal characteristics can be viewed as axiomatic statements with implications to other areas of mathematics, where the independence of a statement A is established by showing that A is a consequence of one cardinal characteristic inequality and $\neg A$ is the consequence of another, both of which are known to be consistent. A detailed overview of this area can be found in [3] and [4].

Questions about definability of sets of reals have led to the development of *descriptive set theory*. This rich sub-field of set theory studies sets of reals from the *definable* point of view, usually using the Borel and the projective hierarchy as a measure of complexity, as well as variations of it such as the effective hierarchy. Questions about *regularity properties* (generalisations of the concept of measurability) have been a motivating factor throughout this development. Typically, it is possible to prove that sets of sufficiently low complexity, such as Borel and analytic sets, are *regular* (although there are interesting exceptions, see, e.g., [25]), and that the Axiom of Choice implies the existence of irregular sets. What happens on all the intermediate levels (even on the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ levels) is typically independent of ZFC.

These two aspects of the real number continuum are not unrelated—in fact, there are numerous connections between the study of regularity properties in descriptive set theory and combinatorial questions concerning cardinal characteristics. One simple connection is via the Solovay and the Judah-Shelah *characterisation theorems* [39, 26]. Due to this characterisation, for instance, the axiom "all $\mathbf{\Delta}^1_2$ sets of reals are measurable" is equivalent to the forcing-theoretic statement "there exists a random-generic real over $L[r]$, for every $r \in \mathbb{R}$". The latter statement is directly related to the *cardinal characteristic* $\mathrm{cov}(\mathcal{N})$ mentioned above (the least number of Lebesgue-null sets needed to cover the set of all reals).

Such connections were heavily employed and studied in my PhD Thesis, [29, Chapter 2]. In particular, building on the general theory of *Idealized Forcing* due to Jindřich Zapletal [41, 42] and work of Ikegami [27], I was able to present Solovay- and Judah-Shelah-style characterisation theorems as well as the theory of (certain types of) cardinal characteristics in this abstract framework. Several papers included in the current *Habilitationsschrift* continue this line of research, answering questions posed in my PhD Thesis or introducing new concepts motivated by those ideas.

There are also other, more subtle connections between regularity properties and cardinal characteristics: for instance, the existence of a *definable wellorder* of the reals (e.g., lightface Δ^1_3-definable) was long seen as conflicting with certain cardinal inequalities on the reals, since traditional forcing arguments destroy the definable wellorder. However, a recent line of work carried out by Fischer, Friedman and Zdomskyy [21, 13, 17, 18] established the consistency of a Δ^1_3-wellorder together with combinatorial properties of the reals, and this line of research was continued in two papers included in this *Habilitationsschrift*.

Finally, an area that has attracted the attention of set theorists in recent years is the study of *generalised Baire spaces*. While the real number continuum is typically represented by ω^ω in set theory (the space of functions from ω to ω), here one attempts to generalist results to the space κ^κ, where κ is a regular uncountable cardinal (sometimes, with additional properties). The area has gained momentum with three recent workshops, dedicated specifically to generalised Baire spaces, held in Amsterdam, Hamburg and Bonn in 2014, 2015 and 2016. The workshops have also led to the output of a crucial paper [32] of which I am one of the main authors,[1] collecting the most important open questions in this young research area. One publication included in this thesis deals with regularity properties on generalised Baire spaces.

1.3 Overview and background of publications

The publications comprising this *Habilitationsschrift* are listed below (followed by the corresponding citation number for future reference).

All the publications included here concern research carried out exclusively after the completion of my PhD degree. A complete list of all publications can be found in Section 2.2.

A discussion of the *collaboration process*, detailing my own contribution to each of the collaborative works, is included in a header page immediately preceding each publication.

1. Vera Fischer, Sy David Friedman and Yurii Khomskii, *Co-analytic mad families and projective wellorders*, Archive for Mathematical Logic 52:7-8, 2013, pp 809-822. [14]

2. Vera Fischer, Sy David Friedman and Yurii Khomskii, *Cichoń's diagram, regularity properties and* $\mathbf{\Delta}^1_3$ *sets of reals*, Archive for Mathematical Logic 53:5–6, 2014, pp 695-729. [15]

3. Vera Fischer, Sy David Friedman and Yurii Khomskii, *Measure, category and projective well-orders*, Journal of Logic and Analysis 6, Paper 8 (2014) (25 pp). [16]

4. Sy David Friedman, Yurii Khomskii and Vadim Kulikov, *Regularity properties on the generalised reals*, Annals of Pure and Applied Logic 167, 2016, pp 408–430. [20]

5. Giorgio Laguzzi and Yurii Khomskii, *Full-splitting Miller trees and infinitely often equal reals*, Annals of Pure and Applied Logic, 168:8, 2017, pp 14911506. [31]

6. Barnabas Farkas, Yurii Khomskii and Zoltán Vydyánszki, *Almost disjoint refinements and mixing reals*, Fundamenta Mathematicae, to appear. [12]

7. Jörg Brendle, Yurii Khomskii and Wolfgang Wohofsky, *Cofinalities of Marczewski ideals*, Colloquium Mathematicum, to appear. [7]

8. Yurii Khomskii, *Filter-Laver measurability*, Topology and its Applications, to appear. [30]

9. Jörg Brendle and Yurii Khomskii, *Projective Maximal Independent Families*, preprint. [9]

We will now provide a brief overview of each work, highlighting the most important results and providing some additional background information, as well as pointing to various connections between the publications.

[1] This paper is not included in this *Habilitationsschrift* since it only contains a survey of open questions and no original results.

The two papers *Co-analytic mad families and projective wellorders* [14] and *Measure, category and projective well-orders* [16], both joint works with Fischer and Friedman, are closely connected and the results belong to the long-term project of studying definable wellorders of the reals together with other properties. The main motivation is the following: it is well-known that if $V = L$ then there exists a $\mathbf{\Sigma}^1_2$ wellorder of the reals, and, conversely, the existence of a $\mathbf{\Sigma}^1_2$ wellorder of the reals implies that the reals are constructible. The standard constructions in set theory to increase the size of the continuum destroy any nice wellorder of the reals. However, using coding argument originally due to Jensen and Solovay [28], Harrington [24, Theorem A] showed that a $\mathbf{\Delta}^1_3$-wellorder was consistent with the continuum being arbitrarily large. Friedman [19] used more sophisticated coding methods to obtain a model with a *lightface* (i.e., without real parameters) Δ^1_3-wellorder of the reals together with $2^{\aleph_0} = \aleph_2$. Fischer and Friedman [13] went further and obtained models with a Δ^1_3-wellorder, $2^{\aleph_0} = \aleph_2$, and three pairs of cardinal characteristic inequalities,[2] showing for the first time how to combine classical forcing iterations with Harrington-style coding arguments.

Fischer, Friedman and Zdomskyy [17] generalised this argument in a different direction, by providing a model with a Δ^1_3 wellorder and the continuum of size $\aleph_3$, as well as some additional properties.[3] Later, the same authors [18] obtained the consistency of the Δ^1_3-wellorder together with $2^{\aleph_0} = \aleph_3$ and Martin's Axiom, answering a question of Harrington. Another result was the consistency of more cardinal characterstic inequalities[4]

It is still an ongoing project to understand to what extent the structure of the real line can be modified by classical forcing arguments while still obtaining a Δ^1_3-definable wellorder. Our two papers are a contribution to this research. In [14] we focus on the following concept:

Definition 1.1. An infinite family $\mathcal{A}$ of infinite subsets of ω is *almost disjoint* if for all $a, b \in \mathcal{A}$, the intersection $a \cap b$ is finite. A *maximal almost disjoint family* (*mad*) is an almost disjoint family which cannot be extended.

By identifying infinite subsets of ω with elements of 2^ω, one can consider mad families as subsets of the reals and study their complexity in the sense of descriptive set theory. A results of Mathias [34] shows that mad families cannot be analytic. In [6] Brendle and Khomskii studied mad families for other levels of the projective hierarchy. Combining those methods with the ones in [17], our main result is:

[14, Theorem 1.7] (Fischer, Friedman, Khomskii): *It is consistent that $\mathfrak{b} = 2^{\aleph_0} = \aleph_3$, there exists a $\mathbf{\Pi}^1_1$ mad family, and a Δ^1_3-definable wellorder of the reals.*

In [16], we consider the so-called Cichoń's diagram and the cardinal characteristics that appear in it: $\mathfrak{b}, \mathfrak{d}$ and the additivity, covering, uniformity and cofinality numbers of the meager ideal and the ideal of Lebesgue-null sets of reals, see Figure 1 and [3] for the relevant definitions. This classical diagram represents all ZFC-provable relations between these 10 cardinal characteristics (plus the size of the continuum). Any combination of values consistent with this diagram is, in fact, consistent with ZFC. The main result of this paper is that each of these combinations is also consistent with a Δ^1_3-wellorder of the reals. The results in **[16, Theorem 4.1 to Theorem 4.23]** cover all the cases.

[2] $\mathfrak{d} < 2^{\aleph_0}$, $\mathfrak{b} < \mathfrak{a}$ and $\mathfrak{b} < \mathfrak{g}$, see [4] for the definitions.

[3] A Π^1_2 definable mad family (see Definition 1.1) and $\mathfrak{b} = \aleph_3$.

[4] $\mathfrak{p} = \mathfrak{b} = \aleph_2 < \mathfrak{a} = \mathfrak{s} = 2^{\aleph_0} = \aleph_3$, see [4] for the definitions.

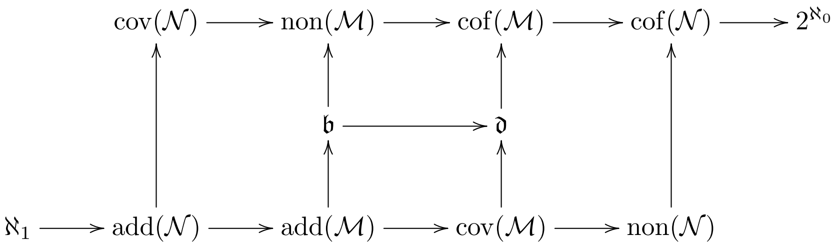

Figure 1: Cichoń's diagram

The paper *Cichoń's diagram, regularity properties and $\boldsymbol{\Delta}^1_3$ sets of reals* [15], also a joint work with Fischer and Friedman, deals with regularity properties naturally related to the cardinal characteristics in Cichoń's diagram for sets of reals higher up in the projective hierarchy, most notably $\boldsymbol{\Delta}^1_3$ set of reals. In this paper we solve a number of questions mentioned in, and motivated by, the work in my PhD Thesis, particularly [29, Chapter 2].

We consider the properties of sets of reals being *Lebesgue-measurable*, having the *Baire property*, being *Laver*- *Miller*- and *Sacks-measurable*, denoted (in correspondence to the forcing partial orders) by $\mathbb{C}, \mathbb{B}, \mathbb{L}, \mathbb{M}$ and $\mathbb{S}$ respectively. The notation used is as follows: for $\mathbb{P}$ one of these properties, $\boldsymbol{\Gamma}(\mathbb{P})$ abbreviates the statement "all sets of complexity $\boldsymbol{\Gamma}$ are $\mathbb{P}$-regular". For the $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$ complexity classes, well-known characterisation theorems due to Solovay [39], Judah and Shelah [26], and Brendle and Löwe [8] establish connections to the existence of certain kinds of reals over L, and thus to the cardinal invariants in Cichoń's diagram. On the $\boldsymbol{\Delta}^1_3$-level, the ZFC-provable implications are summarised in Figure 2.

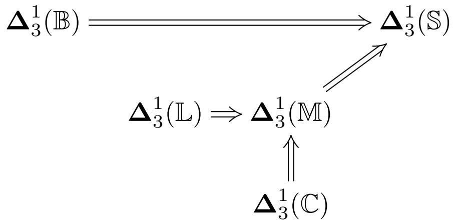

Figure 2: Diagram of implications for $\boldsymbol{\Delta}^1_3$ sets of reals.

The main result of this paper is:

[15, Section 5] (Fischer, Friedman, Khomskii): *Every true/false combination not contradicting this diagram is consistent with ZFC or ZFC with an inaccessible.*

Additional results concern Mathias and Silver forcing in [15, Section 6], partial results about $\boldsymbol{\Delta}^1_4$ sets and about separating $\boldsymbol{\Delta}^1_3$ from $\boldsymbol{\Sigma}^1_3$ measurability, in [15, Section 7]. The most interesting result here is **[15, Theorem 7.12]** and its consequence: that several ZFC-implications which hold on the second level of the projective hierarchy fail to lift to the third and fourth levels.

The publication *Regularity properties on the generalised reals* [20] is an excursion into the area of *generalised Baire spaces*, while remaining within the area of regularity properties for projective sets. The basic idea is as follows: if classical descriptive set theory takes place on the Baire space ω^ω with the topology generated by basic open sets of the form $[\sigma] := \{x \in \omega^\omega : \sigma \subseteq x\}$ for $\sigma \in \omega^{<\omega}$, here we consider a regular uncountable cardinal κ (sometimes with additional assumptions) and look at the *generalised Baire space* κ^κ, with the topology generated by $[\sigma] := \{x \in \kappa^\kappa : \sigma \subseteq x\}$ and $\sigma \in \kappa^{<\kappa}$. An analogous definition can be given for the *generalised Cantor space* 2^κ. This leads to the natural concepts of *generalised-Borel* sets, *generalised-analytic* sets, and the *generalised-projective* hierarchy. We refer the reader to [32] for an overview of the current state of this area.

This work is the first systematic attempt to study regularity properties, such as the ones mentioned above, from the point of view of descriptive set theory in the generalised Baire space. We use the framework from [27] and [29, Chapter 2] to define regularity properties related to *tree-like* forcing notions $\mathbb{P}$, proving some results for general $\mathbb{P}$ as well as deeper results for concrete $\mathbb{P} \in \{ \mathbb{C}_\kappa, \mathbb{S}_\kappa, \mathbb{M}_\kappa, \mathbb{L}_\kappa, \mathbb{R}_\kappa, \mathbb{V}_\kappa\}$ (all the definitions are given in [20, Example 3.2]).

The most important results we obtain are:

[20, Theorem 3.10] (Friedman, Khomskii, Kulikov): *Let $\mathbb{P}$ be a tree-like forcing notion on 2^κ whose conditions are κ-Sacks trees, or a tree-like forcing notion on κ^κ whose conditions are κ-Miller trees. Then $\boldsymbol{\Sigma}^1_1(\mathbb{P})$ fails.*

[20, Corollary 3.14] (Friedman, Khomskii, Kulikov): $\boldsymbol{\Delta}^1_1(\mathbb{P})$ *is consistent for* $\mathbb{P} \in \{\mathbb{C}_\kappa, \mathbb{S}_\kappa, \mathbb{M}_\kappa, \mathbb{L}_\kappa, \mathbb{R}_\kappa\}$*, and if κ is inaccessible, also for $\mathbb{P} = \mathbb{V}_\kappa$.*

[20, Lemma 4.4] (Friedman, Khomskii, Kulikov): *Suppose κ is inaccessible. Then* $\boldsymbol{\Delta}^1_1(\mathbb{M}_\kappa)$ *implies that for every $r \in \kappa^\kappa$ there is an $x \in \kappa^\kappa_\uparrow$ which is unbounded over $\kappa^\kappa_\uparrow \cap L[r]$.*

[20, Lemma 4.9] (Friedman, Khomskii, Kulikov): *Let $\boldsymbol{\Gamma}$ be a class of subsets of κ^κ or 2^κ closed under continuous preimages* (*in particular* $\boldsymbol{\Gamma} = \boldsymbol{\Delta}^1_1$). *Then*

1. $\boldsymbol{\Gamma}(\mathbb{M}_\kappa) \Rightarrow \boldsymbol{\Gamma}(\mathbb{S}_\kappa)$.
2. $\boldsymbol{\Gamma}(\mathbb{V}_\kappa) \Rightarrow \boldsymbol{\Gamma}(\mathbb{S}_\kappa)$.
3. $\boldsymbol{\Gamma}(\mathbb{C}_\kappa) \Rightarrow \boldsymbol{\Gamma}(\mathbb{M}_\kappa)$.
4. $\boldsymbol{\Gamma}(\mathbb{L}_\kappa) \Rightarrow \boldsymbol{\Gamma}(\mathbb{M}_\kappa)$.
5. $\boldsymbol{\Gamma}(\mathbb{R}_\kappa) \Rightarrow \boldsymbol{\Gamma}(\mathbb{M}_\kappa)$.
6. *If κ is inaccessible, then* $\boldsymbol{\Gamma}(\mathbb{C}_\kappa) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V}_\kappa)$.

Next, returning to the classical Baire space ω^ω, we consider another application of the general methods from [29, Chapter 2]. In the paper *Full-splitting Miller trees and infinitely often equal reals* [31] joint with Giorgio Laguzzi, we consider two specific forcing partial order:

Definition 1.2. A tree $T \subseteq \omega^{<\omega}$ is called a *full-splitting Miller tree* iff every $t \in T$ has an extension $s \in T$ such that s is *full-splitting*, i.e., $s^\frown \langle n \rangle \in T$ for every n. Let $\mathbb{FM}$ denote the partial order of full-splitting Miller trees ordered by inclusion.

Definition 1.3. A tree $T \subseteq \omega^\omega$ is called an *infinitely often equal tree*, or simply *ioe-tree*, if for each $t \in T$ there exists $N > |t|$, such that for every $k \in \omega$ there exists $s \in T$ extending t such that $s(N) = k$. Let $\mathbb{IE}$ denote the partial order of ioe-trees ordered by inclusion.

Both forcing partial orders are similar to Cohen forcing but do not have the ccc, and are canonically related to "adding an infinitely often real" (a real $x \in \omega^\omega$ such that for all reals y in the ground model, $\exists^\infty n\ (x(n) = y(n))$. Moreover, there are two interesting Borel-generated σ-ideals, $\mathfrak{D}_\omega$ and $\mathfrak{I}_{\text{ioe}}$, naturally related to $\mathbb{FM}$ and $\mathbb{IE}$, respectively. Some results include:

[31, Theorem 3.4] (Khomskii, Laguzzi): *Let $\mathbf{\Gamma}$ be a pointclass closed under continuous preimages. Then the following are equivalent:*

1. $\mathbf{\Gamma}(\mathbb{C})$
2. $\mathbf{\Gamma}(\mathbb{FM})$
3. $\mathbf{\Gamma}(\mathbb{IE})$

[31, Theorem 3.5] (Khomskii, Laguzzi): $\mathbf{\Delta}^1_2(\mathbb{C}) \Rightarrow \mathbf{\Sigma}^1_2(\text{w}\mathbb{IE})$.

Here w$\mathbb{IE}$ denotes a "local" version of the regularity property related to $\mathbb{IE}$. We also consider *dichotomy properties* "$\mathbb{FM}$-Dich" and "$\mathbb{IE}$-Dich", for the trees from Definitions 1.2 and 1.3 and the respective Borel ideals $\mathfrak{D}_\omega$ and $\mathfrak{I}_{\text{ioe}}$. Here we prove:

[31, Theorem 4.3] (Khomskii, Laguzzi): *The following are equivalent:*

1. $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-Dich})$
2. $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-Dich})$
3. $\forall r \in \omega^\omega\ (\omega_1^{L[r]} < \omega_1)$.

Further results establish that in the *Solovay model* both dichotomy properties hold for all projective sets, and that they follow from the Axiom of Determinacy for all sets of reals. In Section 5 we consider the question whether $\mathbb{IE}$ adds a Cohen real — this is related to an old question of Fremlin: is there a forcing $\mathbb{P}$ such that $\mathbb{P}$ does not add a Cohen real but the two-step iteration $\mathbb{P} * \mathbb{P}$ does? The question was recently solved by Zapletal [43] using rather unorthodox methods. In Theorem 5.3, we prove a property of $\mathbb{IE}$ and argue that this indicates that $\mathbb{IE}$ is a more natural solution.

The paper *Almost disjoint refinements and mixing reals* [12] lies on the intersection between descriptive set theory and combinatorial set theory of the reals. The questions therein have connections both to regularity properties, cardinal characteristics, and forcing properties. It is motivated by the following concept.

Definition 1.4. Let $\mathcal{H} \subseteq [\omega]^\omega$. We say that $\mathcal{H}$ has an *almost-disjoint refinement* $\{A_H : H \in \mathcal{H}\}$ if $A_H \in [H]^\omega$ for all H, and $|A_H \cap A_K| < \omega$ for all $H \neq K \in \mathcal{H}$.

A folklore result states that if $\mathcal{H}$ is a family of size $< 2^{\aleph_0}$, then it has an almost-disjoint refinement, and a theorem due to Balcar and Vojtáš [2] shows that every ultrafilter on ω has an almost-disjoint refinement. More interestingly, Balcar and Pazák [1], and independently Brendle (in [40]), proved the following theorem:

Theorem 1.5 (Balcar-Pazák, Brendle)**.** *If M is a model of set theory with $\mathscr{P}(\omega) \cap M \neq \mathscr{P}(\omega)$, then $[\omega]^\omega \cap \omega$ has an almost disjoint refinement.*

One of the main results of this paper is a generalisation of this theorem to disjoint refinements with respect to an ideal $\mathcal{I}$ on ω. We use $\mathcal{I}^+$ to denote sets which are not in $\mathcal{I}$.

Definition 1.6. Let $\mathcal{I}$ be an ideal on ω. Let $\mathcal{H} \subseteq \mathcal{I}^+$. We say that a family $\{A_H : H \in \mathcal{H}\}$ is an $\mathcal{I}$-almost-disjoint refinement of $\mathcal{H}$ if $A_H \subseteq H$, $A_H \in \mathcal{I}^+$ for every $H \in \mathcal{H}$, and $(A_H \cap A_K) \in \mathcal{I}$ for all $H \neq K \in \mathcal{H}$.

[12, Theorem 1.6] (Farkas, Khomskii, Vydyánszki). *Assume that M is a transitive model with $\omega_1 \subseteq M$, $\mathscr{P}(\omega) \cap M \neq \mathscr{P}(\omega)$, and $\mathcal{I}$ is an analytic or co-analytic ideal with parameters in M. Then there is an $\mathcal{I}$-almost-disjoint refinement of $\mathcal{I}^+ \cap M$.*

Other results include the study of a property of forcing partial orders that we call "adding mixing reals" and "adding mixing injections", closely related to almost disjoint refinements.

The publication *Cofinalities of Marczewski ideals* [7], with Brendle and Wohofsky, answers a question of Miroslav Repický (private communication, cf. [37]).

Definition 1.7. A set $A \subseteq \omega^\omega$ is *Laver-null*, notation $A \in \ell^0$, if and only if for every Laver tree $T \in \mathbb{L}$, there exists a Laver subtree $S \subseteq T$, such that $[S] \cap A = \varnothing$. A set $A \subseteq \omega^\omega$ is *Miller-null*, notation $A \in m^0$, if the same statement holds for Miller trees $T, S \in \mathbb{M}$.

These ideals have been studied, in particular, by Brendle in [5]. Several facts concerning the covering and additivity numbers of these ideals are known, and it was known that the *cofinality numbers* $\mathrm{cof}(\ell^0)$ and $\mathrm{cof}(m^0)$ can consistently be above the continuum (for instance, this holds under CH). Repický (and independently the authors of this paper), asked the following question: is the $\mathrm{cof}(\ell^0) > 2^{\aleph_0}$ and $\mathrm{cof}(m^0) > 2^{\aleph_0}$ provable in ZFC? In this paper we solve this problem, by proving an even stronger result:

[7, Corollary 18] (Brenlde, Khomskii, Wohofsky): $\mathrm{cf}(\mathrm{cof}(\ell^0)) > 2^{\aleph_0}$ *and* $\mathrm{cf}(\mathrm{cof}(m^0)) > 2^{\aleph_0}$.

Next, we return to yet another application of the general methods from [29, Chapter 2]. In the paper *Filter-Laver measurability* [30], we consider the following concept.

Definition 1.8. Let $\mathcal{F}$ be a filter on ω. An *$\mathcal{F}$-Laver tree* is a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending $\mathrm{stem}(T)$, $\mathrm{Succ}_T(\sigma) \in F$. An *$\mathcal{F}^+$-Laver-tree* is a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending $\mathrm{stem}(T)$, $\mathrm{Succ}_T(\sigma) \in \mathcal{F}^+$. We use $\mathbb{L}_\mathcal{F}$ and $\mathbb{L}_{\mathcal{F}^+}$ to denote the partial orders of F-Laver and $\mathcal{F}^+$-Laver trees, respectively, ordered by inclusion.

The forcing partial orders $\mathbb{L}_\mathcal{F}$ and $\mathbb{L}_{\mathcal{F}^+}$ have been used and studied in the literature for various purposes. Notice that if $\mathcal{F}$ is the cofinite filter, then $\mathbb{L}_{\mathcal{F}^+}$ is the standard Laver forcing and $\mathbb{L}_\mathcal{F}$ is (a version of) the standard Hechler forcing. We focus on regularity properties related to these forcing notions, in the sense of [27] and [29]. An important innovation which enables this study is a dichotomy theorem proved recently by Miller [35]. The main results of the paper are of two types: **[30, Lemmas 4.5, 4.6, 4.7, 4.10 and 4.11]** hold for arbitrary filters and describe relationships between various regularity properties depending on reducibility relations between filters. And **[30, Theorem 5.7, 5.15 and 5.16]** yield much stronger results under the assumption that F is an analytic filter (this is because of definability properties of the corresponding forcing and ideal, allowing us to apply results from [27] and [29, Chapter 2].

The last publication, *Projective Maximal Independent Families* [9] deals with the following combinatorial concept, closely related to *maximal almost disjoint families* (cf. Definition 1.1).

Definition 1.9. A family $\mathcal{I} \subseteq [\omega]^\omega$ is called *independent* if whenever we choose finite disjoint $F, G \subseteq \mathcal{I}$, we get

$$\bigcap_{A \in F} A \ \cap \ \bigcap_{B \in G} (\omega \setminus B) \text{ is infinite.}$$

A family $\mathcal{I} \subseteq [\omega]^\omega$ is called a *maximal independent family* (m.i.f.) if it is independent and maximal with regard to this property.

Just as with mad families, maximal independent families can be considered subsets of the reals by identifying the space $[\omega]^\omega$ with the Cantor space. A result of Miller [36] shows that there are no analytic m.i.f.'s. In a way analogous to [6] and [14], one can study in which models there are m.i.f.'s of various complexity in the projective hierarchy. Our main result is the following:

[9, Theorem 4.2] (Brendle, Khomskii): *In the Cohen model, there are no projective m.i.f.'s.*

Here, the Cohen model refers to the ω_1-product of Cohen forcing. In particular, the statement "there are no projective m.i.f.'s" is consistent with ZFC and does not require the strength of an inaccessible cardinal.

1.4 Structure of this *Habilitationsschrift*

This *Habilitationsschrift* is structured as follows. The nine publications are included following a detailed C.V., in the same order as listed above. Each publication is preceded by a *header page*. For joint publications, this page is used to describe the individual contribution of the authors to that publication. In each case, a general description of the type of work carried out and the form of the collaboration is followed by an itemised list of the main results with the appropriate authorship attribution.

1.5 Statement about submission of this habilitation

Hiermit bestätige ich, dass ich mich nicht anderenorts einem Habilitationsverfahren unterzogen habe.

Hamburg, 18.07.2017 *Yurii Khomskii*

1.6 Acknowledgments

First of all, I want to thank Benedikt Löwe for his guidance and support throughout my development as an academic; the support which began twelve years ago in Amsterdam, led me through a Master's degree, a PhD and continued undauntedly afterwards, all the way until this Habilitation. I want to thank Jörg Brendle for being such an inspirational mathematician, always kind, helpful and encouraging. I want to thank Sy David Friedman for playing a crucial role in my professional development and for having so much confidence in my abilities.

This *Habilitationsschrift* would not be possible without my co-authors whom I would like to thank collectively: Jörg Brendle, Barnabas Farkas, Vera Fischer, Sy Friedman, Vadim Kulikov, Giorgio Laguzzi, Zoltán Vydyánszki and Wolfgang Wohofsky.

Countless other mathematicians have contributed to this work through discussion, explanation, motivation or inspiration. Specifically, I would like to mention: Joan Bagaria, Nathan Bowler, Vincenzo Dimonte, Laura Fontanella, Sakae Fuchino, Lorenzo Galeotti, Stefan Geschke, Martin Goldstern, Daisuke Ikegami, Jakob Kellner, Andrea Medini, David Schrittesser, Saharon Shelah, Asger Törnquist, Jouko Väänänen, Jindřich Zapletal and Lyubomyr Zdomskyy.

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2 Curriculum Vitae

2.1 Personal Data

Name:	Yurii Daniilovich Khomskii
Date of Birth:	5 September 1980
Place of Birth:	Moscow, Russia
Nationality:	Dutch and Russian
Family situation:	Married, two children
Languages:	Fluent in: English, Dutch, German, Russian, Bulgarian Some knowledge: French, Latin, Hebrew, Japanese

2.2 Publications

Peer-reviewed:

1. **William of Sherwood, singular propositions and the Hexagon of Opposition**, Jean-Yves Béziau and Gillman Payette (eds.), New Perspectives on the Square of Opposition, Peter Lang, Bern, 2010.

2. **A general setting for the pointwise investigation of determinacy**, Proceedings of the Third Indian Conference on Logic and Applications, (Eds.) R. Ramanujam and Sundar Sarukkai, Springer LNAI 5378, 2009. pp 187–197.

3. (*with Jörg Brendle*) **Polarized partitions on the second level of the projective hierarchy**, Annals of Pure and Applied Logic 163, 2012. pp 1345–1357. (DOI: 10.1016/j.apal.2012.03.003.)

4. (*with Jörg Brendle*) **Mad families constructed from perfect a.d. families**, Journal of Symbolic Logic 78 (4), 2014. pp. 1164-1180. (DOI: 10.2178/jsl.7804070).

5. **Projective Hausdorff gaps**, Archive for Mathematical Logic 53:1-2, 2014. pp 57-64. (DOI: 10.1007/s00153-013-0355-6).

6. (*with Vera Fischer and Sy Friedman*) **Co-analytic mad families and projective wellorders**, Archive for Mathematical Logic 52:7-8, 2013, pp 809-822. (DOI: 10.1007/s00153-013-0345-8).

7. (*with Vera Fischer and Sy Friedman*) **Cichoń's diagram, regularity properties and $\mathbf{\Delta}^1_3$ sets of reals**, Archive for Mathematical Logic 53:5–6, 2014, pp 695-729.

8. (*with Vera Fischer and Sy Friedman*) **Measure, category and projective wellorders**, Journal of Logic and Analysis 6, Paper 8 (2014) (25 pp).

9. (*with Sy Friedman and Vadim Kulikov*) **Regularity properties on the generalised reals**, Annals of Pure and Applied Logic 167, 2016, pp 408–430.

10. (*with Giorgio Laguzzi, Benedikt Löwe and Ilya Sharankou*) **Questions on Generalised Baire Spaces**, Mathematical Logic Quarterly 62:4-5, August 2016, pp 439-456.

11. (*with Giorgio Laguzzi*) **Full-splitting Miller trees and infinitely often equal reals**, Annals of Pure and Applied Logic 168:8, 2017, pp 14911506.

12. (*with Barnabas Farkas and Zoltán Vydyánszki*) **Almost disjoint refinements and mixing reals**, Fundamenta Mathematicae, to appear (arXiv:1510.05699 [math.LO]).

13. (*with Jörg Brendle and Wolfgang Wohofsky*) **Cofinalities of Marczewski ideals**, Colloquium Mathematicum, to appear (arXiv:1611.08143 [math.LO]).

14. **Filter-Laver measurability**, Topology and its Applications, to appear (arXiv:1204.5198 [math.LO]).

Preprints:

15. (*with Jörg Brendle*), **Projective Maximal Independent Families**.

In preparation/in progress:

16. (*with Rachid Atmai and Sy Friedman*) **Regularity Properties from Large Cardinals**.

17. **Eventually different trees**.

18. (*with Sy Friedman, Giorgio Laguzzi, Marlene Koelbing and Wolfgang Wohofsky*) **Uncountable levels of paradise**.

Dissertation:

19. **Regularity properties and definability in the real number continuum**, PhD Thesis, University of Amsterdam, 2012. ILLC Dissertation Series DS-2012-04.

2.3 Education

1992–1999	International School, Maartens College, Groningen Included internationally acknowledged programmes: - International General Certificate of Secondary Education (IGCSE) - International Baccalaureate (IB), 1997-1999
1999–2002	Mathematics, University of Groningen (switched to Leiden)
1999–2001	Computer Science, University of Groningen (incomplete) "Propaedeuse" (2000)
1999–2003	Piano performance: Conservatorium Groningen. Bachelor's Degree "Uitvoerend Musicus" (18 June 2003)
2002–2005	Mathematics, University of Leiden. Bachelor's Degree (18 April 2005) Bachelor Thesis: "Banach Algebras" (Supervisor: Marcel de Jeu)
2005–2007	Master of Logic, University of Amsterdam Master of Science Degree, with distinction ("cum laude") (14 August 2007) Master Thesis: "Regularity Properties and Determinacy" (Supervisor: Benedikt Löwe)
2007–2012	PhD studies, Institute of Logic, Language and Computation (ILLC) University of Amsterdam Supervisor: Benedikt Löwe Co-supervisor: Jörg Brendle (Kobe Univesity, Japan) Mosaic grant, given by the Netherlands Organisation for Scientific Research (NWO)

2.4 Work experience

1. September 2007 – February 2012: *Promovendus* (PhD candididate) at the Institute of Logic, Language and Computation (ILLC), University of Amsterdam.
2. September 2011 – January 2012: Lecturer at the Amsterdam University College (AUC).
3. February 2012 – February 2015: postdoctoral researcher at the Kurt Gödel Research Center for Mathematical Logic (KGRC), University of Vienna.
4. February 2015 – February 2017: postdoctoral researcher ("Wissenschaftlicher Mitarbeiter") at the University of Hamburg, Germany.
5. March 2017 – February 2019: Marie Skłodowska-Curie fellow at the University of Hamburg, Germany.

2.5 Teaching experience

1. At the **University of Amsterdam**:

- Fall 2007: Teaching assistant, *Recursion Theory* (Master of Logic).
- Spring 2008: Teaching assistant, *Axiomatic Set Theory* (Master of Logic and Bachelor of Mathematics).
- Spring 2008: Teaching assistant, *Intuitionistic Logic* (Master of Logic).
- Fall 2008: Lecturer (joint with Brian Semmes), *Capita Selecta: Set Theory, Model Theory* (Master of Logic and Master of Mathematics).
- Spring 2011: Teaching assistant, *Set Theory* ("MasterMath" (Dutch national mathematics Master programme)).
- Spring 2011: Teaching assistant, *Intuitionistic Logic* (Master of Logic).
- Fall 2011: Teaching assistant, *Introduction to Modal Logic* (Master of Logic and Bachelor of Mathematics).

2. At the **Amsterdam University College**:

   - Fall 2011: Lecturer, *Logic, Information Flow and Argumentation* (Liberal Arts Bachelor)

3. At the **KGRC, University of Vienna**:

   - Fall 2013: Lecture/reading course, *Advanced topics in Set Theory* (Mathematics MSc).

4. At the **University of Hamburg**:

   - Spring 2015: Exercise classes, *Mathematik für Informatiker II.*
   - Fall 2015: Exercise classes, *Mathematik für Informatiker I.*
   - Fall 2015: Exercise classes, *Stochastik für Lehramt (LAPSI).*
   - Spring 2016: Lecture *Vertiefung Mengenlehre*, (Mathematics Master).
   - Fall 2016: Exercise classes, *Analysis I.*

5. **Intensive course/invited lecture series**:

   - July 2010: Lecturer of intensive course, *Infinite Games*, Sofia University, Bulgaria (part of *Erasmus teachers' exchange programme*, invitation by prof. Alexandra Soskova).
   - June 2013: Invited lecturer on game theory at the HIM programme *Stochastic Dynamics in Economics and Finance*, Bonn, Germany (invitation by prof. Igor Evstigneev).
   - June 2017: Topics in Set Theory, Master of Logic project, ILLC, Amsterdam.

6. **Student supervision**: I was involved in the supervision of Rogier Jacobsz (MSc thesis, UvA), Adam Lesnikowski (MSc thesis, UvA), Sourav Tarafder (PhD student, exchange in UvA) and Tabea Beese (MSc student, U Hamburg).

7. **Service as examiner**: I served on the MSc thesis committee of Tanmay Inamdar (MSc Logic, 10 August 2013), Hugo Nobrega (MSc Logic, 10 August 2013), Apostolos Tzimoulis (MSc Logic, 10 August 2013), Julia Ilin (MSc Logic, 28 February 2014), Alexander Block (MSc Logic, 28 February 2014) and Lorenzo Galeotti (MSc Logic, 21 July 2015).

2.6 Grants and Distinctions

- Mosaic Award 2007: A grant for a four-year period of doctoral research, given by the Netherlands Organisation for Scientific Research (NWO), aimed at stimulating the participation of ethnic minorities in academic research.
- Marie Skłodowska-Curie fellowship at the university of Hamburg, Germany (1 March 2017 - 28 February 2019).

2.7 Invited talks

1. 4th European Set Theory Conference, 15–18 July 2013.
2. International Conference on Topology and Geometry, joint with the 6th Japan-Mexico Topology Symposium (JAMEX 6), 2–6 September 2013.
3. 1st Pan Pacific International Conference on Topology and Applications, Min Nan Normal University in Zhangzhou, China, 25–30 November 2015.
4. 6th Young Set Theory 2016, Copenhagen, Denmark, 13–17 June 2016.
5. Bonn Set Theory workshop: generalised Baire Spaces, 21–22 September 2016.
6. Arctic Set Theory workshop 3, Kilpisjärvi, Finland, 25–30 January 2017.
7. "The Logic of Infinity", Amsterdam University College, Guest Lecture, 13 April 2017.
8. The 15th Asian Logic Conference, Daejeon, Korea, 10–14 July 2017.

2.8 Organisational and community service

1. Main organiser of the *Amsterdam workshop in set theory*, 26–27 May, 2009 (satellite workshop to the PhD defense of Brian Semmes).
2. Co-organiser of the seminar series *Logic Tea*, University of Amsterdam (2009-2011).
3. Main organiser of the *Amsterdam workshop in set theory*, 1–2 June, 2010 (satellite workshop to the PhD defense of Daisuke Ikegami).
4. Co-organiser of the Set Theory Workshop at the *annual meeting of the Deutsche Mathematiker-Vereinigung (DMV)* in Hamburg, September 2015.
5. Co-organiser of the international conference *Colloquium Logicum* (a bi-annual meeting of the German mathematical logic society DVMLG), Hamburg, September 2016.
6. Served on a PhD hiring committee at the University of Amsterdam, 2011.
7. Assistance at the "Tag der Mathematik", Hamburg University, 2015 and 2016 (supervision and grading).
8. Referee work for:
    - Journal of Symbolic Logic
    - Proceedings of the American Mathematical Society

- Archives for Mathematical Logic
- Mathematical Logic Quarterly
- Journal of Applied Non-Classical Logics
- Journal of Dynamics and Games
- CIE 2012, Cambridge, England
- *Logic, Computation, Hierarchies*, volume 4 of Ontos Mathematical Logic, De Gruyter, (2014)
- *Outstanding Contributions to Logic: Jaakko Hintikka*, edited by G. Sandu and H. van Ditmarsch (Springer-Verlag).

3 Cumulative Habilitation

3.1 Typographical remark

The articles contained in this *Habilitationsschrift* are included in this document. For all articles that have appeared in print, the published paper in the publisher's layout is included in pdf form in the style and formatting with header, footer and page numbers as in the published version. For articles that are yet to appear, the latest preprint version is attached, with page number count starting at 1.

Each publication is preceded by a header page describing in detail my own contribution to it.

The page numbers that appear on these header pages refer to the actual page numbers in this *Habilitationsschrift*. They are thus also visible in the Contents section (as *Subsection* numbers), and may help the reader with orientation throughout the text. However, these page numbers are not visible throughout the rest of the text.

3.2 Co-analytic mad families and definable well-orders

Vera Fischer, Sy David Friedman and Yurii Khomskii, Archive for Mathematical Logic 52:7-8, 2013, pp 809-822.

This paper is the result of combining methods established by Brendle and Khomskii in [6] with those already known to the other authors, e.g., [19, 13]. The publication is the result of intensive collaboration, which took the form of regularly scheduled meetings with all three authors, as well as many informal meetings and discussions between Fischer and Khomskii. Below is a detailed list of the main results of this paper, with an approximate indication of authorship.

- ***Definition 2.2, Lemma 2.3 and Lemma 2.4****: Khomskii, with contribution by Fischer and Friedman*
- ***Lemma 3.2 and paragraphs preceding it****: Fischer, with contribution by Friedman and Khomskii*
- ***Lemma 4.1****: Collaboration*

The paper was written by Fischer and Khomskii (Section 3 by Fischer, Sections 1, 2 and 5 by Khomskii and Section 4 jointly).

Arch. Math. Logic (2013) 52:809–822
DOI 10.1007/s00153-013-0345-8

Mathematical Logic

Co-analytic mad families and definable wellorders

Vera Fischer · Sy David Friedman · Yurii Khomskii

Received: 19 August 2012 / Accepted: 4 June 2013 / Published online: 12 June 2013

Abstract We show that the existence of a Π^1_1-definable mad family is consistent with the existence of a Δ^1_3-definable well-order of the reals and $\mathfrak{b} = \mathfrak{c} = \aleph_3$.

Keywords Projective wellorders · Projective mad families · Large continuum

Mathematics Subject Classification (2000) 03E15 · 03E17 · 03E35

1 Introduction

An *almost disjoint* (*a.d.*) family $\mathcal{A}$ is a collection of infinite subsets of ω, such that $|a \cap b| < \omega$ for all $a, b \in \mathcal{A}$. A *maximal almost disjoint* (*mad*) family is an infinite a.d. family which is maximal with regard to this property, i.e., $\forall a\ \exists b \in \mathcal{A}\ (|a \cap b| = \omega)$. Mad families have been studied from a variety of perspectives: for example, the size of the least mad family that can possibly exist is the cardinal characteristic $\mathfrak{a}$, and its value has been shown to be independent of ZFC. Another perspective is the descriptive set-theoretic one, where one looks at the possible complexity of mad families (as subsets of $[\omega]^\omega$). This investigation has been carried out in a number of results, and we briefly summarize its history.

Theorem 1.1 (Mathias [15]) *There are no analytic mad families.*

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In the constructible universe L, it is easy to construct Σ^1_2-definable mad families. Arnold Miller proved a seemingly stronger result.

Theorem 1.2 (Miller [16]) *In L, there is a Π^1_1 mad family.*

The above result has recently been superseded by Asger Törnquist:

Theorem 1.3 (Törnquist [18]) *The following are equivalent:*

1. *There is a Σ^1_2 mad family.*
2. *There is a Π^1_1 mad family.*

Combining this theorem with well-known facts about constructing Σ^1_2-definable mad families in L which are preserved by iterations of some standard forcing notions (among which Cohen, random, Sacks and Miller forcing), one can easily see that the existence of a Π^1_1 mad family is consistent with $\neg$CH. On the other hand, the following was proved in [9] (where $\mathfrak{b}$ is the *bounding number*, i.e., the least size of an unbounded family, and an *ω-mad family* is a mad family satisfying a stronger maximality requirement—see e.g. [13] for a definition).

Theorem 1.4 (Friedman and Zdomskyy) *It is consistent that $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and there exists a Π^1_2 ω-mad family.*

This was further extended in [7]:

Theorem 1.5 (Fischer et al.) *It is consistent that $\mathfrak{b} = \mathfrak{c} = \aleph_3$, there exists a Π^1_2 ω-mad family and a Δ^1_3-definable well-order of the reals.*

Methods for obtaining models with large continuum together with a Δ^1_3-definable wellorder have been developed by Jensen and Solovay [12], by Harrington [10] and by Friedman [8]; it is an ongoing project to determine to what extent the Δ^1_3 wellorder is compatible with certain other properties of the model (such as a cardinal inequality or the existence of other projective objects, cf. [5–7]).

Dropping the "ω"-requirement, Theorem 1.4 was improved in [4]:

Theorem 1.6 (Brendle and Khomskii) *For any regular uncountable cardinal κ, it is consistent that $\mathfrak{b} = \mathfrak{c} = \kappa$ and there exists a Π^1_1 mad family.*

The present paper is concerned with the following question: to what extent can Theorem 1.5 be merged with Theorem 1.6? Note that we have no chance of obtaining a Π^1_1 ω-mad family together with $\mathfrak{b} > \aleph_1$ (the reason is that an ω-family does not contain a perfect set by [17], so a Π^1_1 ω-mad family must be completely contained in L), so the "ω"-requirement must certainly be dropped. Taking that into account, we do indeed succeed in proving an optimal result extending both Theorem 1.5 and Theorem 1.6.

Theorem 1.7 (Main Theorem) *It is consistent that $\mathfrak{b} = \mathfrak{c} = \aleph_3$, there exists a Π^1_1 mad family, and a Δ^1_3-definable well-order of the reals.*

For the proof of this theorem, we use a combination of the techniques for constructing Δ^1_3 well-orders, as presented in [7], and the techniques from [4] for constructing a Π^1_1 mad family in models where $\mathfrak{b}$ is large. Most of the work involves overcoming two main obstacles:

1. Showing that a version of *almost disjoint coding* has a nice preservation property, and
2. Dealing with iterations longer than length $\aleph_1$.

The first obstacle will be solved in Lemma 2.4 and the second one will be solved by making use of the $\diamondsuit$-principle, in Lemma 4.1.

Note that, while our main theorem is formulated as an optimal generalization of Theorems 1.5 and 1.6, it would be easy to modify the proof so that it yields the same result with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ instead of $\aleph_3$. The only difference would be a straightforward simplification of the coding mechanism. We also conjecture that the same result holds for $\mathfrak{b} = \mathfrak{c} = \kappa$ for any uncountable regular κ, although that would require a substantial change to the coding mechanism, and it is still not completely clear whether that can be done.

This paper is structured as follows: in Sect. 2, we give the preliminary definitions, review the main methods of [4] and [7], and introduce a different version of "almost disjoint coding". In Sect. 3, we review the preparatory forcing construction from [7] and prove that the $\diamondsuit$-principle is preserved after the preparation. Finally, in Sect. 4, we combine these efforts and obtain a proof of the Main Theorem.

2 Preliminaries

We start by summarizing the main tools behind the result of [4]. One of the central concepts there was considering mad families constructed from perfect a.d. families, and *preserving the maximality of the re-interpreted family* by forcing, as opposed to the more classical concept of preserving a mad family *directly*.

Definition 2.1 *A set $\mathcal{A} \subseteq [\omega]^\omega$ is called an $\aleph_1$-perfect mad family if $\mathcal{A} = \bigcup_{\alpha<\aleph_1} A_\alpha$ where each A_α is a perfect a.d. set and $\mathcal{A}$ is a mad family. For a forcing $\mathbb{P}$, such a family $\mathcal{A}$ is said to be $\mathbb{P}$-indestructible if in the generic extension $V[G]$ by $\mathbb{P}$, $A^{V[G]} := \bigcup_{\alpha<\aleph_1} A_\alpha^{V[G]}$ is a mad family.*

The method of [4] involved the construction of an $\aleph_1$-perfect mad family in L, which had a Σ^1_2 definition and moreover was indestructible (in the sense of Definition 2.1) by the κ-iteration of Hechler forcing with finite support, for κ being any uncountable regular cardinal. We now briefly review that construction.

For $\alpha < \aleph_1$, let $P^\alpha := \{P^\alpha_\sigma \mid \sigma \in \omega^{<\omega}\}$ be an infinite partition of some (unspecified) domain $D_\alpha \in [\omega]^\omega$ into infinite sets, indexed by finite sequences σ. For each $\sigma \in \omega^{<\omega}$, let $\{p^\alpha_\sigma(0), p^\alpha_\sigma(1), p^\alpha_\sigma(2), \dots\}$ be the increasing enumeration of P^α_σ. For each $f \in {}^\omega\omega$, let $\Phi^\alpha(f) := \{p^\alpha_{f\restriction n}(f(n)) \mid n \in \omega\}$ and let $A_\alpha := \{\Phi^\alpha(f) \mid f \in {}^\omega\omega\}$. Then A_α is an almost disjoint subfamily of $[D_\alpha]^\omega$ of size $2^{\aleph_0}$. Furthermore A_α is a perfect set in the natural topology of $[D_\alpha]^\omega$ (since Φ is a homeomorphism between ω^ω and A_α).

The idea is then to construct, by induction on $\alpha < \aleph_1$, a sequence of such partitions P^α, each of them giving rise to a perfect a.d. set A_α, and to make sure that the union $\mathcal{A} := \bigcup_{\alpha<\aleph_1} A_\alpha$ becomes a mad family. If the construction takes place in L, it is easy to make it Σ^1_2-definable. To guarantee preservation by Hechler forcing, the following essential property was used:

Definition 2.2 *A forcing $\mathbb{P}$ strongly preserves splitting reals (abbreviated by "s.p.s."), if for every $\mathbb{P}$-name $\dot{a}$ for an element of $[\omega]^\omega$, there is a sequence $\{a_n : n \in \omega\}$ of elements of $[\omega]^\omega$, such that if $z \in [\omega]^\omega$ splits all a_n's, then $\Vdash_{\mathbb{P}}$ "$\check{z}$ splits $\dot{a}$".*

The Hechler partial order satisfies the s.p.s.-property by [1]. Moreover, the s.p.s.-property is preserved by iterations of ccc forcings with finite support (see [2, Proposition 3.10]).

We now state the Main Lemma from [4], involved in the induction step of the construction (in the original Lemma, an ideal on ω was also generated for technical reasons, but we leave it out here since it would only make the presentation more difficult). The notations P^β, A_β, Φ^β etc. refer to the objects described above.

Lemma 2.3 (Main Lemma, [4]) *Let M be a countable model of set theory such that $P^\beta \in M$ for all $\beta < \alpha$. Assume that for all $\beta \neq \beta' < \alpha$ and for all $f, g \in {}^\omega\omega$, the set $\Phi^\beta(f) \cap \Phi^{\beta'}(g)$ is finite (i.e., $\bigcup_{\beta<\alpha} A_\beta$ is an a.d. family).*

Then there exists a new partition P^α (of some domain D_α), lying outside M, which satisfies the following properties:

1. *For every $f, h \in {}^\omega\omega$ and every $\beta < \alpha$, $\Phi^\beta(f) \cap \Phi^\alpha(h)$ is finite (i.e., $\bigcup_{\beta\leq\alpha} A_\beta$ is still a.d.)*
2. *For every $Y \in M$, if Y is almost disjoint from $\Phi^\beta(f)$ for all $f \in {}^\omega\omega$ and all $\beta < \alpha$, then there exists an $h \in {}^\omega\omega$ such that $\Phi^\alpha(h) \subseteq Y$.*
3. *Suppose $V' \supseteq V$ is a model of set theory, $M' \supseteq M$ is a countable model with $M' \in V'$, and every real in V which is splitting over M is still splitting over M'. Then for every $Y \in M'$, if Y is almost disjoint from $\Phi^\beta(f)$ for every $f \in {}^\omega\omega$ in V' and every $\beta < \alpha$, then there exists an $h \in {}^\omega\omega$ in V' such that $V' \vDash \Phi^\alpha(h) \subseteq Y$ (i.e., condition 2 holds relativized to V' and M'.)*

Clearly, the above lemma can be applied with $V' = V[G]$, and $M' = M[G]$ being generic extensions via some forcing that satisfies the s.p.s. property. This Lemma will be the crucial tool in our inductive construction of the mad family in Sect. 4.

Next, we shift our attention to the Δ^1_3-definable well-order of the reals. As a Σ^1_2-definable well-order implies that every real is constructible (see e.g. [11, Theorem 25.39]), a Δ^1_3 well-order is optimal in the presence of $\neg$CH. A (boldface) $\boldsymbol{\Delta}^1_3$ well-order together with $\neg$CH was first obtained by Harrington [10], and Sy Friedman improved this result by establishing the consistency of a (lightface) Δ^1_3 well-order of the reals together with $\mathfrak{c} = \aleph_2$. Different methods of obtaining large continuum, Δ^1_3 well-orders, and the existence of certain combinatorial objects on the reals have recently been developed in [5–7] (dealing with cardinal inequalities, ω-mad families, and MA, respectively).

In this paper we will mostly be using the methods from [7]. The final model will be obtained as a two-step forcing extension of L. In the first stage (the "preliminary stage") the universe is prepared in a special way, by adding certain subsets of $\aleph_1$ and $\aleph_2$ but no new reals. We will denote this intermediate extension by L^*, and the forcing leading up to it by $\mathbb{P}^*$. Note that since no new reals have been added, L^* still satisfies many properties of L, such as having a Σ^1_2-good well-ordering of the reals.

In the next stage (the "coding stage"), new reals are added to L^*, by a finite support iteration of length $\aleph_3$, consisting of σ-centered forcing posets. This iteration simultaneously makes sure that $\mathfrak{b} = \aleph_3$ and that a Δ^1_3 well-ordering of the reals exists. In [7], the Π^1_2-definable mad family was explicitly added by this forcing as well. In our situation, we would like to preserve an $\aleph_1$-perfect mad family defined in L^* instead. We would like to simulate the proof in [4], but for that we need two ingredients: the s.p.s.-property of the forcing, and a way to deal with iterations of length longer than $\aleph_1$. The next theorem deals with the first ingredient.

Recall that one of the central methods in the "coding stage" is *almost disjoint coding*, a technique which allows subsets of ω_1 to be coded by reals in a generic extension. We show that this can be done by a forcing having the s.p.s.-property.

Let $\vec{C} := \{c_\alpha : \alpha < \aleph_1\}$ be a fixed, definable (e.g. closed) family of a.d. sets, and let $A \subseteq \omega_1$ be an arbitrary set. Let $\mathcal{I}_A$ be the ideal on ω generated by the a.d. family $\{c_\alpha : \alpha \in A\}$, let $\mathcal{I}^+_A$ denote $\mathcal{I}_A$-positive sets and $\mathcal{F}_A$ the corresponding filter. While the standard almost disjoint coding can be seen as a Mathias partial order with the filter $\mathcal{F}_A$, we will use a Laver-like partial order instead. Precisely, we prove the following:

Lemma 2.4 *For any given $\vec{C}$ and $A \subseteq \omega_1$, there exists a σ-centered forcing, which we shall denote by $\mathbb{L}_A(\vec{C})$, such that*

1. *$\mathbb{L}_A(\vec{C})$ adds a dominating real,*
2. *$\mathbb{L}_A(\vec{C})$ satisfies the s.p.s.-property, and*
3. *$\mathbb{L}_A(\vec{C})$ adds a generic real $\dot{x}_G$ with the following property:*
   - *(a) if $\alpha \in A$ then $\Vdash |\mathrm{ran}(\dot{x}_G) \cap c_\alpha| < \omega$, and*
   - *(b) if $\alpha \notin A$ then $\Vdash |\mathrm{ran}(\dot{x}_G) \cap c_\alpha| = \omega$.*

     Consequently, $\Vdash$ "$\dot{x}_G$ encodes A".

Proof Let $\mathbb{L}_A(\vec{C})$ be the Laver partial order with filter $\mathcal{F}_A$, i.e., the partial order consisting of all trees T such that for any $t \in T$ longer than $\mathrm{stem}(T)$, we have $\mathrm{Succ}_T(t) := \{n \mid t^\frown\langle n\rangle \in T\} \in \mathcal{F}_A$; the ordering is inclusion.

It is clear that this forcing is σ-centered, and to see that it adds a dominating real, simply note that for any $t \in T$, if $\mathrm{Succ}_T(t) \in \mathcal{F}_A$ then also $\mathrm{Succ}_T(t) \setminus m \in \mathcal{F}_A$ for any finite m. To verify that it has the s.p.s.-property, we use a result of Brendle and Hrušák [3]. We need some definitions:

- An ideal $\mathcal{I}$ on ω is *countably tall* if for any sequence $\{a_n \mid n < \omega\}$ of infinite subsets of ω, there is $b \in \mathcal{I}$ such that $|a_n \cap b| = \omega$ for every n.
- For two ideals $\mathcal{I}, \mathcal{J}$, write $\mathcal{J} \leq_K \mathcal{I}$ ($\mathcal{J}$ *is Katetov-reducible to* $\mathcal{I}$) iff there is an $f : \omega \to \omega$ s.t. $\forall a\ (a \in \mathcal{J} \to f^{-1}[a] \in \mathcal{I})$.

A recent result from [3, Proposition 1] then states the following: Let $\mathcal{I}$ be an arbitrary ideal and $\mathcal{F}$ the corresponding filter. Then the following are equivalent:

1. For all $X \in \mathcal{I}^+$ and every $\mathcal{J} \leq_K \mathcal{I}\restriction X$, $\mathcal{J}$ is not countably tall.
2. Laver forcing with the filter $\mathcal{F}$ has the s.p.s.-property.

The argument for the proof of this result is quite similar to the one for Hechler forcing—indeed Hechler forcing can be seen as Laver with the cofinite filter. To prove that $\mathbb{L}_A(\vec{C})$ satisfies the s.p.s., it suffices to show that $\mathcal{I}_A$ satisfies clause 1 from above. The main point is that $\mathcal{I}_A$ itself is not countably tall.

Let $X \in \mathcal{I}^+_A$, let $\mathcal{J} \leq_K \mathcal{I}_A\restriction X$ and let $f : X \to \omega$ be the function witnessing this Katetov reduction. We have two cases:

Case 1 The set $\{\alpha \in A \mid |f``(c_\alpha \cap X)| = \omega\}$ is finite. Let $\{\alpha_1, \ldots, \alpha_n\}$ enumerate it. Since $X \notin \mathcal{I}_A$, the remaining set $Y := X \setminus (c_{\alpha_1} \cup \cdots \cup c_{\alpha_n})$ is also not in $\mathcal{I}_A$. Let $Y' := f``Y$. Then $Y' \notin \mathcal{J}$, so, in particular, Y' is infinite. We claim that no infinite subset $Z \subseteq Y'$ can be in $\mathcal{J}$, which will witness the fact that $\mathcal{J}$ is not countably tall (in fact it will not even be tall).

Towards a contradiction, let $Z \subseteq Y'$ be an infinite set in $\mathcal{J}$. Then $f^{-1}[Z] \in \mathcal{I}_A$, and $f^{-1}[Z] \cap Y$ is an infinite set, also in $\mathcal{I}_A$. So $f^{-1}[Z] \cap Y$ must be almost covered by some finitely many $c_{\beta_1}, \ldots, c_{\beta_k}$, none of which can be among the c_{α_i}'s. Therefore, Z is almost covered by finitely many sets of the form $f``(c_{\beta_j} \cap Y)$, where $c_{\beta_j} \neq c_{\alpha_i}$ for any i. But by assumption, all such sets were finite, contradicting that Z is infinite.

Case 2 The set $\{\alpha \in A \mid |f``(c_\alpha \cap X)| = \omega\}$ is infinite. Pick a countable sequence $\{\alpha_n \mid n < \omega\}$ from it, and let $a_n := f``(c_{\alpha_n} \cap X)$. We claim that $\{a_n \mid n < \omega\}$ is a witness to the fact that $\mathcal{J}$ is not countably tall. Let $b \in \mathcal{J}$ be arbitrary. By assumption, $b' := f^{-1}[b] \in \mathcal{I}_A$. This means that there are $\beta_1, \ldots \beta_k \in A$ such that $b' \subseteq^* c_{\beta_1} \cup \cdots \cup c_{\beta_k}$. But then b' cannot have infinite intersection with infinitely many of the c_{α_n}'s, since otherwise some c_{β_i} and some c_{α_n}, with $\beta_i \neq \alpha_n$, would have infinite intersection, contradicting their mutual almost disjointness. Therefore, for some n, b' has only finite intersection with c_{α_n}. But then $b = f``(b')$ has finite intersection with a_n, proving that $\mathcal{J}$ is not countably tall.

It remains to show that $\mathbb{L}_A(\vec{C})$ can be used for a.d. coding purposes, i.e., condition (3) from the theorem.

(a) Let $\alpha \in A$ and $T \in \mathbb{L}_A(\vec{C})$. Inductively let $S \leq T$ be obtained by pruning the tree and removing c_α from every splitting node, i.e., making sure that $\mathrm{Succ}_S(t) := \mathrm{Succ}_T(t) \setminus c_\alpha$ for every $t \in S$. Since $\omega \setminus \mathrm{Succ}_T(t)$ is in $\mathcal{I}_A$ and $\alpha \in A$, $\omega \setminus \mathrm{Succ}_S(t)$ is also in $\mathcal{I}_A$, so the tree S is a valid $\mathbb{L}_A(\vec{C})$-condition. Moreover, for all n above the stem, $S \Vdash n \notin c_\alpha$. Hence $S \Vdash |\mathrm{ran}(\dot{x}_G) \cap c_\alpha| < \omega$.

(b) Let $\alpha \notin A$, $T \in \mathbb{L}_A(\vec{C})$ and $n \in \omega$ be given. Let $t := \mathrm{stem}(T)$ and consider $\mathrm{Succ}_T(t) \in \mathcal{F}_A$. Since $\alpha \notin A$ and the collection $\{c_\beta \mid \beta < \aleph_1\}$ was a.d., clearly $c_\alpha \notin \mathcal{I}_A$. But then $\mathrm{Succ}_T(t) \cap c_\alpha$ is infinite and so we may pick $m \geq n$ from this set. Then letting $S \leq T$ be such that $\mathrm{stem}(S) = t^\frown\langle m \rangle$ we have $S \Vdash m \in \mathrm{ran}(\dot{x}_G)$.

As a result, $\dot{x}_G$ codes A as we wanted. □

3 The preliminary stage, $\diamondsuit$ and $\diamondsuit'$

In this section we review the preliminary forcing construction leading from L to L^*, and verify that the $\diamondsuit$-principle is valid in L^*. Most of the exposition here follows closely that of [7], although many details are left out. We start by defining the preliminary forcing $\mathbb{P}^* = \mathbb{P}^0 * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2$.

A transitive ZF$^-$ model $\mathcal{M}$ is *suitable* if $\omega_3^{\mathcal{M}}$ exists and $\omega_3^{\mathcal{M}} = \omega_3^{L^{\mathcal{M}}}$. If $\mathcal{M}$ is suitable then also $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$ and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Fix a $\diamondsuit_{\omega_2}(cof(\omega_1))$ sequence $\langle G_\xi \mid \xi \in \omega_2 \cap cof(\omega_1)\rangle$ which is Σ_1-definable over L_{ω_2}. For $\alpha < \omega_3$, let W_α be the $<_L$-least subset of ω_2 coding α, and for $1 < \alpha < \omega_3$

let $S_\alpha = \{\xi \in \omega_2 \cap cof(\omega_1) \mid G_\xi := W_\alpha \cap \xi \neq \emptyset\}$. Then $\vec{S} = \langle S_\alpha \mid 1 < \alpha < \omega_3 \rangle$ is a sequence of stationary subsets of $\omega_2 \cap cof(\omega_1)$, which are mutually almost disjoint. Let $S_{-1} := \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) \mid G_\xi = \emptyset\}$. Note that S_{-1} is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ which is disjoint from all S_α's.

Step 0. For every α such that $\omega_2 \leq \alpha < \omega_3$ "shoot a club" C_α disjoint from S_α via the poset $\mathbb{P}^0_\alpha$, consisting of all closed subsets of ω_2 which are disjoint from S_α ordered by end-extension, and let $\mathbb{P}^0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1, where for $\alpha \in \omega_2$, $\mathbb{P}^0_\alpha$ is the trivial poset. Then $\mathbb{P}^0$ is countably closed, ω_2-distributive (the proof of which uses the stationarity of S_{-1}) and ω_3-c.c.

Step 1. We begin by fixing some notation. Whenever $k \in \omega$, X is a set of ordinals and $j \in k$, let $I^k_j(X) = \{\gamma \mid k \cdot \gamma + j \in X\}$. In particular, let $Even(X) = I^2_0(X) = \{\gamma \mid 2 \cdot \gamma \in X\}$. For every $\alpha < \omega_3$ let D_α be a subset of ω_2 which codes the triple $\langle C_\alpha, W_\alpha, W_\gamma \rangle$ where γ is the largest limit ordinal $\leq \alpha$, precisely: $I^3_0(D_\alpha) = C_\alpha$, $I^3_1(D_\alpha) = W_\alpha$ and $I^3_2(D_\alpha) = W_\gamma$. Let

$$E_\alpha = \{\mathcal{M} \cap \omega_2 \mid \mathcal{M} \prec L_{\alpha+\omega_2+1}[D_\alpha],\ \omega_1 \cup \{D_\alpha\} \subseteq \mathcal{M}\}.$$

Then E_α is a club on ω_2. Choose $Z_\alpha \subseteq \omega_2$ such that $Even(Z_\alpha) = D_\alpha$ and if $\beta < \omega_2$ is $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then $\beta \in E_\alpha$. Then we have:

$(*)_\alpha$: If $\beta < \omega_2$, $\mathcal{M}$ is a suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \models \psi(\omega_2, Z_\alpha \cap \beta)$, where $\psi(\omega_2, X)$ is the formula "$Even(X)$ codes a triple $(\bar{C}, \bar{W}, \bar{\bar{W}})$, where $\bar{W}$ and $\bar{\bar{W}}$ are the $<_L$-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Similarly to $\vec{S}$, define a sequence $\vec{A} = \langle A_\xi \mid \xi < \omega_2 \rangle$ of stationary subsets of ω_1 which are mutually almost disjoint, using the "standard" $\diamondsuit$-sequence. Code Z_α by a subset X_α of ω_1 with the poset $\mathbb{P}^1_\alpha$ consisting of all pairs $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$ (note that this is closely related to the a.d. coding discussed in Sect. 2, but deals with coding subsets of ω_2 by subsets of ω_1). Then X_α satisfies the following condition:

$(**)_\alpha$: If $\mathcal{M}$ is a suitable model such that $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X_\alpha)$, where $\phi(\omega_1, \omega_2, X)$ is the formula: "Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, such that $Even(\bar{Z})$ codes a triple $(\bar{C}, \bar{W}, \bar{\bar{W}})$, where $\bar{W}$ and $\bar{\bar{W}}$ are the $<_L$-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Let $\mathbb{P}^1 = \prod_{\alpha<\omega_3} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for all $\alpha \in \omega_2$, with countable support. Then $\mathbb{P}^1$ is countably closed and has the ω_2-c.c.

Step 2. Finally we force a "localization" of the X_α's. Fix ϕ as in $(**)_\alpha$ and define the poset $\mathcal{L}_k(X, X')$ as in [7, Definition 1]. That is, let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$

and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. Then let $\mathcal{L}(X, X')$ be the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$
3. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The ordering is end-extension.

For every $\alpha \in Lim(\omega_3)$ and $m \in \omega$, let $\mathbb{P}^2_{\alpha+m} = \mathcal{L}(X_{\alpha+m}, X_\alpha)$. Let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. In $L^{\mathbb{P}^0 * \mathbb{P}^1}$, the poset $\mathbb{P}^2$ has the ω_2-c.c. Also note that $\mathbb{P}^2_{\alpha+m}$ produces a generic function in the space 2^{ω_1} (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$), which is the characteristic function of a subset $Y_{\alpha+m}$ of ω_1 with the following property:

$(***)_\alpha$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $\mathcal{M}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta)$.

Now we let $\mathbb{P}^* := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ be the result of combining these three generic extensions, and use L^* to denote the intermediary extension $L^{\mathbb{P}^*}$.

Next, we want to show that $\diamondsuit$ holds in L^*. Since we have added new subsets of ω_1, this is not a priori obvious. To prove that this is the case, we use the related $\diamondsuit'$-principle, a version of $\diamondsuit$ due to Kunen, in which we allow countably many possibilities at stage α to capture sets.

Definition 3.1 *A sequence $\{S_\alpha \mid \alpha < \omega_1\}$ is a $\diamondsuit'$-sequence if $S_\alpha = \{S_\alpha^n \mid n < \omega\}$ such that $S_\alpha^n \subseteq \alpha$ for all n, and if for all $S \subseteq \omega_1$, the set $\{\alpha \mid \exists n\, (S \cap \alpha = S_\alpha^n)\}$ is stationary.*

Lemma 3.2 *In L^*, there exists a $\diamondsuit'$-sequence which is Σ_1-definable over L_{ω_1}.*

Proof We define the sequence in L, and show that it is preserved by $\mathbb{P}^*$. For $\alpha < \omega_1 (= \omega_1^L)$, let $\beta(\alpha)$ be the least ordinal β such that $L_\beta \models \mathrm{ZF}^- + (\alpha \text{ is countable})$. Let $D'_\alpha := \{A \subseteq \alpha \mid A \in L_{\beta(\alpha)}\}$. We claim that $\{D'_\alpha \mid \alpha < \omega_1\}$ is a $\diamondsuit'$-sequence even after forcing with $\mathbb{P}^*$.

So, let $p_0 \in \mathbb{P}^*$, let $\dot{X}$ be a $\mathbb{P}^*$-name for a subset of ω_1 and $\dot{C}$ a $\mathbb{P}^*$-name for a closed unbounded subset of ω_1. Let $\mathcal{N}$ be the least countable elementary submodel of some large L_Θ such that $p_0, \dot{X}, \dot{C}$ are elements of $\mathcal{N}$. Let $\bar{\mathcal{N}}$ be the transitive collapse of $\mathcal{N}$.

As in the proof of [7, Lemma 1], get an extension p_1 of p_0 which meets all dense sets in $\mathcal{N}$ by considering a generic filter g over $\mathcal{N}$. Now let $\bar{g}$ be the image of g under the transitive collapse that maps $\mathcal{N}$ to $\bar{\mathcal{N}}$.

Then $\bar{g}$ is definable from an ω-enumeration of $\bar{\mathcal{N}}$ and (as $\mathcal{N}$ is the least countable elementary submodel of some L_Θ containing a certain finite set of parameters) there is such an ω-enumeration in $L_{\beta(\alpha)}$, where $\alpha = \mathcal{N} \cap \omega_1$. So $p_1 \Vdash \dot{X} \cap \alpha \in L_{\beta(\alpha)}$ and $p_1 \Vdash \alpha \in \dot{C}$. But then p_1 forces that the intersection $\{\alpha \mid \dot{X} \cap \alpha \in D'_\alpha\} \cap \dot{C}$ is non-empty, which completes the proof. □

To conclude, note that by [14, Theorem II 7.14], every $\diamondsuit'$-sequence gives rise to a $\diamondsuit$-sequence in a natural way. Consequently, there is a $\diamondsuit$-sequence in L^* which is Σ_1 definable over L_{ω_1}.

4 Constructing an s.p.s.-indestructible mad family in L^*

We are now ready to prove the main theorem, using an inductive construction and Lemma 2.3. In [4], the method was to define a sequence $\{M_\alpha \mid \alpha < \aleph_1\}$ of countable models covering all $\mathbb{D}_{\aleph_1}$-names for reals (where $\mathbb{D}_{\aleph_1}$ stands for the $\aleph_1$-iteration of Hechler forcing), while simultaneously constructing the perfect a.d. families A_α using Lemma 2.3. Condition 3 of the Lemma then guaranteed that the family $\mathcal{A} := \bigcup_{\alpha<\aleph_1} A_\alpha$ thus constructed was not destroyed by $\mathbb{D}_{\aleph_1}$, and an additional argument (involving the fact that Hechler forcing is Suslin ccc) then showed that the same must hold for $\mathbb{D}_\kappa$, where κ is any regular uncountable cardinal. Since we will need to deal with more complicated iterations, which are ccc but not Suslin, we need a different method for dealing with longer iterations, and we use the $\diamondsuit$-sequence for this purpose.

Lemma 4.1 *Let $\mathbb{P}$ be any ccc forcing notion satisfying the s.p.s.-property. Then there exists a $\mathbb{P}$-indestructible, $\aleph_1$-perfect, Σ^1_2-definable mad family in L^*. Moreover, in $(L^*)^{\mathbb{P}}$ this family still has a Σ^1_2 definition.*

Proof First of all, note that we may assume, without loss of generality, that in L^* there exists a definable 5-dimensional version of $\diamondsuit$, namely, a sequence

$$\{(X_\alpha, E_\alpha, <_\alpha) \mid \alpha < \aleph_1\}$$

such that $X_\alpha \subseteq \omega_1$, $E_\alpha, <_\alpha \ \subseteq (\omega_1 \times \omega_1)$, and for every triple $(X, E, <)$, the set

$$\{\alpha \mid X \cap \alpha = X_\alpha, E \cap (\alpha \times \alpha) = E_\alpha \text{ and } (< \cap(\alpha \times \alpha)) = <_\alpha\}$$

is stationary. Fix such a sequence for the rest of the proof.

Definition 4.2 *We say that a triple $(X, E, <)$ "codes a* ZF$^-$ *model" iff*

1. *E and $<$ are binary relations on X,*
2. *(X, E) is well-founded and extensional,*
3. *$<$ well-orders X, and*
4. *$(X, E) \models$* ZF$^-$.

We proceed by defining the $\aleph_1$-mad family, by induction on $\alpha < \aleph_1$, using the ideas described in Sect. 2. At each step, Lemma 2.3 is applied to produce the next partition P^α (of some domain D_α), giving rise to a perfect a.d. set A_α. Simultaneously, a sequence of countable transitive ZF$^-$ models $\{M_\alpha \mid \alpha < \aleph_1\}$ will be defined (note that the transitivity of the models is crucial in the current argument). Inductively, the following conditions will be guaranteed for all α:

1. $\langle M_\beta \mid \beta < \alpha \rangle \in M_\alpha$,
2. $\langle P^\beta \mid \beta < \alpha \rangle \in M_\alpha$,
3. $\bigcup_{\beta<\alpha} A_\beta$ is a.d.

We proceed with the inductive construction. At stage α, assume M_β and P^β have been defined, and the three inductive conditions are satisfied. To define M_α, consider two cases:

- *Case 1.* If $(X_\alpha, E_\alpha, <_\alpha)$ codes a countable ZF$^-$ model (in the sense of Definition 4.2), let M'_α be its transitive collapse. If, additionally, it so happens that $\langle M_\beta \mid \beta < \alpha \rangle \in M'_\alpha$ and $\langle P^\beta \mid \beta < \alpha \rangle \in M'_\alpha$, let $M_\alpha := M'_\alpha$.
- *Case 2.* If the above fails, then simply let M_α be the countable, transitive ZF$^-$ model with $<_L$-least code, such that $\langle M_\beta \mid \beta < \alpha \rangle \in M_\alpha$ and $\langle P^\beta \mid \beta < \alpha \rangle \in M_\alpha$.

After that, we are in the right situation to apply Lemma 2.3 to the model M_α and the collection of partitions $\langle P^\beta \mid \beta < \alpha \rangle$, so we use it to construct a new partition P^α, picking the $<_L$-least one satisfying all the conditions.

This completes the inductive definition. We claim that $\mathcal{A} := \bigcup_{\alpha<\aleph_1} A_\alpha$ thus constructed is a $\mathbb{P}$-indestructible mad family. By the third inductive condition, it follows immediately that $\mathcal{A}$ is a.d., so let's focus on its maximality. Let G be $\mathbb{P}$-generic, and let Y be a new real in $L^*[G]$. Since $\mathbb{P}$ may have added many reals, by basic cardinality arguments we clearly cannot assume that Y is contained in some $M_\alpha[G]$. However, here we will use $\diamondsuit$ to get around this difficulty.

Claim 4.3 *For some $\alpha < \omega_1$, Y belongs to a generic extension of M_α via some forcing which has the s.p.s.-property.*

Proof Let $\dot{Y}$ be a $\mathbb{P}$-name for Y. Let N be a countably closed, elementary submodel of some sufficiently large $\mathcal{H}_\theta$, with $|N| = \aleph_1$, containing $\mathbb{P}$, $\dot{Y}$, the entire sequences $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ and $\langle P^\alpha \mid \alpha < \omega_1 \rangle$, and all the countable ordinals. Let E and $<$ be binary relations on ω_1 so that $(N, \in, <_{\mathcal{H}_\theta}) \cong (\omega_1, E, <)$ (here $<_{\mathcal{H}_\theta}$ refers to some natural well-order of N inherited from $\mathcal{H}_\theta$). Also, let $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ be a continuous sequence of countable elementary submodels of $\mathcal{H}_\theta$, converging to N. Note that if G is $\mathbb{P}$-generic, then, since $\mathbb{P}$ is ccc, $N[G]$ is a generic extension of N via $\mathbb{P} \cap N$ and $N_\alpha[G]$ is a generic extension of N_α via $\mathbb{P} \cap N_\alpha$.

Moreover, by continuity of the sequence $\langle N_\alpha \mid \alpha < \omega_1 \rangle$, there are club-many α so that

$$(N_\alpha, \in, <_{\mathcal{H}_\theta}) \cong (\alpha,\ E \cap (\alpha \times \alpha), < \cap (\alpha \times \alpha)).$$

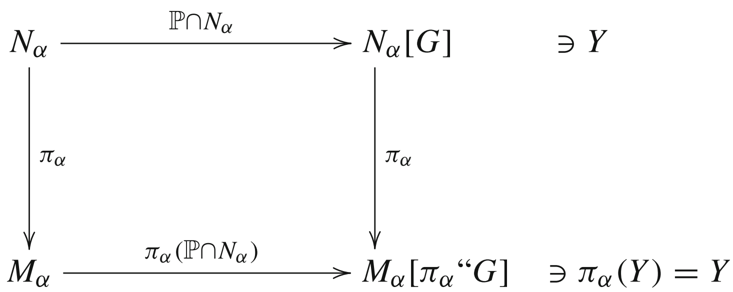
Using $\diamondsuit$, we can then pick an α such that in fact

$$(N_\alpha, \in, <_{\mathcal{H}_\theta}) \cong (X_\alpha, E_\alpha, <_\alpha).$$

Then clearly $(X_\alpha, E_\alpha, <_\alpha)$ codes a model, and by elementarity $\langle M_\beta \mid \beta < \omega_1 \rangle$ and $\langle P^\beta \mid \beta < \omega_1 \rangle$ belong to N_α. Moreover, we may assume that $\dot{Y} \in N_\alpha$.

Let $\bar{N}_\alpha$ be the transitive collapse of N_α, via collapsing function π_α. As all members of the transitive closure of M_β and P^β for $\beta < \alpha$ have rank $< \alpha$ (again, without loss of generality), they are mapped onto themselves by π_α. Also, since N_α contains all the ordinals $< \alpha$, it follows that the initial segments $\langle M_\beta \mid \beta < \alpha\rangle$ and $\langle P^\beta \mid \beta < \alpha\rangle$ are contained in the transitive collapse $\bar{N}_\alpha$. But then, we find ourselves in the situation of Case 1 (from the construction of the models), and it follows that $M_\alpha = \bar{N}_\alpha$.

As $\dot{Y} \in N_\alpha$, $Y \in N_\alpha[G]$. Then $Y = \pi_\alpha(Y)$ is in the transitive collapse of $N_\alpha[G]$ by π_α, which is equal to $M_\alpha[\pi_\alpha``G]$, the generic extension of M_α by the forcing $\pi_\alpha(\mathbb{P} \cap N_\alpha)$.



Since $\mathbb{P} \cap N_\alpha$ has the s.p.s.-property, so does $\pi_\alpha(\mathbb{P} \cap N_\alpha)$. Therefore, Y is indeed in a generic extension of an M_α via a forcing with the s.p.s.-property. □

Now we may apply condition (3) of Lemma 2.3 with $M = M_\alpha$ and $M' = M_\alpha[\pi_\alpha``G]$, and see that Y has infinite intersection with some member of $\bigcup_{\beta\le\alpha} A_\beta$. Therefore, indeed, $\mathcal{A} = \bigcup_{\alpha<\aleph_1} A_\alpha$ is $\mathbb{P}$-indestructible.

It remains only to argue that $\mathcal{A}^{(L^*)^{\mathbb{P}}}$ has a Σ^1_2 definition. For this, first note that $\omega^\omega \cap L^* = \omega^\omega \cap L$, and that, by Lemma 3.2, we may assume that the $\diamondsuit$-sequence we chose in the beginning of the proof is Σ_1 definable over L_{ω_1}. Since the M_α's are chosen so that they are either defined from $\diamondsuit$ or chosen to be $<_L$-least, and the P^α's are also $<_L$-least, we can use a standard argument to show that the set B of (codes for) $\{P^\alpha \mid \alpha < \aleph_1\}$ is a Σ^1_2 set. Then, in $(L^*)^{\mathbb{P}}$, the mad family is given by the formula

$$x \in \mathcal{A} \iff \exists b \in B\ (x \in A_\alpha \text{ for } \alpha \text{ s.t. } b \text{ codes } P^\alpha).$$

Since "$x \in A_\alpha$ for α s.t. b codes P^α" is a recursive computation, the above gives a Σ^1_2 definition of $\mathcal{A}$ in $(L^*)^{\mathbb{P}}$. This completes the proof of Lemma 4.1. □

With this we are almost done with the proof of the Main Theorem. All that remains to be done is forcing a Δ^1_3-definable well-order of the reals, together with $\mathfrak{b} = \mathfrak{c} = \aleph_3$, over L^* (i.e., the "coding stage"). For that, we will define a forcing iteration $\mathbb{P}_{\omega_3}$ following [7, Step 3], with only two essential differences:

1. For a.d. coding purposes, we will use the Laver-like almost disjoint coding from Lemma 2.4 as opposed to the standard a.d. coding, and
2. At stages where no coding is performed, we use a trivial version of the Laver-like coding (or use Hechler forcing).

This way, dominating reals are added cofinally often and the s.p.s.-property is preserved.

So, in L^*, fix a definable (e.g. closed) sequence $\vec{C} = \langle c_\zeta : \zeta < \omega_1\rangle$ of almost disjoint subsets of ω. This will be used for coding purposes. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma \mid \alpha \leq \omega_3, \gamma < \omega_3\rangle$ such that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset which has the s.p.s.-property. Every $\mathbb{Q}_\alpha$ is going to add a generic real whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{u}_\alpha$, and just as in [7] we will have that $L^*[G_\alpha] \cap {}^\omega\omega = L^*[\langle \dot{u}_\xi^{G_\alpha} \mid \xi < \alpha\rangle] \cap {}^\omega\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α. This gives a canonical well-order of the reals in $L^*[G_\alpha]$, which depends only on the sequence $\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha\rangle$. The $\mathbb{P}_\alpha$-name for this well-order will be denoted by $\dot{<}_\alpha$. Additionally, we can make sure that for all $\alpha < \beta$ we have that $\mathbb{P}_\beta$ forces $\dot{<}_\alpha$ to be an initial segment of $\dot{<}_\beta$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L^*, $<^G := \bigcup\{\dot{<}_\alpha^G : \alpha < \omega_3\}$ will be the desired well-order of the reals.

We proceed with the recursive construction of $\mathbb{P}_{\omega_3}$. Along the construction we shall also define a sequence $\langle \dot{A}_\alpha \mid \alpha \in \mathrm{Lim}(\omega_3)\rangle$, where $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $[\alpha, \alpha + \omega)$. For every ν with $\omega_2 \leq \nu < \omega_3$, fix a bijection $i_\nu : \{\langle \zeta, \xi\rangle \mid \zeta < \xi < \nu\} \overset{\sim}{\longrightarrow} \mathrm{Lim}(\omega_2)$. If G_α is $\mathbb{P}_\alpha$-generic over L^*, $<_\alpha = \dot{<}_\alpha^{G_\alpha}$ and x, y are reals in $L^*[G_\alpha]$ such that $x <_\alpha y$, let $x * y := \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}$ and $\Delta(x * y) := \{2n+2 \mid n \in x * y\} \cup \{2n+1 \mid n \notin x * y\}$.

Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Suppose α is a limit ordinal. Write it in the form $\omega_2 \cdot \alpha' + \xi$, where $\xi < \omega_2$. If $\alpha' > 0$, let $i = i_{\mathrm{o.t.}(\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'})}$ and $\langle \xi_0, \xi_1\rangle = i^{-1}(\xi)$. Let $A_\alpha := \dot{A}_\alpha^{G_\alpha}$ be the set $\alpha + (\omega \backslash \Delta(x_{\xi_0} * x_{\xi_1}))$, where x_ζ is the ζ-th real in $L[G_{\omega_2\cdot\alpha'}] \cap [\omega]^\omega$ according to the well-order $\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'}$ (here $G_{\omega_2\cdot\alpha'} = G_\alpha \cap \mathbb{P}_{\omega_2\cdot\alpha'}$).

Then, we define $\mathbb{Q}_\alpha$ as follows: $\mathbb{Q}_\alpha$ is the finite support iteration $\langle \mathbb{P}^n_\alpha, \dot{\mathbb{Q}}^m_\alpha \mid n \leq \omega, m < \omega\rangle$, where

- *Case 1:* if $m \in \Delta(x_{\xi_0} * x_{\xi_1})$ then $\Vdash_m$"$\dot{\mathbb{Q}}^m_\alpha$ is the Laver-like a.d. coding partial order $\mathbb{L}_{Y_{\alpha+m}}(\vec{C})$ from Lemma 2.4", where $\vec{C}$ is the a.d. sequence fixed at the beginning, and $Y_{\alpha+m}$ is the subset of ω_1 whose characteristic function was added by $\mathbb{P}^2_{\alpha+m}$ (see Sect. 2).
- *Case 2:* if $m \notin \Delta(x_{\xi_0} * x_{\xi_1})$ then $\Vdash_m$ $\dot{\mathbb{Q}}^m_\alpha$ is the trivial poset.

Let u^m_α be the generic real added by $\mathbb{Q}^m_\alpha$ in the first case, and the constant 0 function in the second case. Let u_α be a real encoding the u^m_α's for all $m \in \omega$.

If $\alpha < \omega_2$ or α is a successor, let $\mathbb{Q}_\alpha$ be again the Laver-like forcing $\mathbb{L}_{\omega_1}(\vec{C})$, or Hechler forcing (or any other σ-centered forcing that satisfies the s.p.s. property). Notice that what happens at these stages is irrelevant for the purpose of "decoding" the Δ^1_3-well-order.

With this the inductive definition of our finite support iteration $\mathbb{P}_{\omega_3}$ is complete—for more details, we refer the reader to [7]. To complete the proof, first notice that since the sets $\Delta(x, y)$ are always non-empty, Case 1 occurs cofinally often in the iteration, and therefore dominating reals are added cofinally often. It follows that in $(L^*)^{\mathbb{P}_{\omega_3}}$ we have $\mathfrak{b} = \mathfrak{c} = \aleph_3$. To show that in $(L^*)^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable well-order of the reals, we follow the arguments of [7]. Notice that a version of [7, Lemma 3] certainly goes through in our context (in fact it is even easier to prove). Consequently, Lemmas 4 and 5 from [7] hold, and the Σ^1_3 formula defining the well-order can be read off from the statements of these Lemmas.

Finally, note that all the forcing posets in the construction of $\mathbb{P}_{\omega_3}$ (including the trivial ones) satisfy the s.p.s.-property, so by [2, Proposition 3.10], the entire finite support iteration does, as well. Thus we can apply Lemma 4.1 and obtain a Σ^1_2-definable mad family in $(L^*)^{\mathbb{P}_{\omega_3}}$, and, by Theorem 1.3, also a Π^1_1 mad family.

5 Open questions

In the introduction, we mentioned that there is no problem to modify our proof so that it works for $\mathfrak{b} = \mathfrak{c} = \aleph_2$, and a natural open question is whether the same holds for $\mathfrak{b} = \mathfrak{c} = \kappa$ for all regular uncountable κ. We conjecture that the answer is positive, but some work needs to be done on the coding mechanism to make sure it works for larger values of the continuum.

Another question one may ask is whether the existence of a Π^1_1 mad family and a Δ^1_3 well-order is consistent with other values of the cardinal characteristics $\mathfrak{b}$, $\mathfrak{a}$ and $\mathfrak{s}$. For example, is it consistent with $\mathfrak{b} < \mathfrak{c}$ or even $\mathfrak{b} < \mathfrak{a}$?

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3.3 Cichoń's diagram, regularity properties and $\mathbf{\Delta}^1_3$ sets of reals

Vera Fischer, Sy David Friedman and Yurii Khomskii, Archive for Mathematical Logic 53:5–6, 2014, pp 695-729.

In this case, I took most of the initiative to study the questions. The paper was the main output of one years' worth of work. Although the work took on a collaborative form, with the three authors having regularly scheduled meetings together as well as Fischer and Khomskii meeting informally for intensive discussion, most of the results in this paper are due to me. Certain difficulties and obstacles that arose in Section 3 and 4 were solved by intensive collaboration of all three authors, and other sections contain smaller contributions by the other authors as well.

- ***Lemma 3.7***: *Khomskii, with contributions by Fischer and Friedman*
- ***Remark 3.10***: *Khomskii*
- ***Lemma 3.14***: *Khomskii, with contributions by Fischer and Friedman*
- ***Lemma 3.15***: *Khomskii*
- ***Lemma 3.16***: *Khomskii*
- ***Remark 3.17***: *Friedman*
- ***Theorem 3.20***: *Khomskii*
- ***Theorem 4.5***: *Khomskii, with contributions by Fischer and Friedman*
- ***Theorem 4.8***: *Khomskii, with contributions by Fischer and Friedman*
- ***Theorem 4.10***: *Khomskii*
- ***Section 5, pp 25–28***: *Khomskii, with contributions by Fischer*
- ***Section 6, pp 30–33***: *Khomskii, with contributions by Fischer*
- ***Theorem 7.2***: *Khomskii, with contributions by Fischer and Friedman*
- ***Corollary 7.4***: *Khomskii*
- ***Lemma 7.9***: *Khomskii*
- ***Theorem 7.12***: *Khomskii*

The paper was written by Khomskii.

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Mathematical Logic

Cichoń's diagram, regularity properties and $\boldsymbol{\Delta}^1_3$ sets of reals

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Abstract We study regularity properties related to Cohen, random, Laver, Miller and Sacks forcing, for sets of real numbers on the $\boldsymbol{\Delta}^1_3$ level of the projective hieararchy. For $\boldsymbol{\Delta}^1_2$ and $\boldsymbol{\Sigma}^1_2$ sets, the relationships between these properties follows the pattern of the well-known Cichoń diagram for cardinal characteristics of the continuum. It is known that assuming suitable large cardinals, the same relationships lift to higher projective levels, but the questions become more challenging without such assumptions. Consequently, all our results are proved on the basis of ZFC alone or ZFC with an inaccessible cardinal. We also prove partial results concerning $\boldsymbol{\Sigma}^1_3$ and $\boldsymbol{\Delta}^1_4$ sets.

Keywords Cichoń's diagram · Regularity properties · Projective hierarchy

Mathematics Subject Classification (2010) 03E15 · 03E35 · 03E40 · 03E17

1 Introduction

The study of regularity properties in descriptive set theory is closely related to cardinal characteristics of the continuum. By well-known results of Solovay, Judah and Shelah, the statement "all $\boldsymbol{\Sigma}^1_2$ sets of reals are Lebesgue measurable" is equivalent to "for every $r \in \omega^\omega$, the set of random reals over $L[r]$ has measure one", and the statement

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"all $\boldsymbol{\Delta}^1_2$ sets of reals are Lebesgue measurable" is equivalent to "for every $r \in \omega^\omega$, there is a random real over $L[r]$"; analogous results hold for the Baire property and Cohen reals. These characterizations link the statements about projective regularity with the covering and additivity numbers of the meager and null ideals on the reals. Likewise, Brendle and Löwe [8] uncovered a link between the regularity properties naturally connected to Laver-, Miller- and Sacks-forcing for $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$ sets, and the cardinal invariants $\mathfrak{b}$ (the bounding number), $\mathfrak{d}$ (the dominating number) and the size of the continuum.

Thus, if we restrict attention to the second projective level, a very clear picture emerges, in which the relationships between the various regularity statements follow the familiar pattern of Cichoń's diagram (see Fig. 1 in Sect. 2). Other, more exotic, regularity properties have also been extensively studied on the second level, with some important contributions being [6–9,23]. See also [32] for a very detailed and self-contained survey. An abstract approach has been proposed by Ikegami in [24] and developed further in the PhD theses of Laguzzi [36] and of the third author [35].

Far less is known concerning sets higher up in the projective hierarchy, even at the $\boldsymbol{\Sigma}^1_3$ and $\boldsymbol{\Delta}^1_3$ levels. Concerning such questions, there are two, somewhat divergent, methods of approach. According to one of them, adopted e.g., by Ikegami in [24], Judah and Spinas in [31] and a few others, one assumes the existence of certain large cardinals, which imply that all the essential results from the second level lift almost verbatim to higher levels (for the third level, this requires the existence of sharps for sets of ordinals). Although this approach is interesting and certainly worthy of further investigation, it is not the approach we will take in this paper, for reasons that shall be explained in the next section. Here, all results will be proved on the basis of ZFC alone or ZFC with an inaccessible; indeed, we will put special emphasis on eliminating the inaccessible wherever possible (notice that the statement "all $\boldsymbol{\Sigma}^1_3$ sets are Lebesgue measurable" already implies an inaccessible in L by [40]).

Some work in this direction, most of it contained in Chapter 9 of [3], has been carried out by Judah, Shelah, Bagaria and others in the eighties and early nineties, and our methods are related to the ones used there. On the other hand, we have more modern means at our disposal, particularly the theory of "non-elementary proper forcing" (in our case, "Suslin and Suslin$^+$ proper forcing") developed by Judah, Shelah, Goldstern and Kellner, and a result of René David [10] about the existence of a model of set theory in which ω_1 is inaccessible in $L[r]$ for all reals r, but there exists a $\boldsymbol{\Sigma}^1_3$-good

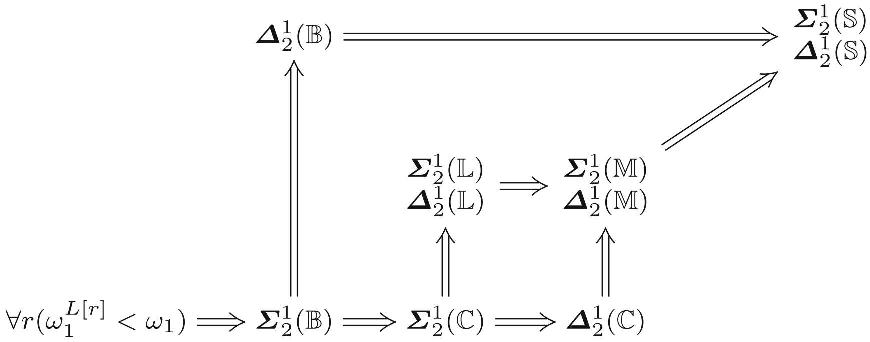


Fig. 1 A complete diagram of implications for $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$ sets of reals

wellorder of the reals. Using these methods, we will provide a complete solution to the situation on the $\boldsymbol{\Delta}^1_3$-level. Although our emphasis will be on the regularity properties corresponding to the cardinal invariants appearing in Cichoń's diagram (i.e., the regularity properties connected to Cohen, random, Laver, Miller and Sacks forcing), our methods are sufficiently general and certainly have many more applications regarding questions of a similar nature.

The paper is structured as follows: in Sect. 2 we introduce the relevant definitions, summarize known results on the second level and provide the motivation for the research carried out in the rest of the paper. In Sect. 3 we recall the basic properties of Suslin and Suslin$^+$ proper forcing, proving several important technical results which may be interesting in their own right and have applications other than those considered in this paper. In the crucial Sect. 4 we develop several methods for obtaining regularity for $\boldsymbol{\Delta}^1_3$ sets of reals in a "minimal" way, using various iterated forcing techniques. In Sect. 5 we use these methods to separate regularity properties on the $\boldsymbol{\Delta}^1_3$-level. In Sect. 6 we briefly consider two additional regularity properties that have received a lot of attention is set theory, and in Sect. 7 we deal with some results concerning $\boldsymbol{\Sigma}^1_3$ and $\boldsymbol{\Delta}^1_4$ sets. Section 8 closes with some open questions.

2 Regularity properties and Cichoń's diagram

2.1 Definitions

We assume that the reader is familiar with the standard definitions of the *Baire property*, *Lebesgue measure*, the ideal $\mathcal{M}$ of *meager* sets and $\mathcal{N}$ of *measure-null* sets, as well as the definitions of *Cohen*, *random*, *Laver*, *Miller* and *Sacks forcing*. Following standard practice, we denote these forcing notions with the letters $\mathbb{C}$, $\mathbb{B}$, $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{S}$. If T is a tree on $\omega^{<\omega}$ or $2^{<\omega}$ then $[T]$ denotes the set of branches through T, and $[t]$ denotes the basic open set for $t \in \omega^{<\omega}$ or $2^{<\omega}$.

Definition 2.1 A set $A \subseteq \omega^\omega$ is

- *Laver-measurable* if $\forall T \in \mathbb{L}\ \exists S \in \mathbb{L}$ s.t. $S \leq T$ and ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).
- *Miller-measurable* if $\forall T \in \mathbb{M} \exists S \in \mathbb{M}$ s.t. $S \leq T$ and ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).
  A set $A \subseteq 2^\omega$ is
- *Sacks-measurable* if $\forall T \in \mathbb{S}\ \exists S \in \mathbb{S}$ s.t. $S \leq T$ and ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).

Sacks-measurability is also known under the term *Marczewski-measurability*. Although contemporary interest in properties such as the ones above is often forcing-related, it is interesting to note that among Polish mathematicians, there had been a considerable interest in them long before the advent of forcing, see e.g., [44].

Both Lebesgue measure and the Baire property can be represented in the style of Definition 2.1, using the following well-known characterizations:

1. A subset A of ω^ω or 2^ω is Lebesgue-measurable iff every closed set C of positive measure has a closed subset $C' \subseteq C$ of positive measure such that $C' \subseteq A$ or $C' \cap A = \varnothing$.

2. A subset A of ω^ω or 2^ω has the Baire property iff every basic open set $[t]$ has a basic open subset $[s] \subseteq [t]$ such that $[s] \setminus A$ is meager or $[s] \cap A$ is meager. Moreover, this holds iff every G_δ non-meager set X has a G_δ non-meager subset $Y \subseteq X$ such that $Y \subseteq A$ or $Y \cap A = \varnothing$.

If we choose to represent random forcing by the partial order of closed sets of positive measure, and Cohen forcing by G_δ (or Borel) non-meager sets, we obtain an exact equivalence between the two classical properties on one hand, and $\mathbb{B}$- and $\mathbb{C}$-measurability in the sense analogous to Definition 2.1 on the other hand. Therefore, we will frequently refer to the Baire property and Lebesgue measure as "$\mathbb{C}$-" and "$\mathbb{B}$-measurability", respectively.

Notation 2.2 If $\boldsymbol{\Gamma}$ is a class of sets (e.g., a projective class), we will use the notation "$\boldsymbol{\Gamma}(\mathbb{P})$" to abbreviate the statement "all sets of complexity $\boldsymbol{\Gamma}$ are $\mathbb{P}$-measurable", with $\mathbb{P}$ ranging over one of the forcing notions considered above.

2.2 The second level

While ZFC proves that analytic sets are $\mathbb{P}$-measurable for all $\mathbb{P}$ as above, statements such as $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ and $\boldsymbol{\Delta}^1_2(\mathbb{P})$ are independent of ZFC. The following results of Solovay [42], Ihoda–Shelah [23] and Brendle–Löwe [8] provide an exact characterization of regularity statements for $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$ sets of reals.

Theorem 2.3 [42]

1. $\boldsymbol{\Sigma}^1_2(\mathbb{B}) \iff \forall r\ \{x \mid x \text{ is not random over } L[r]\} \in \mathcal{N}$.
2. $\boldsymbol{\Sigma}^1_2(\mathbb{C}) \iff \forall r\ \{x \mid x \text{ is not Cohen over } L[r]\} \in \mathcal{M}$.

Theorem 2.4 [23]

1. $\boldsymbol{\Delta}^1_2(\mathbb{B}) \iff \forall r\ \exists x\ (x \text{ is random over } L[r])$.
2. $\boldsymbol{\Delta}^1_2(\mathbb{C}) \iff \forall r\ \exists x\ (x \text{ is Cohen over } L[r])$.

Theorem 2.5 [8]

1. $\boldsymbol{\Sigma}^1_2(\mathbb{L}) \iff \boldsymbol{\Delta}^1_2(\mathbb{L}) \iff \forall r\ \exists x\ (x \text{ is dominating over } L[r])$.
2. $\boldsymbol{\Sigma}^1_2(\mathbb{M}) \iff \boldsymbol{\Delta}^1_2(\mathbb{M}) \iff \forall r\ \exists x\ (x \text{ is unbounded over } L[r])$.
3. $\boldsymbol{\Sigma}^1_2(\mathbb{S}) \iff \boldsymbol{\Delta}^1_2(\mathbb{S}) \iff \forall r\ \exists x\ (x \notin L[r])$.

These three theorems make it possible to compare the strength of various hypotheses of the form $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ and $\boldsymbol{\Delta}^1_2(\mathbb{P})$ with one another. Notice that the right-hand-side statements of Theorem 2.3 are naturally related to the cardinal numbers $\mathrm{add}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M})$; the right-hand-side statement of Theorem 2.4 are related to $\mathrm{cov}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{M})$; and those of Theorem 2.5 to $\mathfrak{b}$, $\mathfrak{d}$ and $2^{\aleph_0}$. So it is not surprising that the relationship between the regularity hypotheses follows a pattern familiar from (part of) the Cichoń diagram—see Fig. 1.

The interpretation of this diagram is as usual: every implication appearing on it is provable in ZFC, as well as the additional implication $\boldsymbol{\Delta}^1_2(\mathbb{L}) + \boldsymbol{\Delta}^1_2(\mathbb{C}) \Rightarrow \boldsymbol{\Sigma}^1_2(\mathbb{C})$ (the counterpart to the cardinal equation $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ established

Table 1 Correspondence between regularity, transcendence and cardinal characteristic

Reg. hypothesis	Transcendence over $L[r]$	Cardinal char.
$\forall r(\omega_1^{L[r]} < \omega_1)$	"Making ground model reals countable"	$\aleph_1$
$\boldsymbol{\Sigma}^1_2(\mathbb{B})$	Measure-one many random reals	$\mathrm{add}(\mathcal{N})$
$\boldsymbol{\Delta}^1_2(\mathbb{B})$	Random reals	$\mathrm{cov}(\mathcal{N})$
$\boldsymbol{\Sigma}^1_2(\mathbb{C})$	Co-meager many Cohen reals	$\mathrm{add}(\mathcal{M})$
$\boldsymbol{\Delta}^1_2(\mathbb{C})$	Cohen reals	$\mathrm{cov}(\mathcal{M})$
$\boldsymbol{\Delta}^1_2(\mathbb{L})$ / $\boldsymbol{\Sigma}^1_2(\mathbb{L})$	Dominating reals	$\mathfrak{b}$
$\boldsymbol{\Delta}^1_2(\mathbb{M})$ / $\boldsymbol{\Sigma}^1_2(\mathbb{M})$	Unbounded reals	$\mathfrak{d}$
$\boldsymbol{\Delta}^1_2(\mathbb{S})$ / $\boldsymbol{\Sigma}^1_2(\mathbb{S})$	New reals	$2^{\aleph_0}$

by John Truss [45]). Any other implication is *not* provable, i.e., any constellation of true/false-assignments to the above statements not contradicting the diagram, is actually consistent with ZFC. We call such an implication diagram "complete". The above facts are well-known, and can be proved by iterating the right type of forcing notions over L and using the fact that certain types of reals are, or are not, added by the iteration, thus forcing the right-hand-side statements of Theorems 2.3, 2.4 and 2.5 to be true or false. This is in perfect analogy to the proofs of the corresponding cardinal inequalities, which can be found e.g., in [3, Chapter 7].

Notice that the statement $\forall r(\omega_1^{L[r]} < \omega_1)$ is a little bit special, since it is the only one that requires the strength of an inaccessible; nevertheless, it is a natural property in this setting because:

(a) it plays the same role as $\aleph_1$ does in the standard Cichoń diagram for cardinal characteristics, and
(b) it is equivalent to a number of projective regularity statements, most notably "all $\boldsymbol{\Sigma}^1_2/\boldsymbol{\Pi}^1_1$ sets have the perfect set property".

The correspondence between regularity hypotheses on the second level, transcendence over L and cardinal characteristics of the continuum is summarized in Table 1 below.

2.3 Beyond the second level

When looking higher up in the projective hierarchy and attempting to generalize the theory to statements like $\boldsymbol{\Sigma}^1_n(\mathbb{P})$ and $\boldsymbol{\Delta}^1_n(\mathbb{Q})$, for $n \geq 3$, we are faced with two distinct methods of approach, as mentioned in the introduction. For example, if $L^\#$ denotes the least inner model closed under sharps for sets of ordinals, Theorems 2.3, 2.4 and 2.5 can be lifted to the next level, so in set-generic extensions of $L^\#$ we obtain characterizations of $\boldsymbol{\Sigma}^1_3(\mathbb{P})$ and $\boldsymbol{\Delta}^1_3(\mathbb{P})$ in terms of transcendence properties over $L^\#$. An immediate consequence is that all the properties of the diagram from Fig. 1 lift to the third projective level as well. For more on this approach, see the work of Ikegami [24, Section 5]. Judah and Spinas [31] also proved results such as: if V is a canonical model with n Woodin cardinals and a measurable above them, then there is a forcing extension in which $\boldsymbol{\Delta}^1_{n+4}(\mathbb{B})$ holds but $\boldsymbol{\Delta}^1_{n+4}(\mathbb{C})$ fails.

In this paper, we do not adopt the "large cardinal approach", for the following reasons:

1. As the consistency of "for all $\mathbb{P}$ and $n < \omega$, $\boldsymbol{\Sigma}^1_n(\mathbb{P})$ holds" is just an inaccessible (it is true in the Solovay model), it seems unnatural to require stronger hypotheses to prove more subtle statements about $\boldsymbol{\Sigma}^1_n(\mathbb{P})$ or $\boldsymbol{\Delta}^1_n(\mathbb{P})$ for low values of n (this view has been expressed by Bagaria, Judah, Shelah and others in the past).
2. Assuming *too* strong large cardinals (for example, enough to yield Projective Determinacy) may trivialize the question. So, for this approach to work properly one must assume *exactly the right* amount of large cardinal strength, which is, arguably, a somewhat artificial requirement.
3. Without large cardinal assumptions, one can obtain results that are not direct analogues of the second level results. In recent work of Friedman and Schrittesser [16], a model for $\mathbf{Proj}(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$ was constructedt ("**Proj**" stands for the class of all projective sets). In particular, this showed that the counterpart to the classical Bartoszyński–Raisonnier–Stern implication "$\boldsymbol{\Sigma}^1_n(\mathbb{B}) \Rightarrow \boldsymbol{\Sigma}^1_n(\mathbb{C})$" fails to lift to higher levels, for all $n \geq 3$ (on the other hand, the existence of a measurable implies $\boldsymbol{\Sigma}^1_3(\mathbb{B}) \Rightarrow \boldsymbol{\Sigma}^1_3(\mathbb{C})$). Other "non-liftings" of implications will follow from our results as well, for example that $\boldsymbol{\Delta}^1_n(\mathbb{L}) + \boldsymbol{\Delta}^1_n(\mathbb{C}) \Rightarrow \boldsymbol{\Sigma}^1_n(\mathbb{C})$ (the analogue of the Truss-implication) consistently fails for $n = 3$ and $n = 4$, see Theorem 7.12. In light of this, it seems more interesting to study such questions in ZFC or at most ZFC with an inaccessible.

So, if we must forgo large cardinal assumptions beyond an inaccessible, we must also forgo beautiful characterization theorems like Theorem 2.3, 2.4 and 2.5. But then, is there anything at all we can say about the relationship between the five regularity properties? Fortunately, a number of simple implications can be proved by straightforward ZFC-arguments. First, an important observation:

Observation 2.6 *(Brendle–Löwe)* Let $\mathbb{P} \in \{\mathbb{B}, \mathbb{L}, \mathbb{M}, \mathbb{S}\}$. For any tree $T \in \mathbb{P}$, there exists a natural homeomorphism φ_T between $[T]$ and the entire space (ω^ω or 2^ω), which preserves the property of "being a $\mathbb{P}$-condition". From this it follows that if $\boldsymbol{\Gamma}$ is a class of sets closed under continuous preimages, and we are only interested in the statement $\boldsymbol{\Gamma}(\mathbb{P})$, then we may safely drop the "below any $\mathbb{P}$-condition"-clause from the definition of $\mathbb{P}$-measurability, and simply say that a set A is $\mathbb{P}$-measurable if and only if there exists a $T \in \mathbb{P}$ such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$. Similarly, A is $\mathbb{C}$-measurable if and only if there is a G_δ non-meager set X such that $X \subseteq A$ or $X \cap A = \varnothing$.

Lemma 2.7 (Brendle–Löwe) *Let $\boldsymbol{\Gamma}$ be a class of sets closed under continuous preimages. Then the following implications hold in ZFC:*

1. $\boldsymbol{\Gamma}(\mathbb{L}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{M}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{S})$.
2. $\boldsymbol{\Gamma}(\mathbb{C}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{M})$.
3. $\boldsymbol{\Gamma}(\mathbb{B}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{S})$.

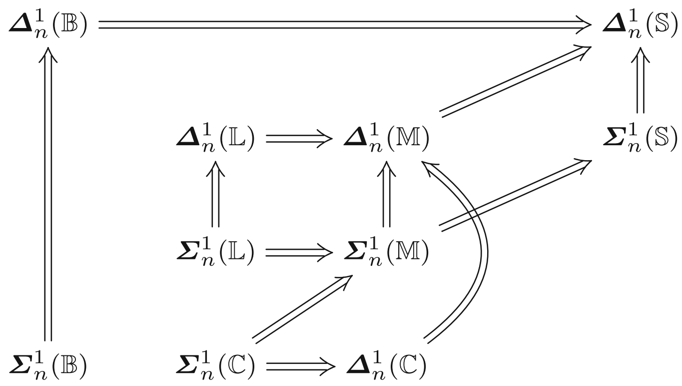


Fig. 2 An incomplete diagram of implications for $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Delta}^1_n$ sets of reals

Proof In view of the previous observation, proving $\boldsymbol{\Gamma}(\mathbb{P}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{Q})$ amounts to finding a $\mathbb{Q}$-object below any $\mathbb{P}$-object. For the first implication, note that a Laver tree is a Miller tree, a Miller tree is a perfect tree in ω^ω, and the imagine of this perfect tree under the natural homeomorphism between ω^ω and a dense G_δ subset of 2^ω, is an uncountable G_δ subset of 2^ω which, by the perfect set theorem, contains the branches of a Sacks tree.

For the second implication, note that a G_δ non-meager set is comeager in a basic open set. It is not hard to inductively construct a Miller tree whose branches are completely contained inside a set that is comeager in a basic open set.

Finally, every closed set of positive measure clearly contains a perfect subset. □

Summarizing the above, we obtain a different implication diagrams for the same regularity properties on the $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Delta}^1_n$ level, for $n \geq 3$, see Fig. 2. Note, however, that unlike Fig. 1, this is not a "complete" diagram, in the sense that it only shows the implications we know to exist so far, but it does not claim that no additional implications exist. Also, notice that the analogue of $\forall r(\omega_1^{L[r]} < \omega_1)$ is missing from the diagram—it is not clear which hypothesis should take its place.

The long-term goal is to "complete" this diagram on the third level, and potentially on all levels $n \geq 3$ (i.e., to find all possible implications and prove that all other implications are consistently false). There are still many obstacles to this goal. However, if we restrict attention exclusively to the $\boldsymbol{\Delta}^1_3$ sets, we obtain a much simpler diagram (see Fig. 3). In Sect. 5 we show that diagram is indeed complete, by constructing models for every combination of "true"/"false"-assignments consistent with the diagram, in ZFC or ZFC with an inaccessible. Partial results related to levels above $\boldsymbol{\Delta}^1_3$ will be discussed in Sect. 7.

We should mention that results concerning the Baire property and Lebesgue measurability were known prior to our work. The consistency of $\boldsymbol{\Delta}^1_3(\mathbb{C}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B})$, for example, follows from [27], and the consistency of the converse, $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$, was first proved by Bagaria in [28] and later (using different methods) by Bagaria and Woodin in [2]. The consistency of $\boldsymbol{\Sigma}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$ had remained open for a long time, until it became a corollary of the much stronger theorem of [16]. To our knowledge, no study of the properties $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{S}$ on higher levels has been carried out so far.

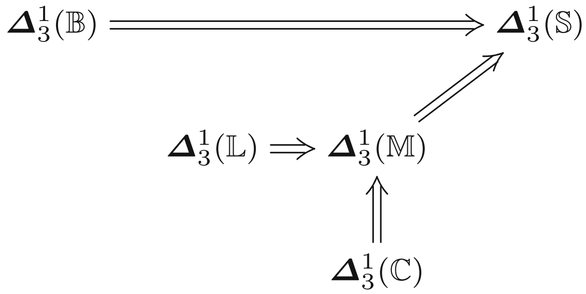


Fig. 3 Diagram of implications for $\mathbf{\Delta}^1_3$ sets of reals

On the technical side, our proofs will involve Suslin and Suslin$^+$ proper forcings, a special case of the general theory of "non-elementary proper forcing" developed by Shelah, cf. [41].

3 Suslin and Suslin$^+$ proper forcing

The theory of Suslin ccc forcings is well-understood, and a detailed summary can be found in [3, Section 3.6]. In the context of forcing notions that are proper but not ccc, there is a closely related concept, developed, among others, by Ihoda and Shelah in [22], Goldstern in [18,19], Shelah in [41] and Kellner in [33,34]. In this section we will give a brief overview of some essential properties of Suslin and Suslin$^+$ proper forcing, and prove some results that will be crucial for the techniques in our paper.

3.1 Basic concepts

The main idea is to replace countable elementary submodels $M \prec \mathcal{H}_\kappa$ for sufficiently large κ in the definition of "proper forcing" by countable transitive (not necessarily collapses of elementary) models of (a sufficient fragment of) ZFC. For that to make sense, the forcing notions need to be definable.

Definition 3.1 Let $\mathbb{P}$ be a forcing partial order whose conditions are (or can be coded by) reals. Assume that $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are definable by projective formulas with a parameter $a \in \omega^\omega$. Let ZFC* denote some (unspecified) sufficiently large finite fragment of ZFC, and let M be a countable transitive model of ZFC* containing the parameter a. Then $\mathbb{P}^M$, $<^M_{\mathbb{P}}$ and $\perp^M_{\mathbb{P}}$ refer to the forcing notion re-interpreted in M. A condition $q \in \mathbb{P}$ is called $(M, \mathbb{P})$-generic if (in V) $q \Vdash$ "$\dot{G} \cap \mathbb{P}^M$ is a $\mathbb{P}^M$-generic filter over M".

Following the terminology introduced by Shelah, countable models of ZFC* which contain the defining parameters will be called "candidates".

Definition 3.2 Let $(\mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}})$ be as above. We say that $\mathbb{P}$ is *proper-for-candidates* if for all candidates M containing the defining parameter of $\mathbb{P}$, and every $p \in \mathbb{P}^M$, there exists a $q \leq p$ which is $(M, \mathbb{P})$-generic.

Some authors call this property "strongly proper", although we will stick to the above terminology in order to avoid confusion with other interpretations of the term "strongly proper".

Note that if $M \prec \mathcal{H}_\kappa$ is a countable elementary submodel of a sufficiently large $\mathcal{H}_\kappa$ such that $\mathcal{H}_\kappa \models \text{ZFC}^*$ and contains all relevant parameters, then a condition q is $(\overline{M}, \mathbb{P})$-generic in the above sense if and only if it is $(M, \mathbb{P})$-generic in the usual sense (with $\mathbb{P}^{\overline{M}} = \mathbb{P} \cap M$). Hence, properness-for-candidates implies ordinary properness.

Usually, properness-for-candidates is coupled with an absoluteness requirement on the definition of the partial order.

Definition 3.3 A forcing $\mathbb{P}$ is *Suslin proper* if $\mathbb{P}$, $\leq_\mathbb{P}$ and $\perp_\mathbb{P}$ are $\boldsymbol{\Sigma}^1_1$-relations, and $\mathbb{P}$ is proper-for-candidates.

If $\mathbb{P}$ is Suslin proper, then $\mathbb{P}^M = \mathbb{P} \cap M$, $\leq^M_\mathbb{P} = \leq_\mathbb{P} \cap M^2$ and $\perp^M_\mathbb{P} = \perp_\mathbb{P} \cap M^2$ by $\boldsymbol{\Sigma}^1_1$-absoluteness. Moreover, the statement "$\{p_i \mid i < \omega\}$ is predense below q" is $\boldsymbol{\Pi}^1_1$ and hence absolute between candidates M and V. Clearly, all Suslin ccc partial orders (i.e., all Suslin partial orders having the ccc) are Suslin proper, and there are some well-known examples of non-ccc forcings that are Suslin proper—most notably Mathias forcing. However, many standard forcing notions (e.g., Sacks, Miller and Laver forcing) are not quite Suslin proper, because $\perp_\mathbb{P}$ fails to be a $\boldsymbol{\Sigma}^1_1$ relation (it is then only $\boldsymbol{\Pi}^1_1$). To fix this problem, an alternative notion was proposed by Shelah and Goldstern:

Definition 3.4 A forcing $\mathbb{P}$ is *Suslin$^+$ proper* if

1. $\mathbb{P}$ and $\leq_\mathbb{P}$ are $\boldsymbol{\Sigma}^1_1$,
2. there is a $\boldsymbol{\Sigma}^1_2$, $(\omega+1)$-place relation $\text{epd}(p_0, p_1, \ldots, q)$ ("effectively predense") such that if $\text{epd}(p_0, p_1, \ldots, q)$ holds for $p_i, q \in \mathbb{P}$, then $\{p_i \mid i < \omega\}$ is predense below q, and
3. for every candidate M containing all relevant parameters, and all $p \in \mathbb{P}^M$, there is a $q \leq p$ such that for every $D \in M$ which is $\mathbb{P}^M$-dense, there exists an enumeration $\{d_i \mid i < \omega\} \subseteq D$ such that $\text{epd}(d_0, d_1, \ldots, q)$ holds. In this case we say that q is an *effective* $(M, \mathbb{P})$-generic condition, and we call this property *effective*-properness-for-candidates.

So Suslin properness implies Suslin$^+$ properness, which in turn implies properness. A sufficient condition for a forcing to be Suslin$^+$ proper is an effective version of Axiom A, where the amalgamation makes sure that epd is defined in a $\boldsymbol{\Sigma}^1_2$-way. All standard definable tree-like forcings which are known to be proper are in fact Suslin$^+$ proper. A good exposition of this phenomenon can be found in Kellner's papers [33,34].

Remark 3.5 In [18, Remark 1.7] it was shown that if $d \in \omega^\omega$ is a code for an analytic set, canonically coding $(\mathbb{P}, \leq_\mathbb{P}, \perp_\mathbb{P})$), then the statement "$d$ codes a Suslin proper forcing" is a $\boldsymbol{\Pi}^1_3$ statement. The same holds for Suslin$^+$ proper forcing, i.e., if d is a code for a $\boldsymbol{\Sigma}^1_2$-set canonically coding $(\mathbb{P}, \leq_\mathbb{P})$ as well as the relation epd, then "d codes a Suslin$^+$ proper forcing" iff

$$\begin{gathered}\leq_\mathbb{P} \text{ is a partial order, and}\\ \forall^1 M\, [M \text{ countable, transitive, } M \models \text{ZFC}^*,\, d \in M \to\\ \forall^0 p \in \mathbb{P}^M \;\; \exists^1 q \leq_\mathbb{P} p \;\; \text{s.t.} \;\; \forall^0 D \in M (M \models \text{“}D \text{ is dense”} \to\\ \exists^1 \{d_i \mid i < \omega\} \subseteq D \text{ s.t. } \text{epd}(d_0, d_1, \ldots, q))]\end{gathered}$$

(where $\forall^0$ and $\exists^0$ refers to natural number quantifiers and $\forall^1$ and $\exists^1$ to real number quantifiers.) As countable, transitive models M can be coded by well-founded relations E on ω, it is not hard to verify that the above statement is $\boldsymbol{\Pi}^1_3$. In particular, if $\mathbb{P}$ is a Suslin (Suslin$^+$) proper forcing then $N \models$ "$\mathbb{P}$ is a Suslin (Suslin$^+$) proper forcing" for any inner model N with $\omega_1 \subseteq N$, by downwards $\boldsymbol{\Pi}^1_3$-absoluteness.

Next, we want to look at the complexity of the forcing relation $\Vdash_{\mathbb{P}}$. First, let us fix the following terminology:

Definition 3.6 Let $\mathbb{P}$ be a forcing notion. We say that τ is a *countable* $\mathbb{P}$*-name for a real* if it is a countable set of pairs $(\check{n}, p)$, where $n \in \omega$ and $p \in \mathbb{P}$.

In the above definition we think of reals as subsets of ω (or members of 2^ω), and if τ is of the above form and G a generic filter, then we think of $\tau[G]$ as the set $\{n \mid \exists p \in G\ ((\check{n}, p) \in \tau)\} \subseteq \omega$ (or the corresponding function in 2^ω).

Although not every name for a real is countable, if $\mathbb{P}$ is proper then for every $\mathbb{P}$-name for a real σ and $p \in \mathbb{P}$ there exists $q \leq p$ and a countable $\mathbb{P}$-name τ for a real such that $q \Vdash \tau = \sigma$. If conditions of $\mathbb{P}$ are reals, each such countable name can be canonically coded by a real. Moreover, if $\mathbb{P}$ is $\boldsymbol{\Sigma}^1_1$ then the statement "x codes a countable name for a real" is $\boldsymbol{\Sigma}^1_1$. We will frequently identify countable $\mathbb{P}$-names for reals with the reals coding them.

The following lemma generalizes [29, Theorem 2.1], and is crucial for computing the complexity of the forcing relation. In its formulation, "$p \Vdash \theta(\tau)$" is to be understood as a formula with real variables p and τ (actually the reals coding them)

Lemma 3.7 *Let $\mathbb{P}$ be Suslin$^+$ proper, $p \in \mathbb{P}$ and τ a countable $\mathbb{P}$-name for a real. Then for all $n \geq 2$:*

1. *If θ is $\boldsymbol{\Pi}^1_n$ then "$p \Vdash \theta(\tau)$" is $\boldsymbol{\Pi}^1_n$.*
2. *If θ is $\boldsymbol{\Sigma}^1_n$ then "$p \Vdash \theta(\tau)$" is $\boldsymbol{\Pi}^1_{n+1}$.*

Proof The proof is by induction on the complexity of θ, with $\boldsymbol{\Pi}^1_2$ being the base case. So first, assume θ is $\boldsymbol{\Pi}^1_2$.

Claim. The following are equivalent:

1. $p \Vdash \theta(\tau)$,
2. for all candidates M containing τ, p, and any parameters appearing in the definition of $\mathbb{P}$ or θ, we have $M \models p \Vdash \theta(\tau)$.

As candidates are coded by well-founded relations E on ω, the above equivalence gives us a $\boldsymbol{\Pi}^1_2$-definition of "$p \Vdash \theta(\tau)$".

Proof of Claim For (2) $\Rightarrow$ (1), fix p and let M be the transitive collapse of an *elementary* submodel of a sufficiently large $\mathcal{H}_\kappa$, containing all necessary parameters. Then by assumption $M \models p \Vdash \theta(\tau)$, but by elementarity and definability of $\Vdash$ in $\mathcal{H}_\kappa$, this implies $p \Vdash \theta(\tau)$ (note that this direction is trivial and does not require θ to have any particular complexity).

For (1) $\Rightarrow$ (2), assume that $p \Vdash \theta(\tau)$ and, towards contradiction, let M be such that $M \models p \not\Vdash \theta(\tau)$ (note that by absoluteness, $M \models p \in \mathbb{P}$ and $M \models$ "τ is a countable name for a real"). Then there is $p' \leq p$ in M such that $M \models p' \Vdash \neg\theta(\tau)$. Let $q \leq p'$ be an $(M, \mathbb{P})$-generic condition, and let G be $\mathbb{P}$-generic over V with $q \in G$. Then G is also M-generic, and $p' \in G$, hence $M[G] \models \neg\theta(\tau[G])$. But this is a $\boldsymbol{\Sigma}^1_2$ formula, so by upwards absoluteness $V[G] \models \neg\theta(\tau[G])$. This contradicts the assumption that $p \Vdash \theta(\tau)$. □(Claim)

The rest follows by induction.

- For $n \geq 2$, assume inductively that for $\boldsymbol{\Pi}^1_n$ formulas χ, the relation "$p \Vdash \chi(\tau)$" is $\boldsymbol{\Pi}^1_n$. Let θ be $\boldsymbol{\Sigma}^1_n$. Then $p \Vdash \theta(\tau)$ iff $\forall q\ (q \in \mathbb{P} \wedge q \leq p \ \rightarrow\ q \nVdash \neg\theta(\tau))$, which is easily seen to be $\boldsymbol{\Pi}^1_{n+1}$.
- For $n \geq 2$, assume inductively that for $\boldsymbol{\Sigma}^1_n$ formulas χ, the relation "$p \Vdash \chi(\tau)$" is $\boldsymbol{\Pi}^1_{n+1}$. Let θ be $\boldsymbol{\Pi}^1_{n+1}$, and write $\theta(\tau)$ as $\forall y\ \chi(\tau, y)$ for a $\boldsymbol{\Sigma}^1_n$ formula χ. Then the following are equivalent:
  (1) $p \Vdash \theta(\tau)$, and
  (2) $\forall q \forall \sigma\ ((q \in \mathbb{P}$ and $q \leq p$ and "σ is a countable name for a real") $\rightarrow\ q \Vdash \chi(\tau, \sigma))$.
  (1) $\rightarrow$ (2) is obvious, and for (2) $\rightarrow$ (1), note that if $p \nVdash \theta(\tau)$ then $\exists q \leq p$ such that $q \Vdash \neg\theta(\tau)$, so $q \Vdash \exists y \neg\chi(\tau, y)$. But then there is a countable name σ and $q' \leq q$ such that $q' \Vdash \neg\chi(\tau, \sigma)$, which contradicts (2).
  As "$q \Vdash \chi(\tau, \sigma)$" is $\boldsymbol{\Pi}^1_{n+1}$ by induction, the statement in (2) is also $\boldsymbol{\Pi}^1_{n+1}$. □

3.2 Iterations

Next, we consider iterations of Suslin and Suslin$^+$ proper forcing notions. This is somewhat tricky, since, in general, even a two-step iteration of Suslin$^+$ forcing notions is not Suslin$^+$ (see [34, Remark 4.12]), so the definition of the iteration cannot be absolute between countable models M and V. However, following [22] and [18], adequate preservation results can still be proved, and that is sufficient for our purposes. In this paper we will only consider iterations of length at most ω_1 with countable support. Most of our technical results just involve proper initial segments of the ω_1-iteration, which simplifies many things.

Definition 3.8 Let $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ be a countable (i.e., full) support iteration of length $\gamma < \omega_1$. We call this a *Suslin* (*Suslin*$^+$) *proper iteration* of length γ if each iterand is Suslin (Suslin$^+$) proper, i.e., for every $\alpha < \gamma$, $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha$ is Suslin (Suslin$^+$) proper".

Since the iteration $\mathbb{P}_\gamma$ is uniquely determined by the sequence $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ of iterands, any candidate M containing the names for the defining parameters of all the $\dot{\mathbb{Q}}_\alpha$'s can uniquely reconstruct the iteration (see e.g., [18, p. 350ff] for details). We will refer to this M-reconstruction of the iteration as $\mathbb{P}^M_\gamma$. In general, $\mathbb{P}^M_\gamma$ is not the same as $\mathbb{P}_\gamma \cap M$. Later we will prove that being a $\mathbb{P}_\gamma$-condition is $\boldsymbol{\Pi}^1_2$, so by downward absoluteness $\mathbb{P}^M_\gamma \supseteq \mathbb{P}_\gamma \cap M$ does hold. However, $\mathbb{P}^M_\gamma$ might contain objects which M believes to be $\mathbb{P}_\gamma$-conditions but which actually (in V) are not.

Definition 3.9 *(Judah–Shelah; Goldstern; Kellner)* If G is a $\mathbb{P}_\gamma$-generic filter over V, and M is a candidate, we can define $G^M = G^M_\gamma$, the "potential $\mathbb{P}^M_\gamma$-generic filter over M induced by G", by induction on $\alpha \leq \gamma$, following [18, Definition 2.6] (see also [34, Definition 4.3.]).

- If $\alpha = \beta + 1$ then $G^M_\alpha = \{p \in \mathbb{P}^M_\alpha \mid p{\restriction}\beta \in G^M_\beta \text{ and } p(\beta)[G^M_\beta] \in G(\beta)\}$.
- If α is limit then $G^M_\alpha = \{p \in \mathbb{P}^M_\alpha \mid \forall\beta < \alpha\ (p \in G^M_\beta)\}$.

(here $G(\beta)$ is the β-th component of G). Then $G^M := G^M_\gamma$ is the result of this induction.

Remark 3.10 1. The object G^M is not always well-defined: for example, if at some stage $\alpha < \gamma$, G^M_α is not $\mathbb{P}^M_\alpha$-generic over M then it does not make sense to evaluate $p(\alpha)[G^M_\alpha]$, so we cannot define $G^M_{\alpha+1}$ either. Therefore, we allow the possibility that G^M is undefined; but when we say "G^M is $\mathbb{P}^M_\gamma$-generic over M", we mean that, inductively, every G^M_α is $\mathbb{P}^M_\alpha$-generic over M for $\alpha < \gamma$, and hence every G^M_α is properly defined (and G^M is $\mathbb{P}^M_\gamma$-generic over M).

2. If G^M is well-defined then, as a filter on $\mathbb{P}^M_\gamma$, it takes the role that "$G \cap M$" would in the usual situation (i.e., where M is a collapse of an elementary submodel). In general, G^M and $G \cap M$ are different. However, their difference arises only from the difference between $\mathbb{P}^M_\gamma$ and $\mathbb{P}_\gamma \cap M$. In particular, if $p \in G^M$ and p is really a $\mathbb{P}_\gamma$-condition, then in fact $p \in G$. This follows inductively from the definition of G^M (if all initial segments of p are real $\mathbb{P}_\alpha$-conditions, then, inductively, it follows that the definition of G^M_α corresponds to the standard definition of the iterated generic filter G_α). We will need this fact several times in our arguments.

See [34] and [41] for a more detailed treatment of these issues.

Definition 3.11 Following [18, Remark 2.13] and [34, Definition 4.4], we define:

- A condition $q \in \mathbb{P}_\gamma$ is $(M, \mathbb{P}_\gamma)$-*generic* if $q \Vdash$ "$\dot{G}^M$ is a $\mathbb{P}^M_\gamma$-generic filter over M".
- If $p \in \mathbb{P}^M_\gamma$, then q is $(M, \mathbb{P}_\gamma, p)$-*generic* if it is $(M, \mathbb{P}_\gamma)$-*generic* and, additionally, $q \Vdash p \in \dot{G}^M$.

The purpose of the "$(M, \mathbb{P}_\gamma, p)$-generic condition" is that we would like to say "for $p \in \mathbb{P}^M_\gamma$, there is $q \leq p$ which is $(M, \mathbb{P}_\gamma)$-generic", but we cannot say this since p might not be in $\mathbb{P}_\gamma$. Instead, saying that "q is $(M, \mathbb{P}_\gamma, p)$-generic" is the desired analogue.

The following theorem, proved by Judah–Shelah and by Goldstern, shows that a property that is almost "properness-for-candidates" is preserved by countable support iterations of Suslin and Suslin$^+$ forcings.

Theorem 3.12 *(Judah–Shelah; Goldstern) Let $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ be a Suslin$^+$ proper iteration of length $\gamma < \omega_1$. Then for every candidate M containing the parameters of all $\dot{\mathbb{Q}}_\alpha$ and containing γ, and for every $p \in \mathbb{P}^M_\gamma$, there exists a q which is $(M, \mathbb{P}_\gamma, p)$-generic.*

Proof This is a specific instance of a more general preservation theorem, where the iteration can have length $\gamma \leq \omega_2$. In that case, we must first make sense of the way a countable model M reflects the iteration. This general result is proved in detail for Suslin proper forcings in [22, Lemma 2.8] and in [18, Theorem 2.16, Corollary 2.17], and in [19] it is also mentioned that analogous results hold for Suslin$^+$. Even stronger results are proved by Shelah in [41], and also by Kellner in [34, Lemma 4.8]. □

Remark 3.13 As we are only dealing with countable iterations, the following holds for $\mathbb{P}_\alpha$ by induction on $\alpha < \omega_1$:

1. Since by Theorem 3.12, each $\mathbb{P}_\alpha$ is proper, $\mathbb{P}_\alpha$-names for reals have countable names (modulo strengthening of the condition).
2. It follows that, inductively, we can assume that all components of $p \in \mathbb{P}_\alpha$ are represented by countable names for reals.
3. As countable names are coded by reals and α is countable, an entire condition $p \in \mathbb{P}_\alpha$ can be coded by a single real. As before, we will identify $\mathbb{P}_\alpha$-conditions and countable $\mathbb{P}_\alpha$-names for reals with the reals coding them.

Now that we can treat $\mathbb{P}_\alpha$ as a forcing with real number conditions, we can also analyze the complexity of $\mathbb{P}_\alpha$, $\leq_\alpha$ and the forcing relation $\Vdash_\alpha$. We already mentioned that $\mathbb{P}_\alpha$ is not Suslin or Suslin$^+$, i.e., neither $\mathbb{P}_\alpha$ nor $\leq_\alpha$ are $\boldsymbol{\Sigma}^1_1$. However, we can prove the following result, inductively on $\alpha < \omega_1$.

Lemma 3.14 *Let $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ be a Suslin$^+$ proper iteration. Then the following holds for all $\alpha < \omega_1$:*

1. *$\mathbb{P}_\alpha$ is $\boldsymbol{\Pi}^1_2$,*
2. *$\leq_\alpha$ is $\boldsymbol{\Pi}^1_2$,*
3. *being a countable $\mathbb{P}_\alpha$-name for a real is $\boldsymbol{\Pi}^1_2$, and*
4. *for any $p \in \mathbb{P}_\alpha$ and a countable $\mathbb{P}_\alpha$-name for a real τ, we have for all $n \geq 2$:*
   (a) *If θ is $\boldsymbol{\Pi}^1_n$ then "$p \Vdash_\alpha \theta(\tau)$" is $\boldsymbol{\Pi}^1_n$.*
   (b) *If θ is $\boldsymbol{\Sigma}^1_n$ then "$p \Vdash_\alpha \theta(\tau)$" is $\boldsymbol{\Pi}^1_{n+1}$.*

Proof The case $\alpha = 1$ follows from the definition of Suslin$^+$ properness and Lemma 3.7. Assume (1)–(4) holds for $\beta < \alpha$. Then:

1. If $\alpha = \beta + 1$ then $p \in \mathbb{P}_{\beta+1}$ iff $p{\restriction}\beta \in \mathbb{P}_\beta$ and "$p(\beta)$ is a countable $\mathbb{P}_\beta$-name for a real" and $(p{\restriction}\beta) \Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$. By induction, this is a conjunction of three $\boldsymbol{\Pi}^1_2$ sentences, where the last one is so due to point (4) and the fact that "$p(\beta) \in \dot{\mathbb{Q}}_\beta$" is $\boldsymbol{\Sigma}^1_1$. If α is limit then (since we are dealing with countable support iterations and $\alpha < \omega_1$) $p \in \mathbb{P}_\alpha$ iff $\forall \beta < \alpha\ (p{\restriction}\beta \in \mathbb{P}_\beta)$. Again, this statement is $\boldsymbol{\Pi}^1_2$ by the induction hypothesis.
2. If $\alpha = \beta + 1$ then $p \leq_{\beta+1} q$ iff $(p{\restriction}\beta) \leq_\beta (q{\restriction}\beta)$ and $(p{\restriction}\beta) \Vdash_\beta p(\beta) \leq_{\dot{\mathbb{Q}}_\beta} q(\beta)$, which is again a conjunction of $\boldsymbol{\Pi}^1_2$ formulas, by induction. If α is limit then $p \leq_\alpha q$ iff $\forall \beta < \alpha\ (p{\restriction}\beta) \leq_\beta (q{\restriction}\beta)$ which is likewise $\boldsymbol{\Pi}^1_2$.
3. The complexity of the set of countable $\mathbb{P}_\alpha$-names is the same as the complexity of $\mathbb{P}_\alpha$, so this follows from point (1).
4. Similarly to the proof of Lemma 3.7, we prove this by induction on the complexity of θ, starting with $\boldsymbol{\Pi}^1_2$. As before:

Claim. The following are equivalent:
(a) $p \Vdash_\alpha \theta(\tau)$,
(b) for all candidates M containing τ, p and α, and any parameters appearing in the definition of any $\dot{\mathbb{Q}}_\beta$ for $\beta < \alpha$ or in θ, we have $M \models p \Vdash_\alpha \theta(\tau)$.

The proof of this equivalence is as in Lemma 3.7, using the "almost-properness-for-candidates"-property satisfied by $\mathbb{P}_\alpha$ (i.e., Theorem 3.12). However, since $\mathbb{P}_\alpha$ is not absolute between V and M, the argument must proceed with some more care. Notice that by downward $\boldsymbol{\Pi}^1_2$-absoluteness, we now already know that $\mathbb{P}_\alpha \cap M \subseteq \mathbb{P}^M_\alpha$. The $(b) \Rightarrow (a)$ direction is exactly as before, i.e., we simply take M to be the collapse of an elementary submodel of $\mathcal{H}_\kappa$. For $(a) \Rightarrow (b)$, assume $p \Vdash_\alpha \theta(\tau)$ and let M be a candidate containing the relevant parameters, and, towards contradiction, suppose $M \models p \not\Vdash_\alpha \theta(\tau)$. By downward $\boldsymbol{\Pi}^1_2$-absoluteness we know that $p \in \mathbb{P}^M_\alpha$, and also M knows that τ is a countable name for a real.

Then $M \models \exists p' \leq_\alpha p\ (p' \Vdash_\alpha \neg\theta(\tau))$ (note that p' may not be in $\mathbb{P}_\alpha$). Now use Theorem 3.12, and find a condition $q \in \mathbb{P}_\alpha$ which is $(M, \mathbb{P}_\alpha, p)$-generic. Then, if G is $\mathbb{P}_\alpha$-generic over V, and $q \in G$, the derived object G^M (see Definition 3.9 (1)) is $\mathbb{P}^M_\alpha$-generic over M, and $p' \in G^M$. Therefore, $M[G^M] \models \neg\theta(\tau[G])$, and by upward $\boldsymbol{\Sigma}^1_2$-absoluteness, $\neg\theta(\tau[G])$ holds in $V[G]$. But also $M \models p' \leq_\alpha p$ and $G^M \subseteq \mathbb{P}^M_\alpha$ is a filter, so also $p \in G^M$. But p was in $\mathbb{P}_\alpha$, so by Remark 3.10 (2) p must in fact be in G. That contradicts $p \Vdash_\alpha \theta(\tau)$. □(Claim).

For the rest, proceed inductively as before: if θ is $\boldsymbol{\Sigma}^1_n$ for $n \geq 2$ then $p \Vdash_\alpha \theta(\tau)$ iff $\forall q\ (q \in \mathbb{P}_\alpha$ and $q \leq_\alpha p \ \rightarrow\ q \not\Vdash_\alpha \neg\theta(\tau))$, which is $\boldsymbol{\Pi}^1_{n+1}$, using the fact that $\mathbb{P}_\alpha$ and $\leq_\alpha$ are $\boldsymbol{\Pi}^1_2$, i.e., points (1) and (2) of the theorem. Likewise, if $\theta(\tau) \equiv \forall y \chi(\tau, y)$ is $\boldsymbol{\Pi}^1_{n+1}$ for $n \geq 2$ and χ is $\boldsymbol{\Sigma}^1_n$, then, as before, $p \Vdash_\alpha \theta(\tau)$ iff $\forall q \forall \sigma$ (($q \in \mathbb{P}_\alpha$ and $q \leq_\alpha p$ and "σ is a countable $\mathbb{P}_\alpha$-name for a real") $\rightarrow\ q \Vdash_\alpha \chi(\tau, \sigma)$). Again this is $\boldsymbol{\Pi}^1_{n+1}$ using the fact that being a countable $\mathbb{P}_\alpha$-name for a real is $\boldsymbol{\Pi}^1_2$, i.e., point (3) of the theorem. □

From this theorem it follows that $\mathbb{P}_\alpha$, $\leq_\alpha$, being a countable $\mathbb{P}_\alpha$-name for a real, and the relation "$p \Vdash_\alpha \theta(\tau)$" for $\boldsymbol{\Pi}^1_2$ formulas θ, are all downwards absolute between V and countable models M (containing the relevant parameters), and absolute in both directions between V and models $W \subseteq V$ with $\omega_1 \subseteq W$.

3.3 Suslin$^+$ proper iterations and inaccessibles

We end this section with two further useful results about Suslin$^+$ proper forcing, under the assumption that $\forall r\ (\omega_1^{L[r]} < \omega_1)$. The first result shows that this assumption is preserved by Suslin$^+$ iterations of countable length.

First, a preliminary Lemma.

Lemma 3.15 *Suppose $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$, $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ is a Suslin$^+$ proper iteration of length $\gamma < \omega_1$, and p is a $\mathbb{P}_\gamma$-condition. Suppose further that p and all the defining parameters of $\mathbb{P}_\gamma$ are coded by a real z. Then there exists $q \in \mathbb{P}_\gamma$ such that $q \Vdash_\gamma p \in \dot{G}$ and $q \Vdash_\gamma$ "$\dot{G}$ is $L[\check{z}]$-generic".*

Proof As ω_1^V is inaccessible in $L[z]$, we can find a candidate $M \subseteq L[z]$ containing all the reals and dense sets of $L[z]$, and moreover reflecting all the relevant properties of $L[z]$ (for example, let $M := L_\lambda[z]$ for some sufficiently large $\lambda < \omega_1^V$ such that $L_\lambda[z] \prec L_{\omega_1^V}[z]$). As p and the parameters of $\mathbb{P}_\gamma$ are now in M, by Theorem 3.12 we can find an $(M, \mathbb{P}_\gamma, p)$-generic condition q. Then $q \Vdash_\gamma$ "$\dot{G}^M$ is $\mathbb{P}^M_\gamma$-generic over M" and, since M has the same reals and dense sets as $L[z]$, also $q \Vdash_\gamma$ "$\dot{G}$ is $L[\check{z}]$-generic". On the other hand, $q \Vdash_\gamma p \in \dot{G}^M$, and since $p \in \mathbb{P}_\gamma$, this implies $q \Vdash_\gamma p \in \dot{G}$ by Remark 3.10 (2). □

Theorem 3.16 *Suppose* $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$ *and* $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ *is a Suslin*$^+$ *proper iteration of length* $\gamma < \omega_1$. *Then* $V^{\mathbb{P}_\gamma} \models \forall r\ (\omega_1^{L[r]} < \omega_1)$.

Proof Suppose, towards contradiction, that the conclusion is false, and let $\dot{r}$ be a countable $\mathbb{P}_\gamma$-name for a real and $p \in \mathbb{P}_\gamma$ such that $p \Vdash_\gamma \omega_1^{L[\dot{r}]} = \omega_1$. Let z be a real in V, coding $p, \dot{r}$ and all the defining parameters of $\mathbb{P}_\gamma$. By Lemma 3.15, there is a $q \in \mathbb{P}_\gamma$ such that $q \Vdash_\gamma p \in \dot{G}$ and $q \Vdash_\gamma$ "$\dot{G}$ is $L[\check{z}]$-generic".
By Remark 3.5, we know that every iterand occurring in $\mathbb{P}_\gamma$ is Suslin$^+$ proper in $L[z]$ as well, so $L[z] \models$ "$\mathbb{P}_\gamma$ is proper". Therefore $q \Vdash_\gamma$ "$L[\check{z}][\dot{G}]$ is a proper forcing extension of $L[\check{z}]$", so in particular

$$q \Vdash_\gamma \omega_1^{L[\check{z}][\dot{G}]} = \omega_1^{L[\check{z}]}.$$

Now notice that in any $\mathbb{P}_\gamma$-extension $V[G]$ of V, since $\dot{r}[G]$ is constructible from $\dot{r}$ and G, and $\dot{r}$ is coded in z, we know that $L[\dot{r}[G]] \subseteq L[z][G]$, which implies $\omega_1^{L[\dot{r}[G]]} \leq \omega_1^{L[z][G]}$. On the other hand, $\omega_1^{L[z]}$ was countable by assumption. It follows that

$$q \Vdash_\gamma \omega_1^{L[\dot{r}]} \leq \omega_1^{L[\check{z}][\dot{G}]} = \omega_1^{L[\check{z}]} < \omega_1$$

which, together with $q \Vdash_\gamma p \in \dot{G}$ and $p \Vdash_\gamma \omega_1^{L[\dot{r}]} = \omega_1$, leads to a contradiction. □

Remark 3.17 The definability of the forcing is essential in the preceding result, since, in general, the assumption $\forall r\ (\omega_1^{L[r]} < \omega_1)$ is not preserved even by ccc forcings. For example, assuming that ω_1 is not Mahlo in L, one can find $A \subseteq \omega_1$ is such that $L[A] \models \forall r\ (\omega_1^{L[r]} < \omega_1)$. Then, using the technique of *almost disjoint coding* (see e.g., [25]), one can construct a ccc forcing notion coding A by the generic real r_G. So $L[A][r_G] = L[r_G]$ will be a generic extension of $L[A]$ by a ccc forcing notion, while obviously satisfying $\omega_1^{L[r_G]} = \omega_1$.

Our second result (which uses the previous result) is a strong absoluteness property of extensions by Suslin$^+$ iterations.

Definition 3.18 Let $\mathbb{P}$ be a forcing notion. Then

1. V is $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-*absolute* iff for all $\boldsymbol{\Sigma}^1_n$ formulas ϕ, all $\mathbb{P}$-generic G over V, and all reals $x \in V$:

$$V \models \phi(x) \iff V[G] \models \phi(x).$$

2. V is $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-*correct* iff for all $\boldsymbol{\Sigma}^1_n$ formulas ϕ, all $\mathbb{P}$-generic G over V, and all reals $x \in V[G]$:

$$V[x] \models \phi(x) \iff V[G] \models \phi(x).$$

$\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-correctness implies $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-absoluteness, but not vice versa. In fact, $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-correctness is much stronger. The following is clear:

Fact 3.19 *If V is $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-correct, then $\boldsymbol{\Sigma}^1_3$-absoluteness holds between any two models W and W' with $V \subseteq W \subseteq W' \subseteq V[G]$.*

Proof Let ϕ be $\boldsymbol{\Sigma}^1_3$ and $x \in W$. If $W \models \phi(x)$ then $W' \models \phi(x)$ by upwards $\boldsymbol{\Sigma}^1_3$-absoluteness (i.e., Shoenfield absoluteness). Conversely, if $W' \models \phi(x)$ then by upwards $\boldsymbol{\Sigma}^1_3$-absoluteness $V[G] \models \phi(x)$, so by $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-correctness $V[x] \models \phi(x)$, so by upwards-absoluteness again $W \models \phi(x)$. □

$\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-correctness and $\boldsymbol{\Sigma}^1_n$-$\mathbb{P}$-absoluteness for all set-forcings $\mathbb{P}$ have been investigated before, by Woodin, Bagaria and Friedman among others. For instance, in [1] and [13] it is shown that $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-absoluteness for all set-forcings $\mathbb{P}$ can be obtained from a reflecting cardinal, whereas $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-correctness for all set-forcings $\mathbb{P}$ implies the existence of sharps for sets of ordinals by [46]).

If we restrict attention to Suslin$^+$ proper forcing notions, $\boldsymbol{\Sigma}^1_3$-correctness can be obtained just from an inaccessible. In fact, in [3, Lemma 9.5.4] it is proved that if $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}$ is Suslin ccc, then V is $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}$-correct. We now extend this result to all Suslin$^+$ proper forcings $\mathbb{P}$ (and their iterations of length ω_1), relying on Theorem 3.16.

Theorem 3.20 *Suppose $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is a Suslin$^+$ proper iteration. Then V is $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}_{\omega_1}$-correct.*

Proof Since upwards $\boldsymbol{\Sigma}^1_3$-absoluteness always holds, it remains to prove the converse. First we do it for countable iterations $\mathbb{P}_\gamma$, $\gamma < \omega_1$.
Suppose, towards contradiction, that downwards-$\boldsymbol{\Sigma}^1_3$-$\mathbb{P}_\gamma$-correctness fails. Then there is a $\boldsymbol{\Sigma}^1_3$ formula ϕ, a countable $\mathbb{P}_\gamma$-name for a real τ, and a condition $p \in \mathbb{P}_\gamma$ such that

$$p \Vdash_\gamma (\phi(\tau) \ \wedge\ V[\tau] \models \neg\phi(\tau)). \tag{*}$$

Our goal is to contradict $(*)$. Let θ be a $\boldsymbol{\Pi}^1_2$ formula and σ a (without loss of generality countable) $\mathbb{P}_\gamma$-name for a real, such that

$$p \Vdash_\gamma (\theta(\tau, \sigma) \ \wedge\ V[\tau] \models \neg\phi(\tau)).$$

Let z be a real coding τ, σ, p and all the defining parameters appearing in $\mathbb{P}_\gamma$ and in ϕ. By Lemma 3.15, there is a q forcing "$\dot{G}$ is $L[\check{z}]$-generic" and "$p \in \dot{G}$".
Let G_γ be any such generic filter with $q \in G_\gamma$ and let us work in $V[G_\gamma]$ for the time being.
Let $x := \tau[G_\gamma]$ and $y = \sigma[G_\gamma]$. Since $p \in G_\gamma$, by $(*)$ we know that $V[G_\gamma] \models \theta(x, y)$. By Shoenfield absoluteness, we also know that $L[z][x][y] \models \theta(x, y)$. As

$L[z][G_\gamma]$ is a generic extension of $L[z]$, we know that the intermediary models $L[z] \subseteq L[z][x] \subseteq L[z][x][y] \subseteq L[z][G_\gamma]$ can all be represented by generic extensions. Let $\mathbb{Q}$ be the forcing leading from $L[z][x]$ to $L[z][x][y]$ (to find $\mathbb{Q}$, first look at the quotient of $\mathbb{P}_\gamma$ modulo the sub-forcing generated by τ, and then take the sub-forcing of that generated by σ). It follows that

$$L[z][x] \models \exists q \in \mathbb{Q}\, (q \Vdash_{\mathbb{Q}} \theta(\check{x}, \sigma)).$$

But by Theorem 3.16, $V[G_\gamma] \models \forall r\, (\omega_1^{L[r]} < \omega_1)$. Therefore also $V[x] \models \forall r\, (\omega_1^{L[r]} < \omega_1)$. Therefore, in particular, $V[x] \models$ "ω_1 is inaccessible in $L[z][x]$". So, in $V[x]$, we can find an *internal* $\mathbb{Q}$-generic filter H over $L[z][x]$, so $V[x] \models (L[z][x][H] \models \theta(x, \sigma[H]))$. By upwards-absoluteness, $V[x] \models \exists y' \theta(x, y')$, i.e., $V[x] \models \phi(x)$. But this is a contradiction with $(*)$, since we had $p \Vdash V[\tau] \models \neg\phi(\tau)$.
To complete the proof of the theorem, it only remains to verify $\boldsymbol{\Sigma}^1_3$-correctness for the entire iteration of length ω_1. But obviously, if $V[G_{\omega_1}] \models \phi(x)$ for some $\boldsymbol{\Sigma}^1_3$ formula ϕ, then actually $V[G_{\omega_1}] \models \exists y \theta(x, y)$, and since $\mathbb{P}_{\omega_1}$ is proper, x and y must both appear at some stage $\gamma < \omega_1$, so by Shoenfield absoluteness $V[G_\gamma] \models \theta(x, y)$. Then, by what we have proved above, $V[x] \models \phi(x)$. □

4 Methods for obtaining regularity

The purpose of this section is to develop methods for obtaining regularity for $\boldsymbol{\Delta}^1_3$ sets of reals, but doing this with "as little damage as possible", i.e., using forcing iterations that preserve certain properties of the ground model. In total, we will present three separate methods of achieving this goal. The first one is due to Judah:

Theorem 4.1 (Judah) 1. *If $V \models \boldsymbol{\Sigma}^1_2(\mathbb{B})$ and $\mathbb{B}_{\omega_1}$ denotes the ω_1-product of random forcing, then $V^{\mathbb{B}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{B})$.*

2. *If $V \models \boldsymbol{\Sigma}^1_2(\mathbb{C})$ and $\mathbb{C}_{\omega_1}$ denotes the ω_1-product of Cohen forcing, then $V^{\mathbb{C}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{C})$.*

Proof See [3, Theorem 9.4.6]. □

It is not clear whether the above can be generalized beyond Cohen and random: the proofs depend on properties of the meager and null ideals (such as the Fubini property), as well as a strong homogeneity of Cohen- and random-products.

Next, we present our second method for obtaining $\boldsymbol{\Delta}^1_3$-regularity. This is inspired by Shelah's original proof that $\boldsymbol{\Delta}^1_3(\mathbb{B})$ does not require an inaccessible, see [40, Sect. 6]. Although we are primarily interested in the regularity properties mentioned in Sect. 2, we would like our proofs to be sufficiently uniform and general, i.e., we would like them to be applicable to many forcing notions $\mathbb{P}$ at once. We could require that $\mathbb{P}$ has trees as conditions (see e.g., *arboreal forcing* from [6,24]), but in some cases (e.g., Cohen) we prefer to work with G_δ sets instead. The reason is that, otherwise, we would need to work "modulo an ideal" which would only complicate the proofs unnecessarily. So we relax the requirement somewhat and adopt the following:

Convention and Notation 4.2

- Let us say that $\mathbb{P}$ is a *real forcing notion* if the conditions are G_δ sets of reals, ordered by inclusion. For conditions $p \in \mathbb{P}$, we will generally use "p" to refer to the real number coding the condition (i.e., G_δ code), and "$[p]$" to refer to the corresponding set of reals. When $\mathbb{P}$-conditions are closed sets, we may identify p with a tree and $[p]$ with the set of branches through that tree.
- We assume that, as usual, $\mathbb{P}$ adds a generic real, denoted by $\dot{g}$, and that for all $p \in \mathbb{P}$ we have $\Vdash_{\mathbb{P}} (p \in \dot{G} \leftrightarrow \dot{g} \in [p])$ (so the generic filter and real are mutually reconstructible).
- We will also assume that for projective pointclasses $\boldsymbol{\Gamma}$, the statements
  - "$\forall A \in \boldsymbol{\Gamma}\ \forall p \in \mathbb{P}\ \exists q \leq p\ ([q] \subseteq A$ or $[q] \cap A = \varnothing)$" and
  - "$\forall A \in \boldsymbol{\Gamma}\ \exists p\ ([p] \subseteq A$ or $[p] \cap A = \varnothing)$"

  are equivalent, so that, as in Observation 2.7, in order to prove $\boldsymbol{\Gamma}(\mathbb{P})$ it will suffice to prove the latter statement (this is achieved by using suitable homeomorphisms between $[p]$ for conditions $p \in \mathbb{P}$ and the entire space of reals).

This level of generality will certainly take care of everything we are interested in, and potentially much more. Note that we could be even more lenient in the convention and allow the conditions of $\mathbb{P}$ to be Borel sets that are large with respect to an ideal, following the approach of Zapletal [47].

Before stating the next theorem we introduce *amoebas* and *quasi-amoebas* for real forcing notions.

Definition 4.3 Let $\mathbb{P}$ be a real forcing notion, and $\mathbb{Q}$ another forcing. We say that

1. $\mathbb{Q}$ is a *quasi-amoeba for* $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{Q}$-generic G, there is a $q \in \mathbb{P}^{V[G]}$ such that $q \leq_{\mathbb{P}} p$ and

$$V[G] \models \forall x \in [q]\ (x \text{ is } \mathbb{P}\text{-generic over } V).$$

2. $\mathbb{Q}$ is an *amoeba for* $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{Q}$-generic G, there is a $q \in \mathbb{P}^{V[G]}$ such that $q \leq_{\mathbb{P}} p$ and for any larger model $W \supseteq V[G]$,

$$W \models \forall x \in [q]\ (x \text{ is } \mathbb{P}\text{-generic over } V).$$

There is a subtle difference between amoebas and quasi-amoebas, which is not visible in the Cohen and random (and, in general, ccc) case, because the assertion "$[q]$ consists of Cohen/random reals over V" is upwards absolute for Cohen/random-conditions q. For non-ccc forcing this is not always the case: for example, "T is a perfect tree of Sacks reals over V" is not upwards absolute, as shown in the next example.

Example 4.4

1. $\mathbb{A}$ (the standard *amoeba for measure*) is an amoeba for $\mathbb{B}$ (see [3, Section 3.4]).
2. $\mathbb{UM}$ (the standard *amoeba for category*) is an amoeba for $\mathbb{C}$ (see [40, Sect. 4]). Also, if $\mathbb{D}$ is Hechler forcing, then the two-step iteration $(\mathbb{D} * \mathbb{D})$ is an amoebas for $\mathbb{C}$ (see [3, Theorem 3.5.1]).

3. Mathias forcing $\mathbb{R}$ is an amoeba for itself [38, Corollary 2.5].
4. $\mathbb{S}$ is a quasi-amoeba, but not an amoeba, for itself [4, Theorem 4, Corollary 5].
5. $\mathbb{M}$ is a quasi-amoeba, but not an amoeba, for itself [4, Proposition 7].
6. $\mathbb{L}$ is *not* a quasi-amoeba for itself [4, Theorem 5].

One might expect quasi-amoebas to be quite useless in iterated forcing constructions, since the property of adding large sets of generic reals is only temporary. Nevertheless, the success of our methods is in part due to the realization that quasi-amoebas are, in fact, sufficient for the following argument.

Theorem 4.5 *Suppose $\mathbb{P}$ is a real forcing notion and $\mathbb{AP}$ a quasi-amoeba for $\mathbb{P}$. Furthermore, assume that both $\mathbb{P}$ and $\mathbb{AP}$ are Suslin$^+$ proper. Let $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1\rangle$ be a countable support iteration whose iterands are $\mathbb{P}$ and $\mathbb{AP}$ interlaced* (*i.e., for even α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \cong \mathbb{P}$ and for odd α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \cong \mathbb{AP}$*). *Then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$.*

Proof Let G_{ω_1} be $\mathbb{P}_{\omega_1}$-generic over V, let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$ be a $\boldsymbol{\Delta}^1_3$ set in $V[G_{\omega_1}]$, defined by $\boldsymbol{\Sigma}^1_3$-formulas ϕ and ψ. As our iteration is proper, we may assume, without loss of generality, that the parameters appearing in ϕ and ψ are in the ground model V (otherwise, they are in some $V[G_{\alpha_0}]$, and we repeat the same argument with $V[G_{\alpha_0}]$ as the ground model).
Our goal is to find a $\mathbb{P}$-condition p in $V[G_{\omega_1}]$ such that $[p] \subseteq A$ or $[p] \cap A = \varnothing$.
Let x_0 be the $\mathbb{P}$-generic real over V, added at the first step of the iteration. In $V[G_{\omega_1}]$, either $\phi(x_0)$ or $\psi(x_0)$ must hold, so without loss of generality we assume that $\phi(x_0)$ holds. Then $\exists y\theta(x_0, y)$ holds for some $\boldsymbol{\Pi}^1_2$ formula θ such that $\phi(x_0) \equiv \exists y\theta(x_0, y)$. By properness, there is an $\alpha < \omega_1$ such that $y \in V[G_\alpha]$, and by Shoenfield absoluteness $V[G_\alpha] \models \theta(x_0, y)$. In V, let p be a $\mathbb{P}_\alpha$-condition and τ a countable $\mathbb{P}_\alpha$-name for a real, such that

$$p \Vdash_\alpha \theta(\dot{g}_0, \tau)$$

where $\dot{g}_0$ is the name for the first $\mathbb{P}$-generic real.
Let us adopt the following notation: let $\mathbb{P}_{1,\alpha}$ be the quotient of the iteration (i.e., such that $\mathbb{P}_1 * \mathbb{P}_{1,\alpha} \cong \mathbb{P}_\alpha$), and when x is a $\mathbb{P}$-generic real over V, "$p[x]$" refers to the $\mathbb{P}_{1,\alpha}$-condition that remains of p after evaluating it according to x (i.e., the filter G_x generated by x), and "$\tau[x]$" refers to the $\mathbb{P}_{1,\alpha}$-name that remains of τ after evaluating it according to x. Here by "$\mathbb{P}_{1,\alpha}$" we are, of course, referring to the *definition* of the iteration. It is well-known that, if we consider τ and p as coded by reals (in some explicit way), then there are Borel functions mapping $\tau \mapsto \tau[x]$ and $p \mapsto p[x]$, in any model that contains x (this is similar to, e.g., [47, Proposition 2.3.1]).
Let $\tilde{\theta}(x, p, \tau)$ be a conjunction of the following statements:

– "$p[x]$ is a $\mathbb{P}_{1,\alpha}$-condition",
– "$\tau[x]$ is a countable $\mathbb{P}_{1,\alpha}$-name for a real", and
– $p[x] \Vdash_{1,\alpha} \theta(\check{x}, \tau[x])$.

Since the quotient $\mathbb{P}_{1,\alpha}$ is a Suslin$^+$ proper iteration, using Lemma 3.14 (1), (3) and (4), we conclude that $\tilde{\theta}$ is a $\boldsymbol{\Pi}^1_2$ statement. For convenience, we will suppress the parameters p and τ from $\tilde{\theta}$ (remember that they are in the ground model V).

As we have $p \Vdash \theta(\dot{g}_0, \tau)$ and x_0 is $\mathbb{P}$-generic over V, we have

$$V[x_0] \models \tilde{\theta}(x_0).$$

Therefore, going back to V, we have

$$p(0) \Vdash_{\mathbb{P}} \tilde{\theta}(\dot{g}_0).$$

But by Lemma 3.14 (4), the above statement is again $\boldsymbol{\Pi}^1_2$, so by Shoenfield absoluteness, $V[x_0] \models p(0) \Vdash_{\mathbb{P}} \tilde{\theta}(\dot{g}_0)$. Let H_1 be the next $\mathbb{AP}$-generic over $V[x_0]$ (i.e., $V[x_0][H_1] = V[G_2]$). By the definition of a quasi-amoeba, in $V[x_0][H_1]$ there is a $\mathbb{P}$-condition q, such that $q \leq p(0)$ and

$$V[x_0][H_1] \models \forall x \in [q]\ (x \text{ is } \mathbb{P}\text{-generic over } V[x_0]).$$

Then

$$V[x_0][H_1] \models \forall x \in [q]\ (V[x_0][x] \models \tilde{\theta}(x)),$$

and by $\boldsymbol{\Pi}^1_2$-absoluteness between $V[x_0][x]$ and $V[x_0][H_1]$:

$$V[x_0][H_1] \models \forall x \in [q]\ (\tilde{\theta}(x)).$$

Let $\Theta(q)$ abbreviate "$\forall x \in [q]\ (\tilde{\theta}(x))$", and notice that, again, it is $\boldsymbol{\Pi}^1_2$. This is the key step of our proof, since now, in all larger models $V[G_\beta]$, $2 \leq \beta < \omega_1$, we have

$$V[G_\beta] \models \Theta(q).$$

It remains to show that $V[G_{\omega_1}] \models [q] \subseteq A$, which will complete the proof. So, in $V[G_{\omega_1}]$, let z be any real in $[q]$. Let $\beta < \omega_1$ be such that $z \in V[G_\beta]$, and assume β is odd (so that $\beta + 1$ is even). Since $V[G_{\beta+1}] \models \Theta(q)$, in particular, $V[G_{\beta+1}] \models \tilde{\theta}(z)$. But looking at the meaning of $\tilde{\theta}$, in particular it says "$p[z] \Vdash_{\mathbb{P}_{1,\alpha}} \theta(\check{z}, \tau)$", which implies "$p[z] \Vdash_{\mathbb{P}_{1,\alpha}} \exists y' \theta(\check{z}, y')$" and hence "$p[z] \Vdash_{\mathbb{P}_{1,\alpha}} \phi(\check{z})$". Notice that, by genericity, we may assume that β was chosen to be sufficiently large so that $p[z]$ in fact belongs to $G_{[\beta+1,\beta+\alpha)}$ (the generic filter restricted to stages $[\beta + 1, \beta + \alpha)$ of the iteration).
It follows that $V[G_{\beta+\alpha}] \models \phi(z)$, and by upwards-absoluteness, $V[G_{\omega_1}] \models \phi(z)$. This completes the proof. □

Corollary 4.6 *If $\mathbb{P}$ is Suslin$^+$ proper and a quasi-amoeba for itself, then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$. In particular $V^{\mathbb{S}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{S})$ and $V^{\mathbb{M}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{M})$.*

If we want to obtain $\boldsymbol{\Delta}^1_3(\mathbb{P})$ for several different $\mathbb{P}$ at the same time, we can alter the above construction somewhat, by interlacing more forcing notions. The only requirement is that the iteration is sufficiently "repetitive", in the sense of the following definition:

Definition 4.7 Suppose $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is a Suslin$^+$ proper iteration, where all iterands have parameters in the ground model. Such an iteration is called *repetitive* if for any $\alpha < \beta < \omega_1$, there are unboundedly many $\gamma < \omega_1$ such that

$$\langle \dot{\mathbb{Q}}_\xi \mid \alpha < \xi \leq \beta \rangle = \langle \dot{\mathbb{Q}}_{\gamma+\xi} \mid \alpha < \xi \leq \beta \rangle.$$

The following theorem is a stronger version of Theorem 4.5:

Theorem 4.8 *Suppose $\mathbb{P}$ and $\mathbb{AP}$ are as in Theorem 4.5, $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is Suslin$^+$ proper with parameters in the ground model and repetitive, and both $\mathbb{P}$ and $\mathbb{AP}$ appear cofinally often in the iteration. Then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$.*

Proof The proof is exactly the same as that of Theorem 4.5. Instead of looking at stages 1 and 2 of the iteration, we look at some stages α_0 and α_1. Then we find the condition q in $V[G_{\alpha_1}]$ and $\Theta(q)$ holds from that point onwards. Later we find a sufficiently large γ so that the segment $\mathbb{P}_{\alpha_1,\alpha_1+\alpha}$ is "copied" after γ, and rely on the same arguments as before. The details are left to the reader. □

In our applications, the last theorem will only be used when we have a finite number of $\mathbb{P}_i$ and quasi-amoeabs $\mathbb{AP}_i$, $i \leq k$. After iterating with $(\mathbb{P}_0 * \mathbb{AP}_0 * \ldots * \mathbb{P}_k * \mathbb{AP}_k)_{\omega_1}$ we obtain a model where $\boldsymbol{\Delta}^1_3(\mathbb{P}_i)$ holds for all $i \leq k$.

Our third method for obtaining $\boldsymbol{\Delta}^1_3$-regularity works under the assumption $\forall r\ (\omega_1^{L[r]} < \omega_1)$. Let us first mention an observation essentially due to Zapletal (cf. [47, Proposition 2.2.2.]), showing that for real forcing notions $\mathbb{P}$, when we have an $(M, \mathbb{P})$-generic condition, we can assume, without loss of generality, that *all* reals in this condition are M-generic.

Lemma 4.9 (Zapletal) *Let $\mathbb{P}$ be a proper, real forcing notion, and M a countable model. If q is an $(M, \mathbb{P})$-generic condition, then there is $q' \leq q$ such that (in V) all $x \in [q]$ are M-generic.*

Proof Let $B := \{x \in [q] \mid x \text{ is } M\text{-generic}\}$. As M is countable, it is easy to see that B is Borel. Let $\dot{g}$ be the name for the $\mathbb{P}$-generic real. Since q is $(M, \mathbb{P})$-generic, $q \Vdash \dot{g} \in B$. But Borel sets are $\mathbb{P}$-measurable (in the sense of Definition 2.1). So either there exists a $q' \leq q$ such that $[q'] \subseteq B$, in which case we are done, or, for every $q' \leq q$ there exists $q'' \leq q'$ such that $[q''] \cap B = \varnothing$. But the latter case implies that $\{q' \mid [q'] \cap B = \varnothing\}$ is dense below q, hence $q \Vdash \dot{g} \notin B$, yielding a contradiction. □

Theorem 4.10 *Suppose $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$, $\mathbb{P}$ is a real forcing notion, and $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is a Suslin$^+$ proper iteration in which $\mathbb{P}$ appears cofinally often. Then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$.*

Proof Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$ be a $\boldsymbol{\Delta}^1_3$ set in $V[G_{\omega_1}]$. As the defining parameter appears at some initial stage of the iteration, and by Theorem 3.16 we know that $\forall r (\omega_1^{L[r]} < \omega_1)$ holds in all $V[G_\alpha]$, let us again assume, without loss of generality, that the parameters are in the ground model V. Also, without loss of generality, we may assume that the first step of the iteration is $\mathbb{P}$.

Let x_0 be the $\mathbb{P}$-generic real over V. Again, let us assume $V[G_{\omega_1}] \models \phi(x_0)$ (without loss of generality). Now by Theorem 3.20 V is $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}_{\omega_1}$-correct, therefore $V[x_0] \models \phi(x_0)$. Then in V, there is a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \phi(\dot{g})$. Then also $p \Vdash_{\mathbb{P}} \theta(\dot{g}, \tau)$ for some countable name τ and a $\boldsymbol{\Pi}^1_2$ formula θ such that $\phi(x) \equiv \exists y \theta(x, y)$. Now let z be a real coding p, τ, and the parameters of $\mathbb{P}$ and θ. By $\boldsymbol{\Pi}^1_2$-absoluteness, $L[z] \models p \Vdash_{\mathbb{P}} \theta(\dot{g}, \tau)$.
Since ω_1^V is inaccessible in $L[z]$, find a countable model $M \subseteq L[z]$ reflecting everything about $\mathbb{P}^{L[z]}$ and containing all the parameters (as in the proof of Lemma 3.15). By Lemma 4.9 there is $q \leq p$ such that all x in $[q]$ are M-generic, hence $L[z]$-generic. So (in V) for all $x \in [q]$ we have $L[z][x] \models \theta(x, \tau[x])$, and by $\boldsymbol{\Pi}^1_2$-absoluteness

$$V \models \forall x \in [q]\, \theta(x, \tau[x]).$$

As this statement is $\boldsymbol{\Pi}^1_2$, it holds in $V[G_{\omega_1}]$, so also the statement $\forall x \in [q]\, \exists y \theta(x, y)$ holds, so $[q] \subseteq A$. □

The advantage of this method over the one before is that we can avoid amoebas, which is useful in situations where no suitable amoebas are available, or those that are available fail to have nice properties. However, to do this we pay the price of using an inaccessible, rather than obtaining a proof on the basis of ZFC alone.

5 Completing the $\boldsymbol{\Delta}^1_3$-diagram

Figure 3 shows the diagram of implications for regularity properties on the $\boldsymbol{\Delta}^1_3$-level.

We will now apply the techniques presented in the previous section to show that this diagram is complete, by constructing models, in ZFC or ZFC with an inaccessible, which separate the regularity statements. We have three methods (Theorems 4.1, 4.8 and 4.10) at our disposal for proving that $\boldsymbol{\Delta}^1_3(\mathbb{P})$ is true in a model. But to separate regularity properties we need another ingredient, namely, a method for showing that $\boldsymbol{\Delta}^1_3(\mathbb{P})$ is *false* in a given model. For this, we note that one direction in the original characterization Theorems 2.3, 2.4 and 2.5 can easily be generalized (for $\mathbb{C}$ and $\mathbb{B}$ this was already mentioned and used in [2]).

Definition 5.1 A wellorder $\preceq$ of a set of reals, of length ω_1, is called $\boldsymbol{\Sigma}^1_n$-*good* if

1. $\preceq$ is a $\boldsymbol{\Sigma}^1_n$-relation, and
2. the statement "x codes the set of $\preceq$-predecessors of y" is $\boldsymbol{\Sigma}^1_n$.

We say that $\preceq$ is a $\boldsymbol{\Sigma}^1_n$-*good wellorder of the reals* if it is a wellorder of the set of all reals.

Fact 5.2 *Suppose M is a model with a $\boldsymbol{\Sigma}^1_n$-good wellorder of the reals. Then:*

1. $\boldsymbol{\Sigma}^1_n(\mathbb{B}) \implies \{x \mid x \text{ is not random over } M\} \in \mathcal{N}$.
2. $\boldsymbol{\Sigma}^1_n(\mathbb{C}) \implies \{x \mid x \text{ is not Cohen over } M\} \in \mathcal{M}$.
3. $\boldsymbol{\Delta}^1_n(\mathbb{B}) \implies \exists x\, (x \text{ is random over } M)$.
4. $\boldsymbol{\Delta}^1_n(\mathbb{C}) \implies \exists x\, (x \text{ is Cohen over } M)$.
5. $\boldsymbol{\Delta}^1_n(\mathbb{L}) \implies \exists x\, (x \text{ is dominating over } M)$.

6. $\boldsymbol{\Delta}^1_n(\mathbb{M}) \implies \exists x$ (x is unbounded over M).
7. $\boldsymbol{\Delta}^1_n(\mathbb{S}) \implies \exists x$ ($x \notin M$).

Proof Points 1–4 follow from the original proofs of Solovay and Judah–Shelah; see also [2, Lemmas 2.3, 2.85 and 2.105]. For 5 and 6, use an argument analogous to the one in [8, Theorems 4.1 and 6.1] replacing $\boldsymbol{\Sigma}^1_2$ by $\boldsymbol{\Sigma}^1_3$ and $\boldsymbol{\Delta}^1_2$ by $\boldsymbol{\Delta}^1_3$ everywhere. 7 is obvious. □

We are going to use the following results about models with $\boldsymbol{\Sigma}^1_3$-good wellorders:

Theorem 5.3 (Bagaria–Woodin) *Assuming just the consistency of ZFC, there is a model, which we will denote by L^*, such that*

1. $L^* \models \boldsymbol{\Sigma}^1_2(\mathbb{B})$ *(and hence also* $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ *for all* $\mathbb{P} \in \{\mathbb{C}, \mathbb{L}, \mathbb{M}, \mathbb{S}\}$*) and*
2. *there is a* $\boldsymbol{\Sigma}^1_3$*-good wellorder of the reals of* L^*.

Proof This model was first constructed in [2]. Easier constructions of models satisfying the above criteria are available using techniques developed recently by Friedman, Fischer, Zdomskyy and others (e.g., [14,15]). □

Theorem 5.4 (David) *Assuming the consistency of ZFC + inaccessible, there is a model, which we will denote by L^d, such that*

1. $L^d \models \forall r\ (\omega_1^{L[r]} < \omega_1)$*, and*
2. *there is a* $\boldsymbol{\Sigma}^1_3$*-good wellorder of the reals of* L^d.

Proof This was proved by René David in [10]. □

We should note that the $\boldsymbol{\Sigma}^1_3$-good wellorder of the reals of L^* and L^d from the above results remains a $\boldsymbol{\Sigma}^1_3$-good wellorder (of the ground-model reals) in forcing extensions. All the models we construct will be forcing extensions of L, L^* or L^d, with the methods from the previous section guaranteeing that $\boldsymbol{\Delta}^1_3(\mathbb{P})$ holds for certain $\mathbb{P}$, while using Fact 5.2, together with known preservation results, to guarantee that $\boldsymbol{\Delta}^1_3(\mathbb{Q})$ fails for other $\mathbb{Q}$. The idea to use David's model L^d to separate regularity properties was first used by Judah and Spinas in [30].

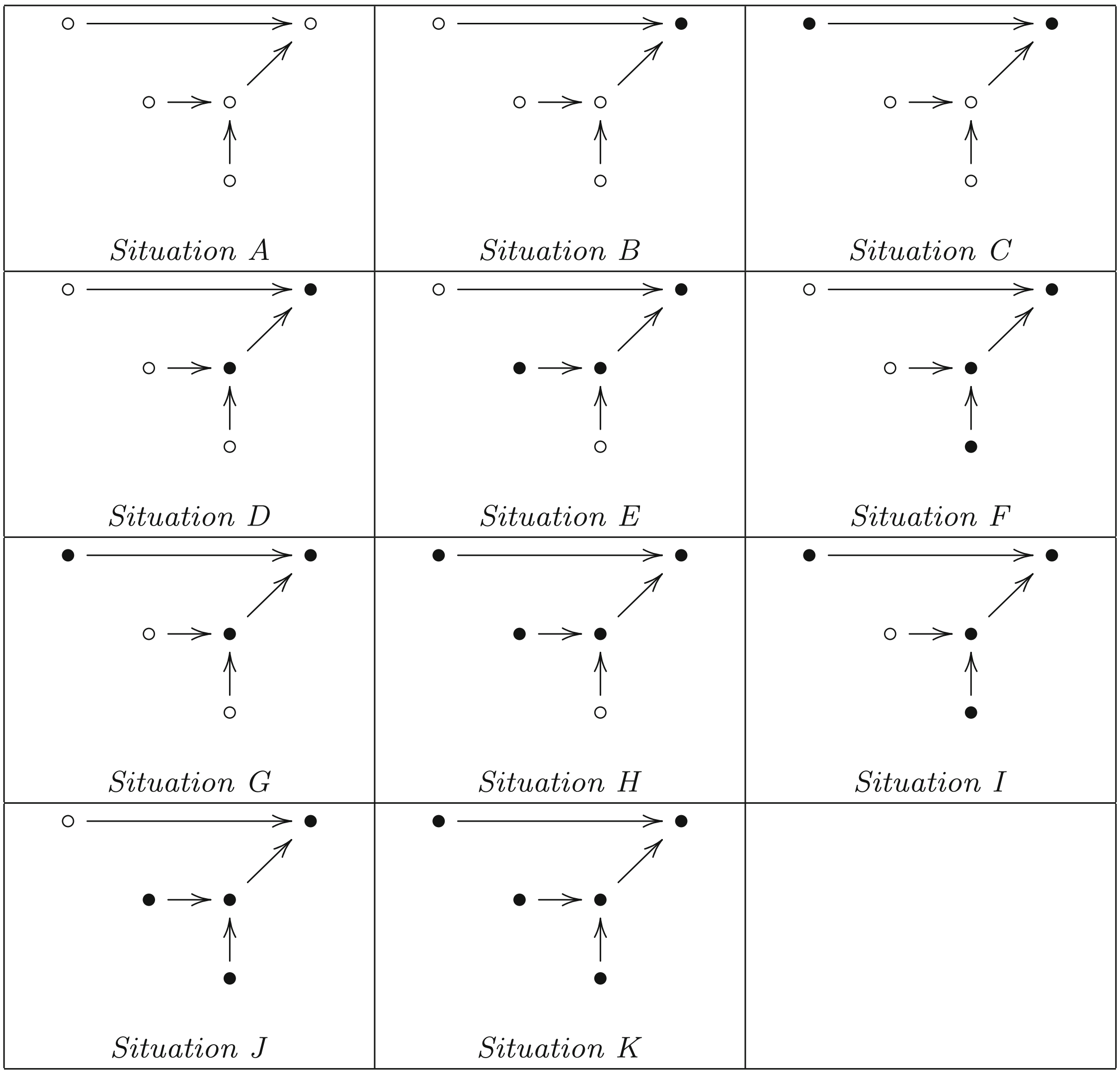
We will use the diagrammatical notation employed by Bartoszyński and Judah in [3, Sections 7.5, 7.6], with empty circles symbolizing "false" and full circles "true". There is a total of eleven possibilities of "true"/"false"-assignments not contradicting the diagram, which we denote with the letters A–K and represent in Table 2.

In the following list we provide models for each situation. Whenever possible, the models will be constructed in ZFC alone. In three cases, namely G, H and I, we will have to make do with an inaccessible (although we conjecture that this hypothesis can be eliminated).

– **Situation A**, *determined by* $\neg\boldsymbol{\Delta}^1_3(\mathbb{S})$.
  This holds in L, L^* and L^d.
– **Situation B**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{S}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{M})$.
  The model for this is $L^{\mathbb{S}_{\omega_1}}$, i.e., the countable support iteration of Sacks forcing of length ω_1 starting from L. Since Sacks forcing is a quasi-amoeba for itself (see Example 4.4), $\boldsymbol{\Delta}^1_3(\mathbb{S})$ follows by Corollary 4.6. Moreover, since $\mathbb{S}_{\omega_1}$ is ω^ω-bounding and does not add random reals (by the Sacks property), it follows that, in this model, even $\boldsymbol{\Delta}^1_2(\mathbb{B})$ and $\boldsymbol{\Delta}^1_2(\mathbb{M})$ fail.

Table 2 Situations A–K in the $\boldsymbol{\Delta}^1_3$-diagram

○ = FALSE ● = TRUE



- **Situation C**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{M})$.
  Take the model $(L^*)^{\mathbb{B}_{\omega_1}}$. By Theorem 4.1 $\boldsymbol{\Delta}^1_3(\mathbb{B})$ holds. Because random forcing is ω^ω-bounding, and because of Fact 5.2 (6), we have $\neg\boldsymbol{\Delta}^1_3(\mathbb{M})$.
- **Situation D**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{M}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Here the model is $L^{\mathbb{M}_{\omega_1}}$. Since Miller forcing is a quasi-amoeba of itself (see Example 4.4) $\boldsymbol{\Delta}^1_3(\mathbb{M})$ follows by Corollary 4.6. On the other hand, Miller forcing does not add Cohen or random reals because of the Laver property [3, Theorem 7.3.45], so both $\boldsymbol{\Delta}^1_2(\mathbb{B})$ and $\boldsymbol{\Delta}^1_2(\mathbb{C})$ fail. Also, Miller forcing does not add dominating reals [3, Theorem 7.3.46], so $\boldsymbol{\Delta}^1_2(\mathbb{L})$ fails.
- **Situation E**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Here, let us provide two models. Spinas, in [43], constructs a version of "amoeba for Laver" forcing, which he denotes by $\mathbb{A}(\mathbb{L})$, and proves that it is an amoeba for Laver in the sense of Definition 4.3 and, at the same time, satisfies the Laver property. It follows that the iteration $(\mathbb{L} * \mathbb{A}(\mathbb{L}))_{\omega_1}$ (i.e., the countable support

iteration of length ω_1 where $\mathbb{L}$ appears at even stages and $\mathbb{A}(\mathbb{L})$ at odd stages) has the Laver property, hence $L^{(\mathbb{L}*\mathbb{A}(\mathbb{L}))_{\omega_1}} \models \neg\boldsymbol{\Delta}^1_2(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_2(\mathbb{C})$. But $\boldsymbol{\Delta}^1_3(\mathbb{L})$ holds by Theorem 4.5.

Another model is the one given in [29, Theorem 3.1], namely, the ω_1-iteration of Mathias forcing starting from $\mathbb{L}$. Here an even stronger assertion holds, namely "all $\boldsymbol{\Delta}^1_3$-sets are Ramsey" which implies $\boldsymbol{\Delta}^1_3(\mathbb{L})$ (see Sect. 6).

– **Situation F**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{C}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{L})$.
Here we take $(L^*)^{\mathbb{C}_{\omega_1}}$. Then $\boldsymbol{\Delta}^1_3(\mathbb{C})$ holds by Theorem 4.1. On the other hand, Cohen forcing adds neither dominating nor random reals, so by Fact 5.2 (3) and (5), neither $\boldsymbol{\Delta}^1_3(\mathbb{B})$ nor $\boldsymbol{\Delta}^1_3(\mathbb{L})$ holds.

– **Situation G**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{M}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
Use the model $(L^d)^{(\mathbb{B}*\mathbb{M})_{\omega_1}}$, i.e., the ω_1-iteration, with countable support, of $\mathbb{B}$ and $\mathbb{M}$ interlaced starting from David's model L^d (which requires an inaccessible). By Theorem 4.10, both $\boldsymbol{\Delta}^1_3(\mathbb{B})$ and $\boldsymbol{\Delta}^1_3(\mathbb{M})$ hold. Since $\mathbb{B}$ and $\mathbb{M}$ do not add dominating reals, $\boldsymbol{\Delta}^1_3(\mathbb{L})$ fails by Fact 5.2 (5). To show that $\boldsymbol{\Delta}^1_3(\mathbb{C})$ also fails we can use a weaker version of the Laver property, namely the property of being "(F, g)-preserving" as defined in [3, Definition 7.2.23]. Both random and Miller forcing satisfy this property [3, Lemma 7.2.25 and Theorem 7.2.26], it is preserved in countable support iterations [3, Theorem 7.2.29], and it implies that no Cohen reals are added [3, Theorem 7.2.24]. Therefore $\neg\boldsymbol{\Delta}^1_3(\mathbb{C})$ follows by Fact 5.2 (4).

– **Situation H**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
Here we use $(L^d)^{(\mathbb{B}*\mathbb{L})_{\omega_1}}$, an ω_1-iteration of random and Laver forcing starting from David's model. By Theorem 4.10 $\boldsymbol{\Delta}^1_3(\mathbb{B})$ and $\boldsymbol{\Delta}^1_3(\mathbb{L})$ hold, and $\boldsymbol{\Delta}^1_3(\mathbb{C})$ fails for the same reason as above, namely, because both random and Laver forcing satisfy the "(F, g)-preserving" property.

– **Situation I**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{C}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{L})$.
Here we use $(L^d)^{(\mathbb{B}*\mathbb{C})_{\omega_1}}$. Again by Theorem 4.10 we have $\boldsymbol{\Delta}^1_3(\mathbb{B})$ and $\boldsymbol{\Delta}^1_3(\mathbb{C})$. But neither random nor Cohen forcing adds dominating reals, so $\neg\boldsymbol{\Delta}^1_3(\mathbb{L})$ fails by Fact 5.2 (5).

– **Situation J**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{L}) + \boldsymbol{\Delta}^1_3(\mathbb{C}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B})$.
Using our methods, we can easily see that $(L^d)^{(\mathbb{C}*\mathbb{L})_{\omega_1}}$ is a model for this, where the fact that no random reals are added follows as in [3, Model 7.6.9]. However, in [26, Theorem 3.2] a model was constructed starting just from ZFC. The method there was similar to an application of our Theorem 4.5, iterating what was essentially a mixture of $\mathbb{C}$, $\mathbb{UM}$ (amoeba for category) and $\mathbb{R}_{\mathcal{F}}$—Mathias forcing with a Ramsey ultrafilter $\mathcal{F}$—with *finite* support, starting in L. Since the use of the Ramsey ultrafilters makes the iteration non-definable, one cannot use the arguments from Sect. 4 directly. Instead, the iteration was done in such a way that each segment $\mathbb{P}_{\alpha,\beta}$ of the iteration would appear again as $\dot{\mathbb{Q}}_\delta$, for cofinally many $\delta < \omega_1$ (using a bookkeeping argument like in standard MA-proofs). In [26] it was shown that such an iteration, starting from L, yields a model in which $\boldsymbol{\Delta}^1_3(\mathbb{R})$ (the Ramsey property; see Sect. 6) as well as $\boldsymbol{\Delta}^1_3(\mathbb{C})$ hold. The former implies $\boldsymbol{\Delta}^1_3(\mathbb{L})$. On the other hand, the iteration remains σ-centered implying that no random reals are added, hence $\boldsymbol{\Delta}^1_2(\mathbb{B})$ fails.

– **Situation K**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{C}) + \boldsymbol{\Delta}^1_3(\mathbb{L})$.

Of course, the Solovay model satisfies this statement, so our only interest here is in constructing a model in ZFC. But this is easy: since we do not have to worry about preserving anything, we can freely apply Corollary 4.6. For example, we can use the model $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{UM}*\mathbb{L}*\mathbb{A}(\mathbb{L}))_{\omega_1}}$, or $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{UM}*\mathbb{R})_{\omega_1}}$. In fact, even $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ is sufficient, because, by the Bartoszyński–Raisonnier–Stern argument, $\mathbb{A}$ already adds a comeager set of Cohen reals (cf. [3, Theorem 2.3.1]).

6 Silver and Mathias

Clearly, the techniques we developed in Sect. 4 are sufficiently general and can be applied to many other regularity properties related to forcing notions on the reals. In this section, we apply our techniques to two additional properties which, though not related to the cardinal numbers in Cichoń's diagram, have nevertheless received a lot of attention.

Definition 6.1 1. A subset $A \subseteq [\omega]^\omega$ has the *Ramsey property* if $\exists a \in [\omega]^\omega$ $([a]^\omega \subseteq A$ or $[a]^\omega \cap A = \varnothing)$.

2. For $a, b \in [\omega]^\omega$ with $|b \setminus a| = \omega$, let $[a, b]^\omega := \{c \in [\omega]^\omega \mid a \subseteq c \subseteq b\}$. We call $[a, b]^\omega$ the *(a, b)-doughnut*. A subset $A \subseteq [\omega]^\omega$ has the *doughnut property* if $\exists a, b$ $([a, b]^\omega \subseteq A$ or $[a, b]^\omega \cap A = \varnothing)$.

The Ramsey property is well-known, and the doughnut property was introduced by DiPrisco and Henle in [11] as a generalization of the Ramsey property. It is not hard to see that the Ramsey and doughnut properties are equivalent to Mathias- and Silver-measurability, respectively. Therefore, we will denote them with the letters $\mathbb{R}$ and $\mathbb{V}$, which typically abbreviate the Mathias and the Silver forcing partial orders. Mathias and Silver forcing are clearly Suslin$^+$ proper.

On the $\boldsymbol{\Delta}^1_2$- and $\boldsymbol{\Sigma}^1_2$-levels, the relationship between these and other properties has been studied in [6,20,23]. The following are particularly interesting:

Fact 6.2 (Judah–Shelah; Halbeisen; Brendle–Halbeisen–Löwe).

1. $\boldsymbol{\Sigma}^1_2(\mathbb{R}) \iff \boldsymbol{\Delta}^1_2(\mathbb{R})$.
2. $\boldsymbol{\Delta}^1_2(\mathbb{C}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{V})$.
3. $\boldsymbol{\Sigma}^1_2(\mathbb{V}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{M})$.

Proof For 1 see [23, Theorem 2.7]. For 2 see [20, Lemma 2.1], and for 3 see [6, Proposition 3.5]. □

Unlike the situation with the properties we previously considered, now there are still some open questions on the second level:

Question 6.3 *Does $\boldsymbol{\Delta}^1_2(\mathbb{L}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{V})$ hold? Or, at least, does $\boldsymbol{\Delta}^1_2(\mathbb{L}) \implies \boldsymbol{\Delta}^1_2(\mathbb{V})$ hold?*

As in Lemma 2.7 we have the following:

Lemma 6.4 (Folklore) *Let $\boldsymbol{\Gamma}$ be closed under continuous pre-images. Then:*

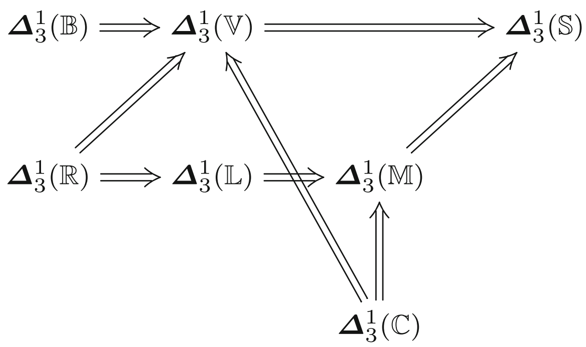


Fig. 4 (Incomplete) implication diagram for $\boldsymbol{\Delta}^1_3$ sets of reals, including $\mathbb{R}$ and $\mathbb{V}$

1. $\boldsymbol{\Gamma}(\mathbb{R}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{S})$.
2. $\boldsymbol{\Gamma}(\mathbb{R}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{L})$.

Proof For the first implication, note that $[a]^\omega$ is a $(\varnothing, a)$-doughnut, and the set of characteristic functions of $x \in [a, b]^\omega$ is a perfect tree in 2^ω. For the second implication, use the fact that for any $a \in [\omega]^\omega$ it is easy to find a Laver tree T such that $\forall x \in [T]\ (\mathrm{ran}(x) \subseteq a)$. □

In his Ph.D. thesis, Laguzzi proved two additional relationships of this kind.

Lemma 6.5 (Laguzzi) *Let $\boldsymbol{\Gamma}$ be closed under continuous pre-images. Then:*

1. $\boldsymbol{\Gamma}(\mathbb{C}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V})$.
2. $\boldsymbol{\Gamma}(\mathbb{B}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V})$.

Proof See [36, Fact 39 and Fact 55]. □

As an illustration of the application of our methods, let us repeat what we did in Sect. 5, i.e., look at the $\boldsymbol{\Delta}^1_3$-diagram with the additional properties $\mathbb{V}$ and $\mathbb{R}$ (Fig. 4). There are now eighteen situations, represented in Table 3 (we have subdivided the situations from the previous section).

To find models for these situations we need the following additional facts (cf. Fact 5.2).

Definition 6.6 Let M be a model. A real $c \in [\omega]^\omega$ is

– *splitting over M* if for all $a \in M \cap [\omega]^\omega$ $(|a \cap c| = |a \setminus c| = \omega)$, and
– *unsplit over M* if for all $a \in M \cap [\omega]^\omega$ $(|c \cap a| < \omega$ or $|c \setminus a| < \omega)$.

Fact 6.7 *Suppose M is a model with a $\boldsymbol{\Sigma}^1_n$-good wellorder of the reals. Then:*

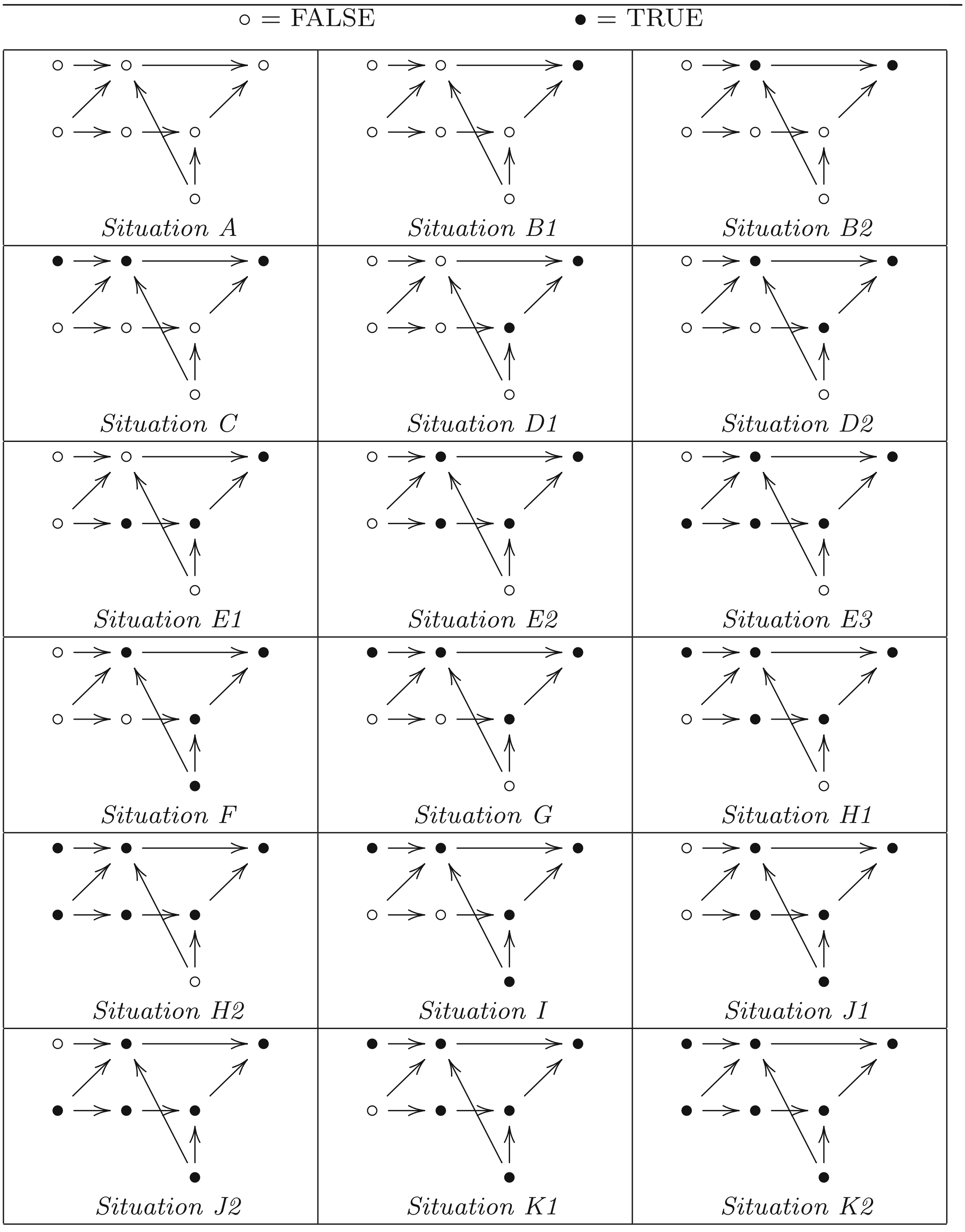
1. $\boldsymbol{\Delta}^1_n(\mathbb{V}) \implies \exists c$ (c is splitting over M).
2. $\boldsymbol{\Delta}^1_n(\mathbb{R}) \implies \exists c$ (c is splitting over M) *and* $\exists c$ (c is unsplit over M).

Proof For the first implication, use the argument in [6, Proposition 2.5], and for the second one, use [21, Theorem 2.2]. □

Splitting and unsplit reals are related to the well-known cardinal characteristics $\mathfrak{s}$ and $\mathfrak{r}$ (the *splitting* and *reaping* number, respectively), in a way similar to the relationship shown in Table 1 (although they do not *characterize* $\mathbb{R}$ and $\mathbb{V}$ in any way).

We can now find models for the following situations from Table 3 (we only list the ones that do not automatically follow from our results in Sect. 5).

Table 3 Situations A–K in the $\boldsymbol{\Delta}^1_3$-diagram



– **Situation B1**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{S}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{V}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{M})$.
  Here the model is $L^{\mathbb{S}_{\omega_1}}$. Sacks forcing preserves P-points, which is an iterable property (see [17, Lemma 2.9] and [3, Theorem 6.2.6]), so in particular no splitting reals are added, hence $\boldsymbol{\Delta}^1_2(\mathbb{V})$ fails by Fact 6.7 (1).
– **Situation B2**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{V}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{M})$.
  Use $(L^d)^{\mathbb{V}_{\omega_1}}$. By the Sacks property of Silver forcing neither random nor unbounded reals are added.

- **Situation D1**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{M}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{V}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Here we use $L^{\mathbb{M}_{\omega_1}}$. Again $\boldsymbol{\Delta}^1_2(\mathbb{V})$ fails because Miller forcing preserves P-points [3, Lemma 7.3.48].
- **Situation D2**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{V}) + \boldsymbol{\Delta}^1_3(\mathbb{M}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Here we can use $(L^d)^{(\mathbb{V}*\mathbb{M})_{\omega_1}}$. Both $\mathbb{V}$ and $\mathbb{M}$ have the Laver property, and both do not add dominating reals.
- **Situation E1**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{V})$.
  We don't know if this situation is consistent!
- **Situation E2**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{L}) + \boldsymbol{\Delta}^1_3(\mathbb{V}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{R}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Use $(L^d)^{(\mathbb{L}*\mathbb{V})_{\omega_1}}$. Both $\mathbb{L}$ and $\mathbb{V}$ have the Laver property, implying that neither random nor Cohen reals are added. To show that $\boldsymbol{\Delta}^1_3(\mathbb{R})$ fails, recall the preservation property called "preserving $\sqsubseteq^{\text{random}}$" in [3, Definition 6.3.7]. Both $\mathbb{L}$ and $\mathbb{V}$ satisfy this property (for $\mathbb{L}$ see [3, Theorem 7.3.39] and for $\mathbb{V}$ it follows from an even stronger result, namely [3, Lemma 6.3.39]), it is preserved by countable support iteration [3, Theorem 6.1.13] and implies that the ground model reals have positive measure [3, Thorem 6.3.13]. From this, one can infer that there are no unsplit reals over the ground model, in a way analogous to the well-known proof of the cardinal inequality $\mathfrak{s} \leq \text{non}(\mathcal{N})$ (i.e., for every $a \in [\omega]^\omega$, the set $X_a := \{b \mid b$ does not split $a\}$ has measure zero). Hence, the result follows from Fact 6.7 (2).
- **Situation E3**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{R}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Clearly $L^{\mathbb{R}_{\omega_1}}$ works here.
- **Situation H1**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{R}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Use $(L^d)^{(\mathbb{B}*\mathbb{L})_{\omega_1}}$. Both $\mathbb{B}$ and $\mathbb{L}$ have the "(F, g)-preserving" property, implying that no Cohen reals are added. To show that $\boldsymbol{\Delta}^1_3(\mathbb{R})$ fails use again the "preserving $\sqsubseteq^{\text{random}}$"-property. Random forcing satisfies this by [3, Lemma 6.3.12], so, as before, we are done by Fact 6.7 (2).
- **Situation H2**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{R}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Use $(L^d)^{(\mathbb{B}*\mathbb{R})_{\omega_1}}$. Both $\mathbb{B}$ and $\mathbb{R}$ have the "(F, g)-preserving" property, implying that no Cohen reals are added.
- **Situation J1**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{C}) + \boldsymbol{\Delta}^1_3(\mathbb{L}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{R})$.
  Use $(L^d)^{(\mathbb{C}*\mathbb{L})_{\omega_1}}$. As in [3, Model 7.6.9] we can show that no random reals are added by the iteration. To show that $\boldsymbol{\Delta}^1_3(\mathbb{R})$ fails, we note that both $\mathbb{C}$ and $\mathbb{L}$ satisfy a strong iterable property implying that no unsplit reals are added: see e.g., [12, Lemma 8, 9] and [5, Main Lemma 1.11], and apply Fact 6.7 (2).
- **Situation J2**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{C}) + \boldsymbol{\Delta}^1_3(\mathbb{R}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{B})$.
  Here we can either use $(L^d)^{(\mathbb{C}*\mathbb{R})}$ or the ZFC-model from [26, Theorem 3.2] which we also used in Situation J in Sect. 5.
- **Situation K1**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{L}) + \boldsymbol{\Delta}^1_3(\mathbb{C}) + \neg\boldsymbol{\Delta}^1_3(\mathbb{R})$.
  Use $(L^d)^{(\mathbb{B}*\mathbb{L}*\mathbb{C})_{\omega_1}}$. To show that no unsplit reals are added, use the iterable version for $\mathbb{C}$ and $\mathbb{L}$ (as in Situation J1) and preservation of $\sqsubseteq^{\text{random}}$ for $\mathbb{B}$. Again, $\boldsymbol{\Delta}^1_3(\mathbb{R})$ fails by Fact 6.7 (2).
- **Situation K2**, *determined by* $\boldsymbol{\Delta}^1_3(\mathbb{B}) + \boldsymbol{\Delta}^1_3(\mathbb{R}) + \boldsymbol{\Delta}^1_3(\mathbb{C})$.
  Here $L^{(\mathbb{B}*\mathbb{A}*\mathbb{R}*\mathbb{C})_{\omega_1}}$ clearly suffices.

To conclude: all situations except E1 are consistent. Moreover, we have ZFC-models for B1, D1, E3, J2 and K2, whereas for the other cases we need an inaccessible. The difficulty concerning Situation E1 lies in the fact that we do not know whether $\boldsymbol{\Delta}^1_3(\mathbb{L}) \Rightarrow \boldsymbol{\Delta}^1_3(\mathbb{V})$ holds (cf. Question 6.3).

7 Beyond $\boldsymbol{\Delta}^1_3$

Although our techniques were primarily developed to deal with the $\boldsymbol{\Delta}^1_3$-level of the projective hieararchy, there are some applications to higher levels as well. In this section we summarize what can be said about higher levels using our techniques. We have two applications: concerning the diagram on the $\boldsymbol{\Delta}^1_4$-level, and concerning the separation of $\boldsymbol{\Delta}^1_3$-regularity from $\boldsymbol{\Sigma}^1_3$-regularity, as well as $\boldsymbol{\Delta}^1_4$-regularity from $\boldsymbol{\Sigma}^1_4$-regularity.

7.1 The $\boldsymbol{\Delta}^1_4$-diagram

Consider the analogue of Fig. 3 but with $\boldsymbol{\Delta}^1_3$ replaced by $\boldsymbol{\Delta}^1_4$. It turns out that, under the assumption $\forall r\,(\omega_1^{L[r]} < \omega_1)$, both Theorems 4.1 and 4.8 have suitable generalizations. The first generalization is due to Judah and Spinas:

Theorem 7.1 (Judah–Spinas) *Assuming $\forall r\,(\omega_1^{L[r]} < \omega_1)$, there exists a model N_0, which has a $\boldsymbol{\Sigma}^1_4$-good wellorder of the reals, and, moreover, such that $N_0^{\mathbb{B}_{\omega_1}} \models \boldsymbol{\Delta}^1_4(\mathbb{B})$ and $N_0^{\mathbb{C}_{\omega_1}} \models \boldsymbol{\Delta}^1_4(\mathbb{C})$.*

For the proof, see [30]. The method is, in essence, an analogue of Theorem 4.1, but starting from David's model L^d instead of L, and using some additional tricks.

The following is a direct generalization of our own Theorem 4.5.

Theorem 7.2 *Suppose $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$, $\mathbb{P}$ is a real forcing notion and $\mathbb{AP}$ a quasi-amoeba for $\mathbb{P}$, and assume that both $\mathbb{P}$ and $\mathbb{AP}$ are Suslin$^+$ proper. Let $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1\rangle$ be a Suslin$^+$ proper iteration whose iterands are $\mathbb{P}$ and $\mathbb{AP}$ interlaced* (*i.e., the same conditions hold as in Theorem* 4.5). *Then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_4(\mathbb{P})$.*

Proof The proof is exactly the same as that of Theorem 4.5. The reader can verify that every step in that proof is valid if we:

1. Replace $\boldsymbol{\Delta}^1_3$ by $\boldsymbol{\Delta}^1_4$, $\boldsymbol{\Sigma}^1_3$ by $\boldsymbol{\Sigma}^1_4$ and $\boldsymbol{\Pi}^1_2$ by $\boldsymbol{\Pi}^1_3$ everywhere.
2. Use $\boldsymbol{\Sigma}^1_3$-$\mathbb{P}_{\omega_1}$-correctness instead of Shoenfield absoluteness everywhere, which is valid by Theorem 3.20. Notice that in the proof we only used Shoenfield absoluteness between models that lay between V and $V[G_{\omega_1}]$, so by Fact 3.19 we are safe.
3. Use Lemma 3.14 (4) to conclude that $\tilde{\theta}$ is $\boldsymbol{\Pi}^1_3$. □

Just as before, we actually have a stronger version which allows us to mix different partial orders $\mathbb{P}$.

Theorem 7.3 *Suppose* $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$, $\mathbb{P}$ *and* $\mathbb{AP}$ *are as before,* $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ *is Suslin*$^+$ *proper with parameters in the ground model and repetitive, and both* $\mathbb{P}$ *and* $\mathbb{AP}$ *appear cofinally often in the iteration. Then* $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_4(\mathbb{P})$.

Using Theorems 7.1 and 7.3, it follows that we can construct a model for any of the situations from Sect. 5 which was obtained by an application of the first two methods (Theorems 4.1 and 4.8) and *not* by the third method (Theorem 4.10).

Corollary 7.4 *Situations A, B, C, D, E, F and K in the* $\boldsymbol{\Delta}^1_4$*-diagram are consistent relative to ZFC + inaccessible.*

7.2 Separating $\boldsymbol{\Delta}$ from $\boldsymbol{\Sigma}$

Recall that, in the long-run, we would like to find "complete" diagrams on the combined $\boldsymbol{\Delta}^1_n$- and $\boldsymbol{\Sigma}^1_n$-levels, for $n \geq 3$ (cf. Fig. 2). But there are many obstacles, and the most urgent one seems to be the following:

Question 7.5 *Does* $\boldsymbol{\Sigma}^1_n(\mathbb{P}) \iff \boldsymbol{\Delta}^1_n(\mathbb{P})$ *hold for* $\mathbb{P} \in \{\mathbb{L}, \mathbb{M}, \mathbb{S}\}$ *and* $n \geq 3$*?*

Further progress in the study of the joint $\boldsymbol{\Sigma}^1_n/\boldsymbol{\Delta}^1_n$-diagram seems to depend largely on the solution to the above question.

Nevertheless, there are a few interesting things we can prove. Recall that, after Shelah proved that $\boldsymbol{\Sigma}^1_3(\mathbb{B})$ implies an inaccessible in [40], Raisonnier [39] provided an alternative and simpler proof, based on the following:

Definition 7.6 Let $\mathcal{F}$ be a non-principal filter on ω. $\mathcal{F}$ is called a *rapid filter* if

$$\forall c \in [\omega]^\omega\ \exists a \in \mathcal{F}\ \forall n\ (|c(n) \cap a| \leq n),$$

or, equivalently, if $\mathcal{F}$ considered as a subset of $\omega^{\uparrow\omega}$ (the space of strictly increasing functions from ω to ω) is a dominating family in $\omega^{\uparrow\omega}$.

The point is that rapid filters provide natural counterexamples to several regularity properties. The following is a folklore result:

Fact 7.7 *If* $\mathcal{F}$ *is a rapid filter, then* $\mathcal{F}$ *(considered as a subset of* $\omega^{\uparrow\omega}$*) is not measurable and does not have the Baire property.*

Rapid filters also provide counterexamples to the Ramsey and doughnut properties from Sect. 6, albeit using a derived construction:

Definition 7.8 For $a \in [\omega]^\omega$, let

$$\dddot{a} := [0, a(0)) \cup [a(1), a(2)) \cup [a(3), a(4)) \cup \ldots$$

where $\{a(n) \mid n < \omega\}$ is the increasing enumeration of a. If $\mathcal{F}$ is a filter on ω, let $\dddot{\mathcal{F}} := \{a \in [\omega]^\omega \mid \dddot{a} \in \mathcal{F}\}$.

It is clear that the operation $\mathcal{F} \mapsto \dddot{\mathcal{F}}$ does not increase the complexity. In [37] Mathias proved that if $\mathcal{F}$ is a rapid filter then $\dddot{\mathcal{F}}$ does not have the Ramsey property. We improve this result as follows:

Lemma 7.9 *If $\mathcal{F}$ is a rapid filter then $\dddot{\mathcal{F}}$ does not have the doughnut property.*

Proof Let $a, b \in [\omega]^\omega$ be arbitrary and assume $|b \setminus a| = \omega$. It is easy to see that $[a, b]^\omega$ cannot be a subset of $\dddot{\mathcal{F}}$: pick any $x, y \in [a, b]^\omega$ such that $x = y \setminus \{n\}$ for some n. Then, clearly, $\dddot{x}$ and $\dddot{y}$ have finite intersection, so x and y cannot both be in $\dddot{\mathcal{F}}$ (this argument works for any non-principal filter $\mathcal{F}$).
So it remains to show that $[a, b]^\omega$ cannot be completely disjoint from $\dddot{\mathcal{F}}$. Let f be an enumeration of $b \setminus a$. As $\mathcal{F}$ is rapid, there is a $y \in \mathcal{F}$ be such that for all n, $|f(n) \cap y| \leq n$. We will find an $x \in [a, b]^\omega \cap \dddot{\mathcal{F}}$. The real x is constructed as follow: if $i \in a$ then $i \in x$; if $i \notin b$ then $i \notin x$; and if $i \in b \setminus a$, then, whether i is in x or not will depends on the consideration described below (notice that, in any case, x will be a member of $[a, b]^\omega$). For every $n \geq 1$ and every element $y(n)$, there is always at least one member of $b \setminus a$ which lies strictly between $y(n-1)$ and $y(n)$. Let m_n be the largest of them. Now it is easy to see that by making the right choice of either "$m_n \in x$" or "$m_n \notin x$" we can always make sure that $y(n)$ is in $\dddot{x} = [0, x(0)) \cup [x(1), x(2)) \cup \dots$ (it does not matter what we do with the other $i \in b \setminus a$ which lie between $y(n-1)$ and $y(n)$). If we do this for every n, we obtain a set x which is in $[a, b]^\omega$, and moreover, $y \setminus \{y(0)\} \subseteq \dddot{x}$. Since $y \in \mathcal{F}$ holds by assumption, $x \in \dddot{\mathcal{F}}$ follows. □

Raisonnier's proof of Shelah's theorem is based on the following crucial lemma:

Lemma 7.10 (Raisonnier) *Suppose $\omega_1^L = \omega_1$ and $\boldsymbol{\Sigma}^1_2(\mathbb{B})$ holds. Then there exists a $\boldsymbol{\Sigma}^1_3$ rapid filter (the* Raisonnier filter).

Looking at Raisonnier's argument, it is straightforward to obtain the following generalization to higher projective levels:

Lemma 7.11 *Suppose M is a model with a $\boldsymbol{\Sigma}^1_n$-good wellorder of the reals. If $\omega_1^M = \omega_1$ and for every r there is a measure-one set of random reals over $M[r]$, then there exists a $\boldsymbol{\Sigma}^1_{n+1}$ rapid filter.*

We can use Raisonnier's argument to prove the following separation results:

Theorem 7.12 1. *It is consistent relative to ZFC that $\boldsymbol{\Delta}^1_3(\mathbb{P})$ holds for all $\mathbb{P}$ considered in Sects. 5 and 6, but $\boldsymbol{\Sigma}^1_3(\mathbb{B})$, $\boldsymbol{\Sigma}^1_3(\mathbb{C})$, $\boldsymbol{\Sigma}^1_3(\mathbb{R})$ and $\boldsymbol{\Sigma}^1_3(\mathbb{V})$ fail.*
2. *It is consistent relative to ZFC + inaccessible that $\boldsymbol{\Delta}^1_4(\mathbb{P})$ holds for all $\mathbb{P}$ considered in Sects. 5 and 6, but $\boldsymbol{\Sigma}^1_4(\mathbb{B})$, $\boldsymbol{\Sigma}^1_4(\mathbb{C})$, $\boldsymbol{\Sigma}^1_4(\mathbb{R})$ and $\boldsymbol{\Sigma}^1_4(\mathbb{V})$ fail.*

Proof For 1, take the model for Situation K2 in Sect. 6, i.e., $L^{(\mathbb{B}*\mathbb{A}*\mathbb{R}*\mathbb{C})_{\omega_1}}$. Since both antecedents of Lemma 7.10 are satisfied we are done by Fact 7.7 and Lemma 7.9. For 2, take the model $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$, use Theorem 7.3 to obtain $\boldsymbol{\Delta}^1_4(\mathbb{P})$ for all $\mathbb{P}$, and again note that both antecedents of Lemma 7.11 are satisfied for $M = L^d$ and $n = 3$, so again we are done by Fact 7.7 and Lemma 7.9. □

The above result has a number of interesting consequences regarding "non-lifting" of implications that were true on the second level. We had the following non-trivial implications:

1. $\boldsymbol{\Delta}^1_2(\mathbb{C}) + \boldsymbol{\Delta}^1_2(\mathbb{L}) \Rightarrow \boldsymbol{\Sigma}^1_2(\mathbb{C})$ (analogue of Truss, cf. [45]).
2. $\boldsymbol{\Delta}^1_2(\mathbb{R}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{R})$ (Ihoda–Shelah, cf. [23]).
3. $\boldsymbol{\Delta}^1_2(\mathbb{C}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{V})$ (Halbeisen, cf. [20, Lemma 2.1]).

By Theorem 7.12, all of the above fail to lift to the third and fourth levels of the projective hierarchy.

8 Open questions

Although we have made significant progress in this area of research, many questions are still open. The most urgent question seems to be:

Question 8.1 *Is $\boldsymbol{\Delta}^1_3(\mathbb{P}) + \neg\boldsymbol{\Sigma}^1_3(\mathbb{P})$ consistent for $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}\}$? More generally, is $\boldsymbol{\Delta}^1_n(\mathbb{P}) + \neg\boldsymbol{\Sigma}^1_n(\mathbb{P})$ consistent for these $\mathbb{P}$?*

We conjecture that the answer is positive. Recall that in Theorem 7.12 we proved the consistency of $\boldsymbol{\Delta}^1_3(\mathbb{R}) + \neg\boldsymbol{\Sigma}^1_3(\mathbb{R})$ using the Raisonnier filter. It would seem plausible that a similar method will work to settle Question 8.1 as well. In fact, we conjecture the following:

Conjecture 8.2 *It is consistent, relative to ZFC, that $\boldsymbol{\Delta}^1_3(\mathbb{P})$ holds for all $\mathbb{P}$ but $\boldsymbol{\Sigma}^1_3(\mathbb{S})$ fails (and therefore, $\boldsymbol{\Sigma}^1_3(\mathbb{P})$ fails for all $\mathbb{P}$).*

The next question concerns the use of inaccessibles in our proofs. Of course, when proving results about $\boldsymbol{\Sigma}^1_3(\mathbb{P})$ or higher projective sets, inaccessibles cannot be avoided (at least if our proofs are to work uniformly for all $\mathbb{P}$). However, they are not necessary for $\boldsymbol{\Delta}^1_3$-results, and their use in our proofs seems to arise mostly from a lack of finer methods. Therefore we conjecture the following:

Conjecture 8.3 *All the situations on the $\boldsymbol{\Delta}^1_3$-level (specifically Situations* G, H, *and* I *from Sect.* 5 *and* B2, D2, E2, H1, H2, J1 *and* K1 *from Sect.* 6*) have models based just in ZFC.*

The plan would be to improve Theorem 4.10 by replacing the assumption $\forall r\ (\omega_1^{L[r]} < \omega_1)$ by a weaker assumption (for example, about the existence of many generics over $L[r]$) that can be obtained without inaccessibles but is still sufficiently strong to guarantee similar results. Then we can obtain models using this method, starting with some other ZFC-model instead of L^d.

Other questions involve finding complete diagrams for levels beyond $\boldsymbol{\Delta}^1_3$.

Question 8.4

1. *Find a complete diagram for $\boldsymbol{\Delta}^1_3$- and $\boldsymbol{\Sigma}^1_3$-regularity (cf. Fig. 2).*
2. *Find models for Situations G, H, I and J in the $\boldsymbol{\Delta}^1_4$-diagram.*
3. *Find a complete diagram for regularity properties of all projective sets.*

Finally, the following simple questions are well-known, but have, so far, remained unresolved:

Question 8.5 *What is the consistency strength of* $\boldsymbol{\Sigma}^1_3(\mathbb{R})$ *and* $\boldsymbol{\Sigma}^1_3(\mathbb{L})$*?*

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3.4 Measure, Category and projective well-order

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This publication was a collaborative effort, with the three authors having regular meetings as well as many informal meetings between Fischer and Khomskii. While the methods for coding a Δ^1_3-wellorder of the reals were already available in previous work of Fischer, Friedman and Zdomskyy (see Sections 1.2 and 1.3), the main obstacle was re-doing standard iterated forcing constructions while making sure that the coding-forcing would not interfere with the relevant preservation theorems. The precise contributions are:

1. ***Lemma 3.4**: Fischer, with contributions by Khomskii and Friedman*
2. ***Lemma 3.5**: Fischer, with contributions by Khomskii and Friedman*
3. ***Lemma 3.6**: Fischer, with contributions by Khomskii and Friedman*
4. ***Lemma 3.7**: Collaborative effort of all three authors*
5. ***Theorems 4.1 – 4.23**: Collaborative effort of all three authors*

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Measure, category and projective wellorders

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Abstract: We show that each admissible assignment of $\aleph_1$ and $\aleph_2$ to the cardinal invariants in the Cichoń Diagram is consistent with the existence of a projective wellorder of the reals.

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1 Introduction

There are various ways of forcing Δ^1_3 wellorders of the reals. In [13], relying on the method of *almost disjoint coding*, L. Harrington produces a generic extension in which there is a boldface Δ^1_3 wellorder of the reals and MA holds. Similar techniques can be found in J. Bagaria and H. Woodin [2]. Later work by R. David [4] and the second author [10, Theorem 8.52] made use of the method of *Jensen coding* to obtain such wellorders when ω_1 is inaccessible to reals. More recently, the present authors, A. Törnquist and L. Zdomskyy have developed and used further techniques to produce generic extensions in which there are lightface Δ^1_3 wellorders of the reals in the presence of a large continuum, as well as other combinatorial properties hold. For example, in V. Fischer and S. D. Friedman [5] the method of *coding with perfect trees* is used to obtain the consistency of the existence of a lightface Δ^1_3 wellorder on the reals with each of the following inequalities between some of the well-known combinatorial cardinal characteristics of the continuum: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$. In V. Fischer, S. D. Friedman and L. Zdomskyy [7] the method of *almost disjoint coding* is used to show that the existence of a lightface Δ^1_3 wellorder of the reals is consistent with $\mathfrak{b} = \mathfrak{c} = \aleph_3$ and the existence of a Π^1_2 definable ω-mad subfamily of $[\omega]^\omega$. The same method has been used in V. Fischer, S. D. Friedman and A. Törnquist [6] to show the existence of a generic extension in which there is a lightface Δ^1_3 wellorder of the reals, there is a Π^1_2 definable maximal family of orthogonal measures,

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while $\mathfrak{b} = \mathfrak{c} = \omega_3$ and there are no Σ^1_2-definable maximal families of orthogonal measures. The method of *Laver-like almost disjoint coding which strongly preserves splitting reals* is used in V. Fischer, S. D. Friedman and Y. Khomskii [9] to obtain the consistency of a Π^1_1 definable mad family in the presence of a lightface Δ^1_3 wellorder of the reals and $\mathfrak{b} = \mathfrak{c} = \aleph_3$, thus improving some of the results of [7]. In V. Fischer, S. D. Friedman and L. Zdomskyy [8] the method of *specializing Suslin trees* is used to obtain further applications to the combinatorial cardinal characteristics of the continuum, more precisely to obtain the consistency of $\mathfrak{p} = \mathfrak{b} = \aleph_2 < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \aleph_3$ with a lightface Δ^1_3 wellorder, as well as to answer a question of L. Harrington by showing that a lightface Δ^1_3 wellorder of the reals is consistent with MA and $\mathfrak{c} = \aleph_3$. Even though finite support iterations of ccc posets are often preferred, since they can produce for example models with arbitrarily large continuum, there are cases as we will see shortly in which such iterations cannot be used and we must make use of countable support iterations.

In this paper we study the classical cardinal characteristics associated to the ideals of measure and category, and the Cichoń diagram, which completely describes the ZFC inequalities between those characteristics. An excellent introduction to the subject can be found in T. Bartoszynski and H. Judah [3]. We will show that every admissible assignment of $\aleph_1$-$\aleph_2$ to these cardinal characteristics can be realized in a model in which there is a Δ^1_3 wellorder of the reals. The fact that such assignments can be realized in forcing extensions (without the wellorder) is well known (see [3]). Given any such admissible constellation, our strategy will be to provide an iteration of length $\aleph_2$ simultaneously forcing the constellation and the Δ^1_3 wellorder. Note that with every invariant in the Cichoń diagram one can associate a forcing notion which increases its value without affecting the values of the other invariants. Thus to a certain extent the problem of realizing such $\aleph_1$-$\aleph_2$ assignments in a generic extension and adding a projective wellorder to the reals reduces to iterating certain posets, on the one hand posets which control the corresponding invariants and on the other hand posets which provide the wellorder, *without introducing undesirable reals.*

Finite support iterations of ccc posets are known to add Cohen reals. This implies that constellations in which the covering of the meager ideal, $\mathrm{cov}(\mathcal{M})$, has size $\aleph_1$ while $\mathfrak{c} = \aleph_2$ remain beyond the reach of such finite support ccc iterations. If we are to provide indeed a uniform method of adding a projective wellorder, which can be used in all 23 cases which we have to consider, the posets which we iterate to force the wellorder should add no unbounded reals (for constellations in which $\mathfrak{d} = \aleph_1$), no dominating reals (for constellations in which $\mathfrak{b} = \aleph_1$), no Cohen reals (for constellations in which $\mathrm{cov}(\mathcal{M}) = \aleph_1$), no random reals (for constellations in which $\mathrm{cov}(\mathcal{N}) = \aleph_1$), etc.

Furthermore it is well-known that the iterations of posets which do not add a certain type of real, for example dominating reals, might very well add such reals (see U. Abraham [1]). Thus we need a poset with strong combinatorial properties which guarantee not only that the poset but also that its iterations do not add undesirable reals.

To achieve our goal, we use the method of coding with perfect trees. The method was introduced in V. Fischer and S. D. Friedman [5], which to the best knowledge of the authors is the first work discussing cardinal characteristics in the context of projective wellorders of the reals. As shown in [5], the poset of coding with perfect trees $\mathcal{C}(Y)$ is ${}^{\omega}\omega$-bounding and proper (see also Lemma 3.3) and so its countable support iterations preserve the ground model reals as a dominating family. As we will see in this paper, $\mathcal{C}(Y)$ has other strong combinatorial properties which guarantee for example that its iterations do not add Cohen and random reals (see Lemmas 3.4 and 3.6). The fact that the combinatorial properties of the coding with perfect trees poset are strong enough to obtain every admissible constellation is one of the main results of this paper.

Of course there are cases in which other methods can be used as well. For example it is well-known that finite support iterations of σ-centered posets do not add random reals. Relying on this fact, in two instances we provide alternative proofs for obtaining the corresponding admissible assignments in the presence of a Δ^1_3 wellorder using the method of almost disjoint coding (see also [7]). However, we have to point out that whenever we choose to use a different method to force the projective wellorder of the reals, we have to guarantee that the corresponding iteration does not add undesirable reals, and so guarantee that the iterands themselves satisfy a number of strong combinatorial properties. The task of verifying what kind of reals are added by a certain partial order, and what kind of reals are not added is in general highly nontrivial and lies at the heart of many open problems in the field.

The poset which forces the definable wellorder of the reals and is introduced in [5] can be presented in the form $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ where $\mathbb{Q}_\alpha = \mathbb{Q}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ is a two-step iteration: an arbitrary S-proper poset $\mathbb{Q}^0_\alpha$ of size at most $\aleph_1$, for some stationary $S \subseteq \omega_1$ chosen in advance, followed by a three step iteration $\mathbb{Q}^1_\alpha = \mathbb{K}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$. The poset $\mathbb{K}^0_\alpha$ shoots closed unbounded sets through certain components of a countable sequence of stationary sets (see [5, Definition 3]), $\mathbb{K}^1_\alpha$ is a poset known as localization (see [5, Definition 1]), and $\mathbb{K}^2_\alpha$ is the forcing notion for coding with perfect trees (see [5, Definition 3]). The poset $Q(T)$ for shooting a club through a stationary, co-stationary set T is $\omega_1 \backslash T$-proper and ω-distributive. The localization poset $\mathcal{L}(\phi)$ is proper and does not add new reals. The only poset of these three forcing notions which does add a real is the coding with perfect trees partial order. The freedom at each stage α of using an arbitrary S-proper poset $\mathbb{Q}^0_\alpha$ allows us to provide in addition each admissible

$\aleph_1$-$\aleph_2$ assignment to the characteristics in the Cichoń diagram.

The paper is organized as follows: in section 2 we establish the relevant preservation theorems for S-proper rather than proper iterations, in section 3 we study the combinatorial properties of the coding with perfect trees poset $\mathcal{C}(Y)$ and in section 4 we show that each admissible assignment is consistent with the existence of a Δ^1_3-w.o. on $\mathbb{R}$.

2 Preservation theorems

Throughout this section S denotes a stationary subset of ω_1.

For $T \subseteq \omega_1$ a stationary, co-stationary set let $Q(T)$ denote the poset of all countable closed subsets of $\omega_1 \backslash T$ with extension relation given by end-extension. Note that if G is a $Q(T)$-generic set, then $\bigcup G$ is a closed unbounded subset of ω_1 which is disjoint from T. Thus $Q(T)$ destroys the stationarity of T. One of the main properties of $Q(T)$ which will be used throughout the paper is the fact that $Q(T)$ is ω-distributive and so does not add new reals (see T. Jech [15]).

Since $Q(T)$ destroys the stationarity of T, it is not proper. However $Q(T)$ is $\omega_1 \backslash T$-proper.

Definition 2.1 Let $T \subseteq \omega_1$ be a stationary set. A poset $\mathbb{Q}$ is T-proper, if for every countable elementary submodel $\mathcal{M}$ of $H(\Theta)$, where Θ is a sufficiently large cardinal, such that $\mathcal{M} \cap \omega_1 \in T$, every condition $p \in \mathbb{Q} \cap \mathcal{M}$ has an $(\mathcal{M}, \mathbb{Q})$-generic extension q.

The proofs of the following two statements can be found in M. Goldstern [11].

Lemma 2.2 *If $\mathbb{Q}$ is S-proper then $\mathbb{Q}$ preserves ω_1. Also $\mathbb{Q}$ preserves the stationarity of every stationary subset S' of ω_1 which is contained in S.*

Lemma 2.3 *If $\langle\langle \mathbb{P}_\alpha : \alpha \leq \delta\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle\rangle$ is a countable support iteration of S-proper posets then $\mathbb{P}_\delta$ is S-proper.*

The proofs of the following two statements follow very closely the corresponding "proper forcing iteration" case (see [1, Theorem 2.10 and 2.12]).

Lemma 2.4 *Assume CH. Let $\langle \mathbb{P}_\alpha : \alpha \leq \delta\rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper posets of size ω_1. Then $\mathbb{P}_\delta$ is $\aleph_2$-c.c.*

Lemma 2.5 *Assume CH. Let $\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ be a countable support iteration of length $\delta < \omega_2$ of S-proper posets of size ω_1. Then CH holds in $V^{\mathbb{P}_\delta}$.*

Preserving $V \cap 2^\omega$ as a dominating or as an unbounded family: A forcing notion $\mathbb{P}$ is said to be *${}^\omega\omega$-bounding* if the ground model reals $V \cap {}^\omega\omega$ form a dominating family in $V^{\mathbb{P}}$. This property is preserved under countable support iteration of proper forcing notions. A forcing notion $\mathbb{P}$ is said to be *weakly bounding* if the ground model reals $V \cap {}^\omega\omega$ form an unbounded family in $V^{\mathbb{P}}$. In contrast to the ${}^\omega\omega$-bounding property, this property of weak unboundedness is not preserved under countable support iterations of proper posets. There are well-known examples of two-step iterations of weakly bounding posets, which add a dominating real over V (see [1]). An intermediate property, which preserves the ground model reals as an unbounded family in countable support iterations of proper posets, is the almost ${}^\omega\omega$-boundedness. A forcing notion $\mathbb{P}$ is said to be *almost ${}^\omega\omega$-bounding* if for every $\mathbb{P}$-name for a real $\dot{f}$, ie a $\mathbb{P}$-name for a function in ${}^\omega\omega$, and for every condition $p \in \mathbb{P}$, there is a real $g \in {}^\omega\omega \cap V$ such that for every $A \in [\omega]^\omega \cap V$ there is an extension $q \leq p$ such that $q \Vdash \exists^\infty i \in \check{A}(\dot{f}(i) \leq \check{g}(i))$. These are our main tools in providing that the ground model reals remain a dominating or an unbounded family in the various models which we are to consider in section 4.

The proofs of the two preservation theorems below follow very closely the proofs of the classical preservation theorems concerning preservation of the ${}^\omega\omega$-bounding and the almost ${}^\omega\omega$-bounding properties respectively under countable support iterations of proper forcing notions (see [1] or [11]).

Lemma 2.6 *Let $\langle \langle \mathbb{P}_i : i \leq \delta \rangle, \langle \dot{\mathbb{Q}}_i : i < \delta \rangle \rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, ${}^\omega\omega$-bounding posets. That is, assume that for all $i < \delta$, $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is ${}^\omega\omega$-bounding and S-proper". Then $\mathbb{P}_\delta$ is ${}^\omega\omega$-bounding and S-proper.*

Lemma 2.7 *Let $\langle \langle \mathbb{P}_i : i \leq \delta \rangle, \langle \dot{\mathbb{Q}}_i : i < \delta \rangle \rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, almost ${}^\omega\omega$-bounding posets. That is, assume that for all $i < \delta$, $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is almost ${}^\omega\omega$-bounding and S -proper". Then $\mathbb{P}_\delta$ is weakly bounding and S-proper.*

Keeping $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$ *and* $\mathrm{cof}(\mathcal{N})$ *small:* Recall that with every ideal $\mathcal{I}$ on a set X we can associate the following invariants:

- $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$,
- $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$,
- $\mathrm{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq X \text{ and } Y \notin \mathcal{I}\}$, and

- $\text{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall B \in \mathcal{I} \exists A \in \mathcal{A}(B \subseteq A)\}$.

Following standard notation we denote by $\mathcal{M}$ and $\mathcal{N}$ the ideals of meager and null subsets of the real line, respectively. Thus $\text{add}(\mathcal{M})$, $\text{cov}(\mathcal{M})$, $\text{non}(\mathcal{M})$, $\text{cof}(\mathcal{M})$ and $\text{add}(\mathcal{N})$, $\text{cov}(\mathcal{N})$, $\text{non}(\mathcal{N})$, $\text{cof}(\mathcal{N})$ denote the above defined cardinal invariants for the ideals $\mathcal{M}$ and $\mathcal{N}$.

To preserve small witnesses to $\text{non}(\mathcal{M})$, $\text{non}(\mathcal{N})$ and $\text{cof}(\mathcal{N})$ we will use preservation theorems which follow the general framework developed by M. Goldstern in [12].

Definition 2.8 ([3, Definition 6.1.6]) Let $\sqsubseteq$ be the union of an increasing sequence $\langle \sqsubseteq_n \rangle_{n \in \omega}$ of two place relations on ${}^\omega\omega$ such that

- the sets $\mathbb{C} = \text{dom}(\sqsubseteq)$ and $\{f \in {}^\omega\omega : f \sqsubseteq_n g\}$, where $n \in \omega$, $g \in {}^\omega\omega$, are closed and have absolute definitions, that is, as Borel sets they have the same Borel codes in all transitive models.
- $\forall A \in [\mathbb{C}]^{\leq \aleph_0} \exists g \in {}^\omega\omega \forall f \in A(f \sqsubseteq g)$.

Let N be a countable elementary submodel of $H(\Theta)$ for some sufficiently large Θ containing $\sqsubseteq$. We say that $g \in {}^\omega\omega$ *covers* N if $\forall f \in N \cap \mathbb{C}(f \sqsubseteq g)$.

Following [3, Definition 6.1.7], we say that a poset $\mathbb{P}$ *S-almost-preserves-$\sqsubseteq$* iff the following holds: if N is a countable elementary submodel of $H(\Theta)$ for some sufficiently large Θ, containing $\mathbb{P}$, $\mathbb{C}$, $\sqsubseteq$ and $\omega_1 \cap N \in S$, g covers N, and $p \in \mathbb{P} \cap N$, then there is an $(N, \mathbb{P})$-generic condition q extending p such that $q \Vdash$ "g covers $N[\dot{G}]$". Similarly, we say that the forcing notion $\mathbb{P}$ *S-preserves-$\sqsubseteq$* if $\mathbb{P}$ satisfies [3, Definition 6.1.10] with respect only to countable elementary submodels whose intersection with ω_1 is an element of the stationary set S. More precisely, $\mathbb{P}$ *S-preserves-$\sqsubseteq$* if whenever N is a countable elementary submodel of $H(\Theta)$ for some sufficiently large Θ which contains $\mathbb{P}$ and $\sqsubseteq$ as elements and such that $\omega_1 \cap N \in S$, whenever g covers N and $\langle p_n \rangle_{n \in \omega}$ is a sequence of conditions interpreting the $\mathbb{P}$-names $\langle \dot{f}_i \rangle_{i \leq k} \in N$ for functions in $\mathbb{C}$ as the functions $\langle f_i^* \rangle_{i \leq k}$, then there is an N-generic condition $q \leq p_0$ such that $q \Vdash_{\mathbb{P}}$ "g covers $N[\dot{G}]$" and

$$\forall n \in \omega \forall i \leq k \; q \Vdash_{\mathbb{P}} (f_i^* \sqsubseteq_n g \to \dot{f}_i \sqsubseteq_n g).$$

Furthermore we obtain the following analogue of Goldstern's preservation theorem (see [12] or [3, Theorem 6.1.3]).

Theorem 2.9 *Let S be a stationary set and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ be a countable support iteration such that for all $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves-$\sqsubseteq$". Then $\mathbb{P}_\delta$ S-preserves-$\sqsubseteq$.*

Of particular interest for us are the relations $\sqsubseteq^{random}$, $\sqsubseteq^{Cohen}$ and $\sqsubseteq^{\Delta}$ defined in Definitions 6.3.7, 6.3.15, and on page 303, respectively, of [3]. For convenience of the reader we define these relations below:

$\sqsubseteq^{random}$: Denote by Ω the set of all clopen subsets of 2^ω. Then let

$$\mathbb{C}^{\text{random}} = \{f \in \Omega^\omega : \forall n \in \omega(\mu(f(n)) \leq 2^{-n})\}$$

and for $f \in \mathbb{C}^{\text{random}}$ let $A_f = \bigcap_{n\in\omega} \bigcup_{k\geq n} f(k)$. Now for $f \in \mathbb{C}^{\text{random}}$, $x \in 2^\omega$ and $n \in \omega$ define

$$f \sqsubseteq_n^{\text{random}} x \iff \forall k \geq n(x \notin f(k)).$$

Let $\sqsubseteq^{random} = \bigcup_{n\in\omega} \sqsubseteq_n^{random}$. Note that $f \sqsubseteq^{\text{random}} x$ if and only if $x \notin A_f$ and that x covers N with respect to $\sqsubseteq^{\text{random}}$ if and only if x is random over N.

$\sqsubseteq^{Cohen}$: Let

$$\mathbb{C}^{\text{Cohen}} = \{f \in \Omega^\Omega : \forall U \in \Omega(f(U) \subseteq U)\}.$$

For $f \in \mathbb{C}^{\text{Cohen}}$ let $A_f := \bigcup_{U\in\Omega} f(U)$. Note that A_f is an open dense subset of 2^ω and that for every dense open set $H \subseteq 2^\omega$ there is an $f \in \mathbb{C}^{\text{Cohen}}$ such that $A_f \subseteq H$. Fix some standard enumeration $\{U_n\}_{n\in\omega}$ of Ω and for $f \in \mathbb{C}^{\text{Cohen}}$, $x \in 2^\omega$, $n \in \omega$ define:

$$f \sqsubseteq_n^{\text{Cohen}} x \iff \exists k \leq n(x \in f(U_k)).$$

Let $\sqsubseteq^{Cohen} = \bigcup_{n\in\omega} \sqsubseteq_n^{Cohen}$. Then $f \sqsubseteq^{Cohen} x$ if and only if $x \in A_f$. Therefore x covers N with respect to $\sqsubseteq^{Cohen}$ if and only if x is a Cohen real over N.

$\sqsubseteq^{\Delta}$: Let $\mathbb{Q}_+ = \mathbb{Q} \cap [0, 1]$, let $\Delta = \{f \in \mathbb{Q}_+^\omega : \sum_{n\in\omega} f(n) < 1\}$ and let

$$\mathbb{C}^{\Delta} := \{f \in ((\mathbb{Q}_+)^{<\omega})^\omega : \forall n \sum_{i\in\text{dom}(f(n))} f(n)(i) < 2^{-(n+1)}\}.$$

For $f \in \mathbb{C}^\Delta$ let $\epsilon_f \in \Delta$ be defined by $\epsilon_f = f(0)^\frown f(1)^\frown \cdots$. For $f, g \in \mathbb{C}^\Delta$ define

$$f \sqsubseteq_n^{\Delta} g \iff \forall m \geq n(\epsilon_f(m) \leq \epsilon_g(m)).$$

Let $\sqsubseteq^{\Delta} = \bigcup_{n\in\omega} \sqsubseteq_n^{\Delta}$.

Each of those relations satisfies the properties of Definition 2.8. Thus Theorem 2.9 implies the following two theorems (analogous to Theorems 6.1.13 and 6.3.20, respectively, from [3]).

Theorem 2.10 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ is a countable support iteration and for each $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves- $\sqsubseteq^{random}$ ", then $\mathbb{P}_\delta$ preserves outer measure. That is for every set $A \subseteq 2^\omega$, $V^{\mathbb{P}_\delta} \models \mu^*(A) = \mu^*(A)^V$. In particular $\Vdash_\delta V \cap 2^\omega \notin \mathcal{N}$.*

Theorem 2.11 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ is a countable support iteration and for each $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves- $\sqsubseteq^{Cohen}$ ", then $\mathbb{P}_\delta$ preserves non meager sets. That is for every set $A \subseteq 2^\omega$ which is not meager, $V^{\mathbb{P}_\delta} \models A$ is not meager. In particular $\Vdash_\delta V \cap 2^\omega \notin \mathcal{M}$.*

Recall that a forcing notion $\mathbb{P}$ has the *Sacks property* if and only if for every $\mathbb{P}$-name $\dot{g}$ for a function in ${}^\omega\omega$ there is a slalom $S \in V$, ie a function $S \in ([\omega]^{<\omega})^\omega$ such that $|S(n)| \leq 2^n$ for all n, and such that $\Vdash_{\mathbb{P}}$ "$\forall n(\dot{g}(n) \in S(n))$". By [3, Lemma 6.3.39] a proper forcing notion $\mathbb{P}$ has the Sacks property if and only if $\mathbb{P}$ preserves $\sqsubseteq^\Delta$. By [3, Theorem 2.3.12] if $\mathbb{P}$ has the Sacks property then every measure zero set in $V^{\mathbb{P}}$ is covered by a Borel measure zero set in V and so $\mathbb{P}$ preserves the base of the ideal of measure zero sets. We obtain the following analogue of [3, Theorem 6.3.40].

Theorem 2.12 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ is a countable support iteration and for each $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves- $\sqsubseteq^\Delta$ ", then $\mathbb{P}_\delta$ has the Sacks property and so preserves the base of the ideal of measure zero sets.*

No random and no amoeba reals: Some of the preservation theorems which we use to show that certain iterations do not add amoeba or random reals, are based on a general framework due to H. Judah and M. Repický [14].

Definition 2.13 ([3, Definition 6.1.17]) Let $\sqsubseteq$ be the union of an increasing chain $\langle \sqsubseteq_n \rangle_{n \in \omega}$ of two place relations on ${}^\omega\omega$ such that

- for all $n \in \omega$ and all $h \in {}^\omega\omega$ the set $\{x : h \sqsubseteq_n x\}$ is relatively closed in the range of $\sqsubseteq$,
- for every $A \in [\mathrm{dom}(\sqsubseteq)]^{\leq \aleph_0}$ there is $f \in \mathrm{dom}(\sqsubseteq)$ such that $\forall g \in A \forall n \in \omega \exists k \geq n$ such that $\forall x(f \sqsubseteq_k x) \rightarrow g \sqsubseteq_k x)$, and
- the formula $\forall x \in {}^\omega\omega(f \sqsubseteq_n x \rightarrow g \sqsubseteq_n x)$ is absolute for all transitive models containing f and g.

A real x is said to be $\sqsubseteq$*-dominating* over V if for all $y \in V \cap \mathrm{dom}(\sqsubseteq)$, $y \sqsubseteq x$.

We have the following S-proper analogue of Judah and Repický's preservation theorem (see [3, Theorem 6.1.18]).

Theorem 2.14 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$, δ limit, is a countable support iteration of S-proper posets, such that for all $\alpha < \delta$, $\mathbb{P}_\alpha$ does not add a $\sqsubseteq$-dominating real, then $\mathbb{P}_\delta$ does not add a $\sqsubseteq$-dominating real.*

Note that $x \in 2^\omega$ $\sqsubseteq^{\text{random}}$-dominates V if and only if x is random over V. Furthermore the relation $\sqsubseteq^{\text{random}}$ satisfies the conditions of definition 2.13 and so by the above theorem we obtain the following S-proper analogue of Theorem 6.3.14 from [3].

Theorem 2.15 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$, δ limit, is a countable support iteration of S-proper forcing notions and for each $\alpha < \delta$, $\mathbb{P}_\alpha$ does not add random reals, then $\mathbb{P}_\delta$ does not add a random real.*

Note that $\sqsubseteq^\Delta$ also satisfies the conditions of Definition 2.13. Then by Theorem 2.14 above, as well as [3, Theorem 2.3.12] we obtain the following analogue of [3, Theorem 6.3.41].

Theorem 2.16 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$, δ limit, is a countable support iteration of S-proper posets and for all $\alpha < \delta$, $\Vdash_\alpha$ "$\bigcup(\mathcal{N} \cap V) \notin \mathcal{N}$", then $\Vdash_\delta$ "$\bigcup(\mathcal{N} \cap V) \notin \mathcal{N}$".*

Other preservation theorems: We say that a forcing notion $\mathbb{P}$ is S-(f,h)-*bounding*, if it satisfies [3, Definition 7.2.13] but instead of proper we require that $\mathbb{P}$ is S-proper. That is, we say that $\mathbb{P}$ *is* S-(f,h)-*bounding*, if $\mathbb{P}$ is S-proper, for every $k \in \omega$ $\lim_{n\to\infty} h(n)^k \cdot f^{-1}(n) = 0$ and for every $f' \in V^{\mathbb{P}} \cap \prod_{n\in\omega} f(n)$ there is $S \in V \cap ([\omega]^{<\omega})^\omega$ such that for all $n \in \omega$ $|S(n)| \leq h(n)$ and for all $n \in \omega (f'(n) \in S(n))$. The proof of [3, Lemma 7.2.15] remains true under this modification, and so we obtain that if $\mathbb{P}$ is S-(f,h)-bounding then $\mathbb{P}$ does not add random or Cohen reals. Furthermore we have the following analogue of Shelah's theorem (see S. Shelah [16] or T. Bartoszynski and H. Judah [3, Theorem 7.2.19]).

Theorem 2.17 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$, δ limit, is a countable support iteration such that for all α, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is S-(f,h)-bounding", then $\mathbb{P}_\delta$ is S-(f,h)-bounding.*

We will also use preservation theorems for the so called (F,g)-preserving posets. For convenience of the reader we state the definition of (F,g)-preserving (see [3, Definition 7.2.23]). Let g be a given real and for $n \in \omega$ let $P_n = \{a \subseteq g(n+1) : |a| = g(n+1)/2^n\}$. For a set $A \subseteq P_n$ define $\text{norm}(A) = \min\{|X| : \forall a \in A (X \nsubseteq a)\}$. Let F be a family of strictly increasing functions. For every $f \in F$ choose a function $f^+ \in F$ and assume that for all $f \in F$, $n \in \omega$ we have that $f(n) < g(n)/2^n$. A forcing notion $\mathbb{P}$ is said to be (F,g)-*preserving* if for every $f \in F$ and every $\mathbb{P}$-name

$\dot{S}$ which has the property that for all n, $\Vdash_{\mathbb{P}} \dot{S}(n) \subseteq P_n$ and $\Vdash_{\mathbb{P}} \text{norm}(\dot{S}(n)) < f(n)$, there exists a function $T \in V$ such that for all n, $T(n) \subseteq P_n$, $\text{norm}(T(n)) < f^+(n)$ and $\Vdash_{\mathbb{P}} \dot{S}(n) \subseteq T(n)$. Note that the countable support iteration of (F, g)-preserving posets is (F, g)-preserving (see [3, Theorem 7.2.29]) and that (F, g)-preserving posets do not add Cohen reals (see [3, Theorem 7.2.24]).

3 Coding with perfect trees

Let $Y \subseteq \omega_1$ be such that in $L[Y]$ cofinalities have not been changed, and let $\bar{\mu} = \{\mu_i\}_{i \in \omega_1}$ be a sequence of L-countable ordinals such that μ_i is the least ordinal μ with $\mu > \bigcup\{\mu_j : j < i\}$, $L_\mu[Y \cap i] \models \text{ZF}^-$ and $L_\mu \models$ "ω is the largest cardinal". A real r is said to *code* Y below i if for all $j < i$, $j \in Y$ if and only if $L_{\mu_j}[Y \cap j, r] \models \text{ZF}^-$. Whenever T is a perfect tree, let $|T|$ be the least i such that $T \in L_{\mu_i}[Y \cap i]$.

Fix $L[Y]$ as the ground model. The poset $\mathcal{C}(Y)$, to which we refer as *coding with perfect trees*, consists of all perfect trees $T \subseteq 2^{<\omega}$ such that every branch r through T codes Y below $|T|$. For T_0, T_1 conditions in $\mathcal{C}(T)$ define $T_0 \leq T_1$ if and only if T_0 is a subtree of T_1.[1]

Below we summarize some of the main properties of the poset $\mathcal{C}(Y)$. Note that $T_0 \leq T_1$ if and only if $[T_0] \subseteq [T_1]$, where $[T]$ denotes the set of infinite branches through T. For $n \in \omega$, let $T_0 \leq_n T_1$ if and only if $T_0 \leq T_1$ and T_0, T_1 have the same first n splitting levels. (For the notion of n-splitting level of a tree see for example [15].) For T a perfect tree and $m \in \omega$ let $S_m(T)$ be the set of nodes on the m-splitting level of T (and so $|S_m(T)| = 2^m$), and for $t \in T$ let $T(t) = \{\eta \in T : t \subseteq \eta \text{ or } \eta \subseteq t\}$. Note that by Π^1_1 absoluteness, r codes Y below $|T|$ even for branches through T in the generic extension.

Lemma 3.1 *[5, Lemma 5] If $T \in \mathcal{C}(Y)$ and $|T| \leq i < \omega_1$, then there is $T^* \leq T$ such that $|T^*| = i$.*

Lemma 3.2 *[5, Lemma 6] If G is $\mathcal{C}(Y)$-generic and $\{R\} = \bigcap\{[T] : T \in G\}$, then for all $j < \omega_1$ we have that*

$$j \in Y \text{ if and only if } L_{\mu_j}[Y \cap j, R] \models \text{ZF}^-.$$

That is, R codes Y.

[1] $\mathcal{C}(Y)$ is non-empty, since the full tree $2^{<\omega}$ belongs to it.

Lemma 3.3 *[5, Lemmas 7 and 8] $\mathcal{C}(Y)$ is a proper, ${}^{\omega}\omega$-bounding forcing notion.*

By [3, Lemma 2.2.4] for every meager set $F \subseteq 2^{\omega}$ there are reals $x_F \in 2^{\omega}$ and $f_F \in \omega^{\omega}$ such that

$$F \subseteq \{x : \forall^{\infty} n \exists i \in [f_F(n), f_F(n+1)) x_F(i) \neq x(i)\}.$$

We will refer to x_F and f_F as representatives of the meager set F.

Lemma 3.4 *The coding with perfect trees forcing notion $\mathcal{C}(Y)$ preserves $\sqsubseteq^{Cohen}$.*

Proof Let N be a countable elementary submodel of $L_{\Theta}[Y]$ for some sufficiently large Θ, such that $\mathcal{C}(Y)$, $\bar{\mu}$ are elements of N. Let c be a Cohen real over N. Let T be a condition in $\mathcal{C}(Y) \cap N$. It is enough to show that there is a condition T^* which is a $(N, \mathcal{C}(Y))$-generic extension of T and which forces that "c is Cohen over $N[\dot{G}]$".

Let $\{\dot{x}_n, \dot{f}_n\}_{n \in \omega}$ and $\{D_n\}_{n \in \omega}$ enumerate names for representatives of all meager sets in $N^{\mathcal{C}(Y)}$ and all dense subsets of $\mathcal{C}(Y)$ in N, respectively. Let $\overline{N}$ denote the transitive collapse of N, let $i = \omega_1 \cap N$. Note that $\overline{N} = L_{\mu}[Y \cap i]$ for some μ and since $L_{\mu_i}[Y \cap i] \models$ "i is countable", we have that $L_{\mu}[Y \cap i]$ is an element of $L_{\mu_i}[Y \cap i]$. Let $\bar{\imath} = \{i_k\}_{k \in \omega}$ be an increasing cofinal sequence in i such that $\bar{\imath} \in L_{\mu_i}[Y \cap i]$. Recursively we will define a sequence of conditions $\tau = \{T_n\}_{n \in \omega}$, such that for every n, the condition T_n is an element of N, $T_{n+1} \leq_{n+1} T_n$, $|T_n| \geq i_n$ and

(1) $T_{2n} \Vdash_{\mathcal{C}(Y)}$ "$c \notin F(\dot{x}_n, \dot{f}_n)$", where $F(\dot{x}_n, \dot{f}_n)$ denotes a name for the meager set corresponding to the names $\dot{x}_n, \dot{f}_n$,

(2) $T_{2n+1} \Vdash_{\mathcal{C}(Y)}$ "$\dot{G} \cap N \cap D_n \neq \emptyset$", where $\dot{G}$ is the canonical $\mathcal{C}(Y)$-name for the generic filter.

Furthermore the entire sequence τ will be an element of $L_{\mu_i}[Y \cap i]$, since it will be definable in $L_{\mu_i}[Y \cap i]$. Thus its fusion T^* will also be an element of $L_{\mu_i}[Y \cap i]$, and so a condition in $\mathcal{C}(Y)$ which extends T and has the desired properties.

We will need the following two claims:

Claim *Let $R \in \mathcal{C}(Y) \cap N$ and let $\{\dot{x}, \dot{f}\}$ be $\mathcal{C}(Y)$-names in N (for reals), representing a meager set in $N^{\mathcal{C}(Y)}$, let $n \in \omega$ and let $\alpha \in N \cap \omega_1$ such that $\alpha > |R|$. Then there is a condition R' in N such that $R' \leq_n R$, $|R'| \geq \alpha$ and every branch through R' decides $\dot{x}, \dot{f}$.*

Proof Let N_0 be a sufficiently elementary submodel of N such that $N \models$ "N_0 is countable" and all relevant parameters are elements of N_0, that is R, $\mathcal{C}(Y)$, $\bar{\mu}$, $\dot{f}$, $\dot{x}$, n and α are elements of N_0. Let $\overline{N_0}$ denote the transitive collapse of N_0 and let $j = \omega_1 \cap N_0$. Note that $\overline{N}_0$ is of the form $L_\mu[Y \cap j]$ for some μ, and since $L_\mu[Y \cap j] \models$ "j is uncountable" and $L_{\mu_j}[Y \cap j] \models$ "j is countable" we have that $\overline{N}_0 = L_\mu[Y \cap j] \in L_{\mu_j}[Y \cap j]$. On the other hand, since $L_{\mu_j}[Y \cap j]$ is definable from Y, j, and μ_j, and all of those are in N, we obtain that $L_{\mu_j}[Y \cap j] \in N$. Let $\bar{j} = \{j_m\}_{m \in \omega}$ be an increasing cofinal in j sequence, which is an element of $L_{\mu_j}[Y \cap j]$.

The condition R' will be obtained as the fusion of a sequence $\langle R_m \rangle_{m \in \omega}$ such that the entire sequence is definable in $L_{\mu_j}[Y \cap j]$ and for all m, $R_m \in N_0$ (and so $R_m \in \overline{N}_0$). Let $R_0 = R$. For every $s \in \mathrm{Split}_n(R_0)$ and every $t \in \mathrm{Succ}_s(R_0)$ find $R^0_t \leq R_0(t)$ which decides $\dot{x} \restriction |t|$ and $\dot{f} \restriction |t|$. By elementarity we can assume that $R^0_t \in N_0$ and so $R^0_t \in \bar{N}_0$. Since the set of conditions in $\mathcal{C}(Y)$ of height strictly greater than α and j_0 is dense, again by elementarity we can assume that $|R^0_t| > \alpha, j_0$. Let $R_1 = \bigcup_{s \in \mathrm{Split}_n(R_0)} \bigcup_{t \in \mathrm{Succ}_t(R_0)} R^0_t$. Then in particular $R_1 \in N_0$ and $|R_1| > \alpha, i_0$. Now suppose $R_m \in N_0$ is defined. Then for every $s \in \mathrm{Split}_{n+m}(R_m)$ and $t \in \mathrm{Succ}_s(R_m)$ find $R^m_t \leq R_m(t)$ in $\bar{N}_0$ of height $> \alpha, j_m$, which decides $\dot{x} \restriction |t|$, $\dot{f} \restriction |t|$. Let $R_{m+1} = \bigcup_{s \in \mathrm{Split}_{n+m}(R_m)} \bigcup_{t \in \mathrm{Succ}(s)} R^m_t$. Then $R_{m+1} \leq_{m+n} R_m$, $R_{m+1} \in N_0$ and $|R_{m+1}| > \alpha, j_m$. With this the inductive construction of the fusion sequence is complete. Since $\langle R_m \rangle_{m \in \omega}$ is definable in $L_{\mu_j}[Y \cap j]$, we obtain that $R' = \bigcap_{m \in \omega} R_m \in L_{\mu_j}[Y \cap j]$. Then in particular $|R'| = j$, which implies that R' is indeed a condition in $\mathcal{C}(Y)$. □

Claim *Let R', $\dot{x}$, $\dot{f}$, n, α, N be as above and let c be a Cohen real over N. Then there is a condition $R'' \in N$ such that $R'' \leq_n R'$, $|R''| \geq \alpha, |R'|$ and R'' forces that c does not belong to the meager set determined by $\dot{x}, \dot{f}$.*

Proof Just as in the previous claim let N_0 be a sufficiently elementary submodel of N such that $N \models$ "N_0 is countable" and all relevant parameters are elements of N_0. Let $\overline{N_0}$ denote the transitive collapse of N_0. Let $j = \omega_1 \cap N_0$ and let $\bar{j} = \{j_m\}_{m \in \omega}$ be an increasing and cofinal in j sequence which is an element of $L_{\mu_j}[Y \cap j]$. The condition R'' will be obtained as the limit of a fusion sequence $\langle R_m \rangle_{m \in \omega}$ which is definable in $L_{\mu_j}[Y \cap j]$ and whose elements are in N_0. Let $R_0 = R'$. For every $s \in \mathrm{Split}_n(R_0)$ and every $t \in \mathrm{Succ}_t(R_0)$ find a branch $b_t \in N_0 \cap [R_0]$ such that $t \subseteq b_t$. Then b_t gives an interpretation of the names $\dot{x}, \dot{f}$ as reals x^t and f^t in N_0. Since c is Cohen over N, it is Cohen over N_0 and so there is $j_t > |t|$ such that

$$x^t_n \restriction [f^t(j_t), f^t(j_t + 1)) = c \restriction [f^t(j_t), f^t(j_t + 1)).$$

Take any $k_t > j_t$. Let $R_1 = \bigcup_{s\in \mathrm{Split}_n(R_0)} \bigcup_{t\in \mathrm{Succ}_s(R_0)} R_0(b_t\restriction k_t)$. Thinning out once again we can assume that $|R_0(b_t\restriction k_t)| > j_0, \alpha$. Also, clearly $R_1 \in N_0$.

Suppose R_m is defined. Again, for every $s \in \mathrm{Split}_{n+m}(R_m)$ and $t \in \mathrm{Succ}_s(R_m)$ find a branch $b_t \in [R_m]\cap N_0$ such that $t \subseteq b_t$. Then b_t gives an interpretation x^t, f^t of $\dot{x}, \dot{f}$ as reals x^t, f^t in N_0. Using the fact that c is Cohen over N_0 we can find $\{l^t_a\}_{1\leq a\leq m}$ such that $|t| < l^t_1$, $l^t_a < l^t_{a+1}$ for $a < m$ such that for every $j \in \{l^t_a\}_{1\leq a\leq m}$,

$$x^t\restriction[f^t(j), f^t(j+1)) = c\restriction[f^t(j), f^t(j+1)).$$

Take any $k_t > l^t_m$. Let $R_{m+1} = \bigcup_{s\in \mathrm{Split}_{n+m}(R_m)} \bigcup_{t\in \mathrm{Succ}_s(R_m)} R_m(b_t\restriction k_t)$. Passing to an extension if necessary we can assume that $|R_m(b_t\restriction k_t)| > j_m, \alpha$ and so that $|R_{m+1}| > j_m, \alpha$. Let $R'' = \cap_{m\in\omega} R_m$. Then R'' is a condition in N with the desired properties. □

With this we can proceed with the construction of the fusion sequence $\langle T_n\rangle_{n\in\omega}$. Let $T_0 = T$. Reproducing the proof of [5, Lemma 7] find $T_1 \in N$ such that $T_1 \leq_1 T_0$, $|T_1| \geq i_1$ and $T_1 \Vdash \dot{G}\cap N\cap D_1 \neq \emptyset$. Suppose T_{2n-1} is defined for some $n \geq 1$. Using the previous two claims find a condition $T_{2n} \in N\cap \mathcal{C}(Y)$ such that $|T_{2n}| \geq i_{2n}$, $T_{2n} \leq_{2n} T_{2n-1}$, and T_{2n} forces that c does not belong to the meager set corresponding to $\{\dot{x}_n, \dot{f}_n\}$. Obtain T_{2n+1} as in the base case. With this the fusion sequence $\langle T_n\rangle_{n\in\omega}$ is defined. Let $T^* = \bigcap_{n\in\omega} T_n$. Note that $|T^*| = i$ and so in particular $T \in \mathcal{C}(Y)$. Clearly, T^* is $(N, \mathcal{C}(Y))$-generic and $T^* \Vdash_{\mathcal{C}(Y)}$ "c is Cohen over $N[\dot{G}]$". □

In order to show that the coding with perfect trees forcing notion preserves $\sqsubseteq^{\text{random}}$, we will use the fact that $\mathcal{C}(Y)$ is weakly bounding and that $\mathcal{C}(Y)$ preserves positive outer measure (see below).

Lemma 3.5 *Suppose that A is a set of positive outer measure. Then* $\Vdash_{\mathcal{C}(Y)} \mu^*(A) > 0$.

Proof Suppose not. Then there is a condition $T \in \mathcal{C}(Y)$ such that $T \Vdash \mu^*(A) = 0$. Let N be a countable elementary submodel of $L_\Theta[Y]$ for some sufficiently large Θ such that $T, \mathcal{C}(Y), A$ are elements of N. Then there is a sequence $\langle \dot{I}_n\rangle_{n\in\omega} \in N$ of names for rational intervals such that $T \Vdash \lim_{m\to\infty} \sum_{n>m} \mu(\dot{I}_n) = 0$ and $T \Vdash A \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} \dot{I}_m$. Then in particular, there is a $\mathcal{C}(Y)$-name for a function $\dot{g}$ in ${}^\omega\omega$ such that for all n, $T \Vdash \sum_{m\geq \dot{g}(n)} \mu(\dot{I}_m) < 2^{-(n^2+n)}$. Since $\mathcal{C}(Y)$ is ${}^\omega\omega$-bounidng (see Lemma 3.3), there is $R \leq T$ and a ground model real g, ie function in ${}^\omega\omega$ such that for all $n \in \omega$, $R \Vdash \dot{g}(n) < \check{g}(n)$. Then in particular, for all $n \in \omega$, $R \Vdash \sum_{g(n)\leq i<g(n+1)} \mu(\dot{I}_i) < 2^{-(n^2+n)}$. Let $i = \omega_1 \cap N$ and let $\bar{\imath} = \{i_n\}_{n\in\omega}$ be an increasing and cofinal in i sequence, which belongs to $L_{\mu_i}[Y \cap i]$. Recursively define a fusion sequence $\langle R_n \rangle_{n\in\omega}$ as follows. Let $R_0 = R$. Suppose R_n has been defined. For every n-splitting node t of R_n find $R_t \leq R_n(t)$ such that for some finite sequence $\langle I^n_{t,j} \rangle_{g(n)\leq j<g(n+1)}$ of rational intervals, for all $j : g(n) \leq j < g(n+1)$ we have $R_t \Vdash \dot{I}_j = \check{I}^n_{t,j}$. By elementarity we can assume that R_t is a condition which is an element of N which is also of height $\geq i_n$, and that $\langle I^n_{t,j} \rangle_{g(n)\leq j<g(n+1)} \in N$. Let $R_{n+1} = \bigcup_{t\in \mathrm{Split}_n(R_n)} R_t$ and let $J_n = \bigcup_{t\in \mathrm{Split}_n(R_n)} \bigcup_{g(n)\leq j<g(n+1)} I^n_{t,j}$. Note that $J_n \in N$ and $\mu(J_n) < 2^{-n}$. Let R^* be the fusion of the sequence $\langle R_n \rangle_{n\in\omega}$. Then R^* is a condition in $\mathcal{C}(Y)$ of height i, such that

$$R^* \Vdash \bigcap_n \bigcup_{m\geq n} \dot{I}_m \subseteq \bigcap_n \bigcup_{m\geq n} J_m.$$

Since $J := \bigcap_n \bigcup_{m\geq n} J_m$ is a measure zero set, there is $x \in A\backslash J$. However

$$R^* \Vdash x \in \bigcap_n \bigcup_{m\geq n} \dot{I}_m$$

and so $R^* \Vdash x \in J$, which is a contradiction. □

Lemma 3.6 *The coding with perfect trees forcing notion $\mathcal{C}(Y)$ preserves $\sqsubseteq^{\mathrm{random}}$.*

Proof The proof proceeds similarly to the proof that Laver forcing preserves $\sqsubseteq^{\mathrm{random}}$ (see [3, Theorem 7.3.39]). Let N be a countable elementary submodel of $L_\Theta[Y]$ for some sufficiently large Θ, let $\dot{f}_0$ be an element of $\dot{\mathbb{C}}^{\mathrm{random}} \cap N$, and let $\tau = \langle T_n \rangle_{n\in\omega} \in N$ be an approximating sequence for $\dot{f}_0$ below T for some $T \in \mathcal{C}(Y) \cap N$. Let f^*_0 be the approximation of $\dot{f}_0$ determined by τ. Note that $f^*_0 \in N \cap {}^\omega\Omega$. Let x be a random real over N. We have to show that there is an extension T^* of T which is an $(N, \mathcal{C}(Y))$-generic condition, such that $T^* \Vdash$ "x is random over $N[\dot{G}]$" and such that for all $n \in \omega$, $T^* \Vdash (f^*_0 \sqsubseteq_n x \to \dot{f}_0 \sqsubseteq_n x)$.

Let D be a dense open subset of $\mathcal{C}(Y)$. Denote by $\mathrm{cl}(D) = \{T : \exists n \forall t \in \mathrm{Split}_{\geq n}(T)$ (if there is $R_t \leq_0 T(t)$ such that $R_t \in D$ then $T(t) \in D)\}$. Note that for every $n \in \omega$, $\mathrm{cl}(D)$ is n-dense (ie dense with respect to $\leq_n$) and open. Thus if $\{D_n\}_{n\in\omega}$ is a sequence of dense open sets, then $\bigcap_{n\in\omega} \mathrm{cl}(D_n)$ is n-dense for all n. Also, we have that if $S \leq T \in \mathrm{cl}(D)$, then there is $s \in S$ such that $T(s) \in D$.

Let $\mathcal{D}$ denote the collection of all dense subsets of $\mathcal{C}(Y)$ which are in N. Since x is random over N and $f^*_0 \in N$ there is n_0 such that for all $k \geq n_0$, $x \notin f^*_0(k)$. For every $n \geq n_0$ let Y^n_n be the set of all reals $z \in 2^\omega$ such that there is $Z \leq T_n$ such that $\phi_n(z, Z)$ holds, where $\phi_n(z, Z)$ is the conjunction of the following three formulas:

(1) $\phi_1(Z) \equiv$ for all $D \in \mathcal{D} \cap N \exists R \in \text{cl}(D) \cap N(Z \leq R)$, and

(2) $\phi_2(z, T) \equiv$ for all $\dot{f} \in \dot{\mathbb{C}}^{\text{random}} \cap N \forall^\infty n(Z \Vdash z \notin \dot{f}(n))$,

(3) $\phi_3^n(z, T) \equiv$ for all $k \geq n$, $Z \Vdash z \notin \dot{f}_0(k)$.

Note that $Z \nVdash z \notin \dot{f}(n)$ iff there is $Z' \leq Z$ such that $Z' \Vdash z \in \dot{f}(n)$ iff there is $Z' \leq Z$ such that $z \in \dot{f}(n)[Z']$ which is equivalent to there is $s \in Z$ such that $z \in \dot{f}(n)[Z_s]$ iff there is $R \in \text{cl}(D_n^{\dot{f}}) \cap N$ and there is $s \in Z$ such that $Z \geq R$ and $z \in \dot{f}(n)[R_s]$. Since the quantifiers of ϕ_1, ϕ_2, ϕ_3 are relativized to subsets of N, all three of these formulas are Borel.

For a partial order $\mathbb{P}$ and $p \in \mathbb{P}$ let $\mathbb{P}(p) = \{q \in \mathbb{P} : q \leq p\}$. Recall that a forcing notion $\mathbb{P}$ is weakly homogenous if for every $p, q \in \mathbb{P}$ there are $p' \leq p$ and $q' \leq q$ such that $\mathbb{P}(p') \cong \mathbb{P}(q')$. To see that $\mathcal{C}(Y)$ is weakly homogeneous consider arbitrary T_0 and T_1 in $\mathbb{P}$. Without loss of generality $|T_0| \leq |T_1|$. The properties of $\mathcal{C}(Y)$ imply that T_0 has an extension T_0' such that $|T_0'| = |T_1|$. Then the order preserving bijection between T_0' and T_1 extends to a partial order isomorphism between $\mathcal{C}(Y)(T_0')$ and $\mathcal{C}(Y)(T_1)$, and so $\mathcal{C}(Y)$ is weakly homogenous. Now using this fact and the fact that $\mathcal{C}(Y)$ preserves positive outer measure (see Lemma 3.5), one can easily modify the proof of [3, Lemma 7.3.41] to obtain that for every $n \geq n_0$, the inner measure $\mu_*(Y_n^n) \geq 1 - 2^{-n}$. This implies that $Y^* := \bigcup_{n \geq n_0} Y_n^n$ is a set of measure 1.

Claim *(see [3, Lemma 7.3.42]) There is a sequence $\langle B_k : k \geq n_0 \rangle \in N$ of Borel sets such that for all n, $B_n \in N$ and $B_n \triangle Y_n^n \subseteq \bigcup(\mathcal{N} \cap N)$.*

Proof Fix $z \in 2^\omega$ and let G be an $N[z]$-generic filter for $\text{Coll}(2^{2^{\aleph_0}}, \aleph_0)$ (the algebra for collapsing $2^{2^{\aleph_0}}$ onto $\aleph_0$). Now we have $z \in Y_n^n$ iff $L_\Theta[Y] \models \exists Z \leq T \phi_n(z, Z)$ iff $N[z][G] \models \exists Z \leq T \phi_n(z, Z)$ iff $N[z] \Vdash$ " $\Vdash_{\text{Coll}(2^{2^{\aleph_0}}, \aleph_0)} \exists Z \leq T \phi_n(z, Z)$". The second equivalence follows from absoluteness of Σ^1_1 formulas and the third from homogeneity of $\text{Coll}(2^{2^{\aleph_0}}, \aleph_0)$.

Let $\phi_n^*(z)$ denote the formula " $\Vdash_{\text{Coll}(2^{2^{\aleph_0}}, \aleph_0)} \exists Z \leq T \phi_n(z, Z)$". That is $z \in Y_n^n$ iff $N[z] \models \phi_n^*(z)$. Let B_n be a Borel set in N representing the Boolean value $[\![\phi_n^*(\dot{r})]\!]_{\mathbb{B}}$ where $\dot{r}$ is the canonical name for a random real. For a random real z over N we have,

$$z \in Y_n^n \iff N[z] \models \phi_n^*(z) \iff z \in B_n.$$

Therefore $B_n \triangle Y_n^n \subseteq \bigcup(\mathcal{N} \cap N)$. □

Note that in particular $\mu(B_n) \geq 1 - 2^{-n}$.[2] Using the fact that x is random over N we obtain that there is $n^* \geq n_0$ such that $x \in B_{n^*}$. Again since $B_{n^*} \triangle Y_{n^*}^{n^*} \subseteq \bigcup(\mathcal{N} \cap N)$,

[2]This follows from the facts that $\mu_*(Y_n^n) \geq 1 - 2^{-1}$ and $B_n \triangle Y_n^n$ is null.

$x \in Y^{n^*}_{n^*}$. Let T^* be a witness to $x \in Y^{n^*}_{n^*}$. Then $T^* \leq T_{n^*}$, T^* is $(N, \mathcal{C}(Y))$-generic, $T^* \Vdash$ "x is random over N" and for all $k \geq n^*$, $T^* \Vdash x \notin \dot{f}_0(k)$. Then

$$T^* \Vdash f^*_0 \restriction n^* = \dot{f}_0 \restriction n^* \wedge \forall k \geq n(x \notin \dot{f}_0(k))$$

which implies that for all $n \in \omega$, $T^* \Vdash (f^*_0 \sqsubseteq_n x \to \dot{f}_0 \sqsubseteq_n x)$. □

Recall that a forcing notion $\mathbb{P}$:

- has the *Laver property* if and only if for every function $f \in V \cap \omega^\omega$ and a $\mathbb{P}$-name $\dot{g}$ such that $\Vdash_{\mathbb{P}} \forall n(\dot{g}(n) \leq f(n))$ there is a slalom $S \in V$ such that $\Vdash_{\mathbb{P}} \forall n \dot{g}(n) \in S(n)$.
- has property L_f where $f \in {}^\omega\omega$, if for every $p \in \mathbb{P}$, $n \in \omega$ and $A \in [\omega]^{<\omega}$ the following holds: if $p \Vdash \dot{a} \in A$, then there is $q \leq_n p$ and $B \subseteq A$, $|B| \leq f(n)$ such that $q \Vdash \dot{a} \in B$.

Lemma 3.7 *Sacks coding $\mathcal{C}(Y)$ has the property L_f where $f(n) = 2^n$ for all n, and so has the Laver property. It is ${}^\omega\omega$-bounding and so has the Sacks property. Furthermore it is (F, g)-preserving for some F and g (see [3, Definition 7.2.23]) and is (f, h)-bounding for all f and h.*

Proof Suppose $T \in \mathcal{C}(Y)$, $n \in \omega$ and $A \in [\omega]^{<\omega}$ such that $T \Vdash \dot{a} \in \check{A}$. Let $S_n(T)$ be the n-th splitting level of T. Then $|S_n(T)| = 2^n$ and for every $t_j \in S_n(T)$ there is $T'_j \leq T(t_j)$ such that $T'_j \Vdash \dot{a} = \check{k}_j$ for some $k_j \in A$. Let $B = \{k_j\}_{j \in 2^n} \subseteq A$, $T' = \bigcup_{j \in 2^n} T'_j$. Then $T' \leq_n T$ and $T' \Vdash \dot{a} \in \check{B}$. By [3, Lemma 7.2.2], if $\mathbb{P}$ has the L_f property for some f then $\mathbb{P}$ has the Laver property. Since $\mathcal{C}$ is ${}^\omega\omega$-bounding, by [3, Lemma 6.3.38] it has the Sacks property. The Laver property implies also that $\mathcal{C}(Y)$ is (F, g)-preserving for some F and g (see [3, Lemma 7.2.25] and is (f, h)-bounding for all f and h (see [3, Lemma 7.2.16]). □

4 Measure, category and projective wellorders

The underlying forcing construction is the construction from [5] forcing a Δ^1_3-w.o. of the reals. For completeness of the argument we will give a brief outline of this construction. Recall that a transitive ZF^- model M is *suitable* if ω_2^M exists and $\omega_2^M = \omega_2^{L^M}$. Assume V is the constructible universe L. Let $F : \omega_2 \to L_{\omega_2}$ be a bookkeeping function which is Σ_1-definable over L_{ω_2} and let $\bar{S} = (S_\beta : \beta < \omega_2)$ be a sequence of almost disjoint stationary subsets of ω_1 which is Σ_1-definable over

L_{ω_2} with parameter ω_1, such that $F^{-1}(a)$ is unbounded in ω_2 for every $a \in L_{\omega_2}$ and whenever M, N are suitable models such that $\omega_1^M = \omega_1^N$ then $F^M, \bar{S}^M$ agree with $F^N, \bar{S}^N$ on $\omega_2^M \cap \omega_2^N$. In addition, if M is suitable and $\omega_1^M = \omega_1$, then $F^M, \bar{S}^M$ equal the restrictions of $F, \bar{S}$ to the ω_2 of M. Let S be a stationary subset of ω_1 which is Δ_1-definable over L_{ω_1} and almost disjoint from every element of $\bar{S}$.

Recursively define a countable support iteration $\langle\langle \mathbb{P}_\alpha : \alpha \le \omega_2\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\rangle\rangle$ such that $\mathbb{P} = \mathbb{P}_{\omega_2}$ will be a poset adding a Δ^1_3-definable wellorder of the reals. We can assume that all names for reals are nice in the sense of [5] and that for $\alpha < \beta < \omega_2$ all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals which are not $\mathbb{P}_\alpha$-names. For each $\alpha < \omega_2$ define $<_\alpha$ as in [5]: that is, if x, y are reals in $L[G_\alpha]$ and $\sigma^\alpha_x, \sigma^\alpha_y$ are the $<_L$-least $\mathbb{P}_\gamma$-names for x, y respectively, where $\gamma \le \alpha$, define $x <_\alpha y$ if and only if $\sigma^\alpha_x <_L \sigma^\alpha_y$. Note that $<_\alpha$ is an initial segment of $<_\beta$. If G is a $\mathbb{P}$-generic filter, then $<^G = \bigcup\{<^G_\alpha : \alpha < \omega_2\}$ will be the desired wellorder of the reals.

In the recursive definition of $\mathbb{P}_{\omega_2}$, $\mathbb{P}_0$ is defined to be the trivial poset and $\dot{\mathbb{Q}}_\alpha$ is of the form $\dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$, where $\dot{\mathbb{Q}}^0_\alpha$ is an arbitrary $\mathbb{P}_\alpha$-name for a proper forcing notion of cardinality at most $\aleph_1$ and $\dot{\mathbb{Q}}^1_\alpha$ is defined as in [5] and so carries out the task of forcing the Δ^1_3-w.o. of the reals. Note that $\mathbb{Q}^1_\alpha$ is the iteration of countably many posets shooting clubs through certain stationary, co-stationary sets from $\bar{S}$ (and so each of those is S-proper and ω-distributive), followed by a "localization" forcing which is proper and does not add new reals, followed by coding with perfect trees. In the following we will use the fact that $\dot{\mathbb{Q}}^0_\alpha$ is arbitrary, to force the various $\aleph_1$-$\aleph_2$-admissible assignments to the cardinal characteristics of the Cichón diagram in the presence of a Δ^1_3 wellorder of the reals.

Theorem 4.1 *The constellation determined by* $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$ *and* $\mathfrak{b} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof Perform the countable support iteration described above, which forces a Δ^1_3-w.o. of the reals and in addition specify $\dot{\mathbb{Q}}^0_\alpha$ as follows. If α is even let $\Vdash_\alpha \dot{\mathbb{Q}}^0_\alpha = \mathbb{B}$ be the random real forcing, and if α is odd let $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{C}$ be the Cohen forcing. Then in $V^{\mathbb{P}_{\omega_2}}$ $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$. At the same time, since the countable support iteration of S-proper, almost ${}^\omega\omega$-bounding posets is weakly bounding, the ground model reals remain an unbounded family and so a witness to $\mathfrak{b} = \aleph_1$. □

Theorem 4.2 *The constellation determined by* $\mathfrak{d} = \aleph_2$, $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof In the forcing construction described above, which forces a Δ^1_3-w.o. of the reals, define $\dot{\mathbb{Q}}^0_\alpha$ to be the rational perfect tree forcing PT defined in [3, Definition 7.3.43]. To claim that $\mathfrak{d} = \aleph_2$ in the final generic extension, note that PT adds an unbounded real. It remains to show that $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$. By [3, Theorem 7.3.46] the rational perfect tree forcing preserves $\sqsubseteq^{\mathrm{Cohen}}$, and by Lemma 3.4 the coding with perfect tress $\mathcal{C}(Y)$ also preserves $\sqsubseteq^{\mathrm{Cohen}}$. Therefore by Theorem 2.11 in $V^{\mathbb{P}_{\omega_2}}$ the set $2^\omega \cap V$ is non meager and so $V^{\mathbb{P}_{\omega_2}} \Vdash \mathrm{non}(\mathcal{M}) = \aleph_1$. By [3, Theorem 7.3.47], the rational perfect tree forcing preserves $\sqsubseteq^{\mathrm{random}}$ and by Lemma 3.6 the prefect tree coding $\mathcal{C}(Y)$ preserves $\sqsubseteq^{\mathrm{random}}$. Therefore by Theorem 2.10 in the final extension $2^\omega \cap V$ is a non null set and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{N}) = \aleph_1$. □

Theorem 4.3 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \aleph_2$, $\mathfrak{b} = \mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For even α let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$, and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the Blass-Shelah forcing notion $\mathbb{Q}$ defined in [3, 7.4.D]. Since all iterands are almost ${}^\omega\omega$-bounding, by Lemma 2.7 the ground model reals remain an unbounded family and so a witness to $\mathfrak{b} = \aleph_1$. On the other hand $\mathbb{Q}$ adds an unbounded real and $\Vdash_{\mathbb{Q}}$ "$2^\omega \cap V \in \mathcal{N}$", which implies that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \aleph_2$. Since cofinally often we add random reals, we have that $\mathrm{cov}(\mathcal{N}) = \aleph_2$ in the final extension. To show that no Cohen reals are added by the iteration, use the fact that all iterands are (F, g)-preserving, as well as [3, Theorems 7.2.29 and 7.2.24]. □

Theorem 4.4 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \mathfrak{d} = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha = \mathrm{PT}_{f,g}$, and for α odd let $\dot{\mathbb{Q}}^0_\alpha = \mathrm{PT}$, where $\mathrm{PT}_{f,g}$ and PT are defined in [3, Definition 7.3.43 and Definition 7.3.3] respectively. Since $\Vdash_{PT_{f,g}}$ $2^\omega \cap V \in \mathcal{M}$ and *PT* adds an unbounded real, $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{M}) = \mathfrak{d} = \aleph_2$. All iterands are almost ${}^\omega\omega$-bounding and so $\mathfrak{b}$ remains small. All iterands S preserve $\sqsubseteq^{random}$, and so by Theorem 2.10 $\mathbb{P}_{\omega_2}$ preserves outer measure and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{N}) = \aleph_1$. To see that the iteration does not add random reals, note that *PT* and $\mathcal{C}(Y)$ have the Laver property and so are (f, g)-bounding for all f, g. On the other hand $PT_{f,g}$ is (f, h)-bounding for some appropriate h, which implies that all iterands are S-(f, h)-bounding. Then by Theorem 2.17, $\mathbb{P}_{\omega_2}$ is S-(f, h)-bounding, which implies that is does not add random reals. □

Theorem 4.5 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_2$ *and* $\mathfrak{b} = \mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the rational perfect tree forcing PT, and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$. Then $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{cov}(\mathcal{N}) = \mathfrak{d} = 2^{\aleph_0}$. By [3, Theorem 6.3.12] $\mathbb{B}$ preserves $\sqsubseteq^{\mathrm{random}}$, by [3, Theorem 7.3.47] PT preserves $\sqsubseteq^{\mathrm{random}}$ and by Lemma 3.6 Sacks coding preserves $\sqsubseteq^{\mathrm{random}}$. Then Theorem 2.10, $V^{\mathbb{P}_{\omega_2}} \vDash 2^\omega \cap V \notin \mathcal{N}$. All iterands are almost ${}^\omega\omega$-bounding, and so by Theorem 2.7 the ground model reals remain an unbounded family in $V^{\mathbb{P}_{\omega_2}}$. □

Theorem 4.6 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \aleph_2$ *and* $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be Cohen forcing, and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be $\mathrm{PT}_{f,g}$ (see [3, Definition 7.3.3]). Since $\Vdash_{\mathrm{PT}_{f,g}} 2^\omega \cap V \in \mathcal{M}$, $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{M}) = \aleph_2$. Since cofinally often we add Cohen reals, clearly $\mathrm{cov}(\mathcal{M}) = \aleph_2$ in the final generic extension. All involved partial orders are almost ${}^\omega\omega$-bounding and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{b} = \omega_1$. To see that the iteration does not add random reals, proceed by induction using Theorem 2.15 at limit steps.

Alternative Proof: The result can be obtained using finite support iteration of ccc posets. We will slightly modify the coding stage of the construction of [7]. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ be a finite support iteration such that $\mathbb{P}_0$ is the poset defined in [7, Lemma 1]. Suppose $\mathbb{P}_\alpha$ has been defined. If α is a limit, $\alpha = \omega_1 \cdot \alpha' + \xi$ where $\xi < \omega_1$ and $\alpha' > 0$, define $\mathbb{Q}_\alpha$ as in Case 1 of the original construction. If α is not of the above form, ie α is a successor or $\alpha < \omega_1$, let $\dot{\mathbb{Q}}_\alpha$ be a name for the following poset adding an eventually different real:

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in \omega^{<\omega}, s_1 \in [\mathrm{o.t.}(\dot{<}^G_\alpha)]^{<\omega}\}^3,$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if s_0 is an initial segment of t_0, $s_1 \subseteq t_1$, and for all $\xi \in s_1$ and all $j \in [|s_0|, |t_0|)$ we have $t_0(j) \neq x_\xi(j)$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{<}^{G_\alpha}_\alpha$. The sets $\dot{A}_\alpha$ are defined as in [7]. With this the definition of $\mathbb{P}_{\omega_2}$ is complete. Following the proof of the original construction one can show that $\mathbb{P}_{\omega_2}$ does add a Δ^1_3-definable wellorder of the reals (note that in our case $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{c} = \aleph_2$.) Since the eventually different forcing adds a Cohen real and makes the ground model reals meager, we obtain that $V^{\mathbb{P}_{\omega_2}} \Vdash \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_2$. Since all iterands of our construction are σ-centered, by [3, Theorems 6.5.30 and 6.5.29] $\mathbb{P}_{\omega_2}$ does not add random reals and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{cov}(\mathcal{N}) = \aleph_1$. The ground model reals remain an unbounded family and so a witness to $\mathfrak{b} = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. We should point out that the coding techniques of [7] allow one to obtain the consistency

[3] The relation $\dot{<}^G_\alpha$ was defined in the second paragraph of section 4.

of the existence of a Δ^1_3 wellorder of the reals with $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \aleph_3$ and $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \aleph_1$. □

Theorem 4.7 *The constellation determined by* $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the rational perfect tree forcing PT, and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the poset S_{g,g^*} (see [3, 7.3.C]). Note that $\Vdash_{S_{g,g^*}} 2^\omega \cap V \in \mathcal{N}$ and so $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_2$. On the other hand PT adds an unbounded real, which implies that $(\mathfrak{d} = \aleph_2)^{\mathbb{V}^{\mathbb{P}_{\omega_2}}}$. Also S_{g,g^*}, PT and $\mathcal{C}(Y)$ preserve $\sqsubseteq^{Cohen}$, which by Theorem 2.11 implies that $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{M}) = \aleph_1$. To see that there are no Cohen reals added by the iteration we use the S-(f,g)-bounding property. More precisely, PT and $\mathcal{C}(Y)$ have the Laver property and so are (f,g)-bounding for all f, g. The poset S_{g,g^*} is (g, g^*)-bounding, which implies that all iterands are S-(g, g^*)-bounding. Thus by Theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(g, g^*)-bounding, and so the entire iteration does not add Cohen reals. □

Theorem 4.8 *The constellation determined by* $\mathrm{cov}(\mathcal{M}) = \aleph_2$, $\mathrm{non}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For every $\alpha < \omega_2$ let $\dot{\mathbb{Q}}^0_\alpha$ be Cohen forcing. By [3, Theorem 6.3.18] $\mathbb{C}$ preserves $\sqsubseteq^{\text{Cohen}}$ and by Theorem 3.4 Sacks coding preserves $\sqsubseteq^{\text{Cohen}}$. Then by Theorem 2.11 the entire iteration $\mathbb{P}_{\omega_2}$ preserves non-meager sets and so in particular $V^{\mathbb{P}_{\omega_2}} \models 2^\omega \cap V \notin \mathcal{M}$. □

Theorem 4.9 *The constellation determined by* $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{M}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α an even successor let $\dot{\mathbb{Q}}^0_\alpha$ be the rational perfect tree forcing PT, for α an odd successor let $\dot{\mathbb{Q}}^0_\alpha$ be $\mathrm{PT}_{f,g}$ (see [3, Definition 7.3.3]), and for α a limit let $\dot{\mathbb{Q}}^0_\alpha = S_{g,g^*}$. Clearly $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{M}) = \aleph_2$. To show that $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ use the fact that all forcing notions used in the iteration are S-(f,h)-bounding and so by Theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(f,h)-bounding. Thus no real in $V^{\mathbb{P}_{\omega_2}}$ is Cohen or random over V. To show that $\mathfrak{b} = \aleph_1$ in the final extension, use the facts that all iterands are almost ${}^\omega\omega$-bounding. □

Theorem 4.10 *The constellation determined by* $\mathrm{add}(\mathcal{N}) = \aleph_2$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof Note that if $\mathbb{A}$ is amoeba forcing then $V^{\mathbb{A}} \models \bigcup(\mathcal{N} \cap V) \in \mathcal{N}$. Thus, in order to obtain the desired result it is sufficient to require that for every every $\alpha < \omega_2$, $\dot{\mathbb{Q}}^0_\alpha$ is the amoeba forcing. □

Theorem 4.11 *The constellation determined by* $\mathrm{cof}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof Sacks coding has the Sacks property and so by [3, Lemma 6.3.39] $\mathcal{C}(Y)$ preserves $\sqsubseteq^\Delta$ (and so it S-preserves-$\sqsubseteq^\Delta$). For every α let $\dot{\mathbb{Q}}^\alpha_0$ be the trivial poset. Then by theorem 2.12 $\mathbb{P}_{\omega_2}$ preserves the base of the ideal of measure zero sets, that is $V^{\mathbb{P}_{\omega_2}} \models \mathrm{cof}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})^V = \aleph_1$. □

Theorem 4.12 *The constellation determined by* $\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{add}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α an even successor let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$, for α an odd successor let $\dot{\mathbb{Q}}^0_\alpha$ be Cohen forcing $\mathbb{C}$, and for α a limit let $\dot{\mathbb{Q}}^0_\alpha$ be Laver forcing LT. Then clearly in $V^{\mathbb{P}_{\omega_2}}$ we have that $\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$. To show that there are no amoeba reals in the final generic extension, and so $\mathrm{add}(\mathcal{N}) = \aleph_1$, proceed by induction using Theorem 2.16 at limit stages. □

Theorem 4.13 *The constellation determined by* $\mathrm{cof}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each α let $\mathbb{Q}^0_\alpha$ be the poset $\mathbb{U}$ defined in [3, Page 339]. This poset is ${}^\omega\omega$-bounding, preserves $\sqsubseteq^{random}$, preserves $\sqsubseteq^{Cohen}$ and does not have the Sacks property. By Theorem 2.6, the ground model reals dominate the reals in $V^{\mathbb{P}_{\omega_2}}$ and so $\mathfrak{d} = \aleph_1$. On the other hand since all iterands S-preserves-$\sqsubseteq^{random}$ and S-preserve-$\sqsubseteq^{Cohen}$, in $V^{\mathbb{P}_{\omega_2}}$ we have $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_1$. Thus in particular $\mathbb{V}^{\mathbb{P}_{\omega_2}} \models \mathrm{cof}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$. To see that $\mathrm{cof}(\mathcal{N}) = \aleph_2$ in $V^{\mathbb{P}_{\omega_2}}$ use the fact that $\mathbb{U}$ does not have the Sacks property (see [3]). □

Theorem 4.14 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be random real forcing, for α an odd successor let $\dot{\mathbb{Q}}_\alpha$ be the poset S_{g,g^*} defined in [3, Section 7.3.C], and for α a limit let $\dot{\mathbb{Q}}^0_\alpha$ be Laver forcing. To see that $\mathrm{cov}(\mathcal{M}) = \aleph_1$ in the final generic extension, note that all iterands are (F, g)-preserving and so by [3, Theorems 7.2.29 and 7.2.24] $\mathbb{P}_{\omega_2}$ does not add Cohen reals. □

Theorem 4.15 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each $\alpha < \omega_2$ let $\dot{\mathbb{Q}}^0_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{PT}_{f,g}$. Note that by [3, Theorem 7.3.6] we have that $V^{\mathrm{PT}_{f,g}} \models V \cap {}^\omega\omega \in \mathcal{M}$. Therefore in $V^{\mathbb{P}_{\omega_2}}$ we have that $\mathrm{non}(\mathcal{M}) = \aleph_2$. The poset $\mathrm{PT}_{f,g}$ is (f,h)-bounding for some h, and so all iterands are S-(f,h)-bounding. Then by Theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(f,h)-bounding, which implies that $\mathbb{P}_{\omega_2}$ does not add random reals. Thus $\mathrm{cov}(\mathcal{N}) = \aleph_1$ in the final generic extension. Since all iterands are ${}^\omega\omega$-bounding, by Theorem 2.6 the ground model reals are a witness to $\mathfrak{d} = \omega_1$ in $V^{\mathbb{P}_{\omega_2}}$. By [3, Theorem 7.3.15] the poset $\mathrm{PT}_{f,g}$ preserves $\sqsubseteq^{\text{random}}$, Sacks coding preserves $\sqsubseteq^{\text{random}}$, and so by Theorem 2.10 $\mathbb{P}_{\omega_2}$ preserves outer measure. Thus $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_1$. □

Theorem 4.16 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$ and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be Laver forcing LT. Then we immediately get that $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_2$ in $V^{\mathbb{P}_{\omega_2}}$. By [3, Theorem 7.3.39] LT preserves $\sqsubseteq^{\text{random}}$, by [3, Theorem 6.3.12] $\mathbb{B}$ preserves $\sqsubseteq^{\text{random}}$ and Sacks coding preserves $\sqsubseteq^{\text{random}}$. Then by Theorem 2.10 $V^{\mathbb{P}_{\omega_2}} \models 2^\omega \cap \mathcal{N} \notin \mathcal{N}$ and so $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_1$. □

Theorem 4.17 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each α, let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$. Since $\mathbb{B}$ and the Sacks coding preserve $\sqsubseteq^{\text{random}}$, Theorem 2.10 implies that $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_1$. By Lemma 2.6 $\mathbb{P}_{\omega_2}$ is ${}^\omega\omega$-bounding and so $\mathfrak{d} = \aleph_1$ in the final generic extension. □

Theorem 4.18 *The constellation determined by* $\mathrm{add}(\mathcal{M}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the Cohen forcing $\mathbb{C}$, and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the Laver forcing. Clearly $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} = \aleph_2$ in $V^{\mathbb{P}_{\omega_2}}$. To show that $\mathbb{P}_{\omega_2}$ does not add random reals proceed by induction using Theorem 2.15 at limit steps.

Alternative proof: The result can be obtained using finite support iteration of ccc posets, by slightly modifying the coding stage of the poset forcing a Δ^1_3 definable wellorder of the reals from [7]. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta ; \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a finite support iteration where $\mathbb{P}_0$ is the poset defined in [7, Lemma 1]. Suppose $\mathbb{P}_\alpha$ has been defined. If α is a limit and $\alpha = \omega_1 \cdot \alpha' + \xi$ where $\xi < \omega_1$ and $\alpha' > 0$, define $\mathbb{Q}_\alpha$ as in Case 1 of the original construction. Otherwise, if α is a successor or $\alpha < \omega_1$ let $\mathbb{Q}_\alpha$ be the poset from Case 2 of the same paper. Note that in this case $\mathbb{Q}_\alpha$ adds a dominating real. In either case A_α is defined as in [7]. With this the definition of $\mathbb{P}_{\omega_2}$ is complete. Following the proof of the original iteration, one can show that $\mathbb{P}_{\omega_2}$ adds a Δ^1_3-definable wellorder of the reals. Note that in $V^{\mathbb{P}_{\omega_2}}$ we have $\mathrm{add}(\mathcal{M}) = \aleph_2$, since cofinally often we add dominating and Cohen reals. To show that $\mathrm{cov}(\mathcal{N})$ remains small, ie that random reals are not added, use the fact that all iterands are σ-centered and [3, Theorems 6.5.30, 6.5.29]. We should point out that the coding techniques of [7] allow one to obtain the consistency of the existence of a Δ^1_3 wellorder of the reals with $\mathrm{add}(\mathcal{M}) = \aleph_3$ and $\mathrm{cov}(\mathcal{N}) = \aleph_1$. □

Theorem 4.19 *The constellation determined by* $\mathrm{cof}(\mathcal{M}) = \aleph_1$ *and* $\mathrm{non}(\mathcal{N}) = \aleph_2$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each α let $\dot{\mathbb{Q}}^0_\alpha$ be the poset S_{g,g^*} defined in [3, Section 7.3.C]. Note that $V^{S_{g,g^*}} \models V \cap 2^\omega \in \mathcal{N}$. Thus clearly $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_2$. Now $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$. Thus it is sufficient to show that both $\mathfrak{d}$ and $\mathrm{non}(\mathcal{M})$ remain small in the final generic extension. However S_{g,g^*} is ${}^\omega\omega$-bounding and preserves $\sqsubseteq^{Cohen}$. Then theorems 2.6 and 2.11 imply that $\mathfrak{d} = \mathrm{non}(\mathcal{M}) = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. □

Theorem 4.20 *The constellation determined by* $\mathrm{non}(\mathcal{N}) = \mathfrak{b} = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be S_{g,g^*}, and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the Laver forcing LT. Since all iterands are S-(g, g^*)-bounding, by Theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(g, g^*)-bounding, which implies (see [3, Lemma 7.2.15]) that no real in $V^{\mathbb{P}_{\omega_2}}$ is Cohen or random over V. Therefore $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. Recall also that $\Vdash_{S_{g,g^*}} \Vdash 2^\omega \cap V \in \mathcal{N}$ and LT adds a dominating real. □

Theorem 4.21 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be $\mathrm{PT}_{f,g}$ and for α odd, let $\dot{\mathbb{Q}}^0_\alpha$ be S_{g,g^*}. Since $\Vdash_{\mathrm{PT}_{f,g}} 2^\omega \cap V \in \mathcal{M}$ and $\Vdash_{S_{g,g^*}} 2^\omega \cap V \in N$, we have $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_2$. All iterands are S-(f,h)-bounding and ${}^\omega\omega$-bounding, which implies that in $V^{\mathbb{P}_{\omega_2}}$ there are no random reals over V and the ground model reals form a dominating family. □

Theorem 4.22 *The constellation determined by* $\mathfrak{b} = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For every α let $\dot{\mathbb{Q}}^0_\alpha$ be the Laver forcing LT. Since LT adds a dominating function, clearly $\mathfrak{b} = \aleph_2$. Since LT and Sacks coding S-preserve-$\sqsubseteq^{random}$, by Theorem 2.10 the ground model reals $V \cap 2^\omega$ are not null in $V^{\mathbb{P}_{\omega_2}}$. Since LT and Sacks coding have the Laver property they are (f,g)-bounding, which implies that the iteration does not add random reals. □

Theorem 4.23 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the forcing notion S_{g,g^*} defined in [3, Section 7.3.C], and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$. Since S_{g,g^*} makes the ground model reals a null set, $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_2$. Clearly $\mathrm{cov}(\mathcal{N})$ is large in the final extension, and since all iterands are ${}^\omega\omega$-bounding the ground model reals remain a witness to $\mathfrak{d} = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. □

5 Questions

We would like to conclude with some open questions. It is of interest whether all of the constellations can in fact be obtained without the existence of a Δ^1_3 wellorder of the reals. Note that this would follow if one could simultaneously have that all Δ^1_3 sets enjoy some regularity property that conflicts a Δ^1_3 wellorder. Can we even guarantee that there are no projective wellorders at all? Another direction is the question whether an assignment of larger values to the cardinal invariants in the Cichón diagram is consistent with the existence of a Δ^1_3 wellorder. What about constellations in which the invariants have more than two distinct values? Are those consistent with the existence of a Δ^1_3 wellorder of the reals?

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3.5 Regularity properties on the generalised reals

Sy David Friedman, Yurii Khomskii and Vadim Kulikov, Annals of Pure and Applied Logic 167, 2016, pp 408–430.

This publication began as a joint work of Friedman and Kulikov. Before I joined this project, some basic results had already been established—preliminary versions of Theorems 3.10 and 3.13 and some parts of Theorem 4.9. After joining, we continued having regular meetings with all three authors in addition to multiple informal collaborative meetings between Kulikov and Khomskii. The long technical result concerning Miller forcing (Lemma 4.4) as well as the technical parts of Lemma 4.9 are due to Khomskii. In detail:

- ***Section 3 (Definitions and ideas):*** *Collaborative effort of all three authors*
- ***Idea of Definition 3.6:*** *Collaboration*
- ***Lemma 3.8:*** *Khomskii and Friedman*
- ***Theorem 3.10:*** *Khomskii, based on preliminary results by Friedman and Kulikov*
- ***Lemma 3.12:*** *Khomskii*
- ***Theorem 3.13:*** *Collaboration*
- ***Corollary 3.14:*** *Collaboration*
- ***Observation 3.16:*** *Khomskii*
- ***Lemma 4.1:*** *Khomskii*
- ***Lemma 4.4:*** *Khomskii*
- ***Lemma 4.9:*** *Collaboration*
- ***Theorem 4.11:*** *Collaboration*

The paper was written by Khomskii.

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Regularity properties on the generalized reals

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ABSTRACT

We investigate regularity properties derived from tree-like forcing notions in the setting of "generalized descriptive set theory", i.e., descriptive set theory on κ^κ and 2^κ, for regular uncountable cardinals κ.

1. Introduction

Generalized Descriptive Set Theory is an area of research dealing with generalizations of classical descriptive set theory on the Baire space ω^ω and Cantor space 2^ω, to the *generalized Baire space* κ^κ and the *generalized Cantor space* 2^κ, where κ is an uncountable regular cardinal satisfying $\kappa^{<\kappa} = \kappa$. Some of the earlier papers dealing with descriptive set theory on $(\omega_1)^{\omega_1}$ were motivated by model-theoretic concerns, see e.g. [24] and [30, Chapter 9.6]. More recently, generalized descriptive set theory became a field of interest in itself, with various aspects being studied for their own sake, as well as for their applications to different fields of set theory.

This paper is the first systematic study of *regularity properties* for subsets of generalized Baire spaces. We will focus on regularity properties derived from tree-like forcing partial orders, using the framework introduced by Ikegami in [16] (see Definition 3.1) as a generalization of the Baire property, as well as a num-

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ber of other standard regularity properties (Lebesgue measurability, Ramsey property, Sacks property etc.) In the classical setting, such properties have been studied by many people, see, e.g., [15,3,4,19]. Typically, these properties are satisfied by analytic sets, while the Axiom of Choice can be used to provide counterexamples. On the second projective level one obtains independence results, as witnessed by "Solovay-style" characterization theorems, such as the following:

Theorem 1.1. *(See Solovay [29].) All $\mathbf{\Sigma}^1_2$ sets have the Baire property if and only if for every $r \in \omega^\omega$ there are co-meager many Cohen reals over $L[r]$.*

Theorem 1.2. *(See Judah–Shelah [15].) All $\mathbf{\Delta}^1_2$ sets have the Baire property if and only if for every $r \in \omega^\omega$ there is a Cohen real over $L[r]$.*

These types of theorems make it possible to study the relationships between different regularity properties on the second level. Far less is known for higher projective levels, although some results exist in the presence of large cardinals (see [16, Section 5]) and some other results can be found in [1, Chapter 9] and in the recent works [8,6]. Solovay's model [29] provides a uniform way of establishing regularity properties for all projective sets, starting from ZFC with an inaccessible.

When attempting to generalize descriptive set theory from ω^ω to κ^κ for a regular uncountable κ, at first many basic results remain intact after a straightforward replacement of ω by κ. But, before long, one starts to notice fundamental differences: for example, the generalized $\mathbf{\Delta}^1_1$ sets are not the same as the generalized Borel sets; absoluteness theorems, such as Σ^1_1- and Shoenfield absoluteness, are not valid; and in the constructible universe L, there is a $\mathbf{\Sigma}^1_1$-good well-order of κ^κ, as opposed to merely a $\mathbf{\Sigma}^1_2$-good well-order in the standard setting (see Section 2 for details). Not surprisingly, regularity properties also behave radically different in the generalized context. Halko and Shelah [13] first noticed that on 2^κ, the generalized Baire property provably fails for $\mathbf{\Sigma}^1_1$ sets. On the other hand, it holds for the generalized Borel sets, and is independent for generalized $\mathbf{\Delta}^1_1$ sets. This suggests that some of the classical theory on the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ level corresponds to the $\mathbf{\Delta}^1_1$ level in the generalized setting.

It should be noted that other kinds of regularity properties have been considered before, sometimes leading to different patterns in terms of consistency of projective regularity. For example, in [27] Schlicht shows that it is consistent relative to an inaccessible that a version of the perfect set property holds for all generalized projective sets. By [22], as well as recent results of Laguzzi and the first author, similar results hold for suitable modifications of the properties studied here.

This paper is structured as follows: Section 2 will be devoted to a brief survey of facts about the "generalized reals". In Section 3 we introduce an abstract notion of regularity and prove that, under certain assumption, the following results hold:

1. Borel sets are "regular".
2. Not all analytic sets are "regular".
3. For $\mathbf{\Delta}^1_1$ sets, the answer is independent of ZFC.

In Section 4 we focus on some concrete examples on the $\mathbf{\Delta}^1_1$-level and generalize some classical results from the $\mathbf{\Delta}^1_2$-level. Section 5 ends with a number of open questions.

2. Generalized Baire spaces

We devote this section to a survey of facts about κ^κ and 2^κ which will be needed in the rest of the paper, as well as specifying some definitions and conventions. None of the results here are new, though some are not widely known or have not been sufficiently documented.

Notation 2.1. $\kappa^{<\kappa}$ denotes the set of all functions from α to κ for some $\alpha < \kappa$, similarly for $2^{<\kappa}$. We use standard notation concerning sequences, e.g., for $s, t \in \kappa^{<\kappa}$ we use $s^\frown t$ to denote the *concatenation* of s and t, $s \subseteq t$ to denote that *s is an initial segment of t* etc. $\kappa^\kappa_\uparrow$ denotes the set of strictly increasing functions from κ to κ, and $\kappa^{<\kappa}_\uparrow$ the set of strictly increasing functions from α to κ for some $\alpha < \kappa$. Also, we will frequently refer to elements of κ^κ or 2^κ as "κ-reals" or "generalized reals".

For finite sequences, it is customary to denote the length by $|s|$. In the generalized context, in order to avoid confusion with cardinality, we denote the *length* of a sequence (i.e., the unique α such that $s \in \kappa^\alpha$ or 2^α) by "len(s)".

2.1. Topology

We always assume that κ is an uncountable, regular cardinal, and that $\kappa^{<\kappa} = \kappa$ holds. The *standard topology* on κ^κ is the one generated by basic open sets of the form $[s] := \{x \in \kappa^\kappa \mid s \subseteq x\}$, for $s \in \kappa^{<\kappa}$; similarly for 2^κ. Many elementary facts from the classical setting have straightforward generalizations to the generalized setting. The concepts *nowhere dense* and *meager* are defined as usual, and a set A has the *Baire property* if and only if $A \triangle O$ is meager for some open O. The following classical results are true regardless of the value of κ:

- *Baire category theorem*: the intersection of κ-many open dense sets is dense.
- *Kuratowski–Ulam theorem* (also called *Fubini for category*): if $A \subseteq \kappa^\kappa \times \kappa^\kappa$ has the Baire property then A is meager if and only if $\{x \mid A_x \text{ is meager}\}$ is comeager, where $A_x := \{y \mid (x, y) \in A\}$.

Definition 2.2. A *tree* is a subset of $\kappa^{<\kappa}$ or $2^{<\kappa}$ closed under initial segments. For a node $t \in T$, we write $\mathrm{Succ}_T(t) := \{s \in T \mid s = t^\frown\langle\alpha\rangle \text{ for some } \alpha\}$. A node $t \in T$ is called

- *terminal* if $\mathrm{Succ}_T(t) = \varnothing$,
- *splitting* if $|\mathrm{Succ}_T(t)| > 1$, and
- *club-splitting* if $\{\alpha \mid t^\frown\langle\alpha\rangle \in T)\}$ is a club in κ.

We use the notation Split(T) to refer to the set of all splitting nodes of T.

A $t \in T$ is called a *successor node* if len(t) is a successor ordinal and a *limit node* if len(t) is a limit ordinal. A tree is *pruned* if it has no terminal nodes, and *<κ-closed* if for every increasing sequence $\{s_i \mid i < \lambda\}$ of nodes from T, for $\lambda < \kappa$, the limit $\bigcup_{i<\lambda} s_i$ is also a node of T.

Notice that concepts such as *club-splitting*, *successor* and *limit* node, and *<κ-closed* are inherent to the generalized setting and have no classical counterpart. Most of the trees we consider will be pruned and $<\kappa$-closed.

A *branch through T* is a κ-real $x \in \kappa^\kappa$ or 2^κ such that $\forall\alpha\,(x{\restriction}\alpha \in T)$, and $[T]$ denotes the set of all branches through T. As usual, $[T]$ is topologically closed and every closed set has the form $[T]$ for some tree T.

The Borel and projective hierarchies are defined in analogy to the classical situation: the Borel sets form the smallest collection of subsets of κ^κ or 2^κ containing the basic open sets and closed under complements and κ-unions. A set is $\mathbf{\Sigma}^1_1$ iff it is the projection of a closed (equivalently: Borel) set; it is $\mathbf{\Pi}^1_n$ iff its complement is $\mathbf{\Sigma}^1_n$; and it is $\mathbf{\Sigma}^1_{n+1}$ iff it is the projection of a $\mathbf{\Pi}^1_n$ set, for $n \geq 1$. It is $\mathbf{\Delta}^1_n$ iff it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, and *projective* iff it is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ for some $n \in \omega$.

In spite of the close similarity of the above notions to the classical ones, there are also fundamental differences:

Fact 2.3. Borel $\neq \boldsymbol{\Delta}^1_1$.

A proof of this fact can be found in [11, Theorem 18 (1)], and we also refer readers to Sections II and III of the same paper for a more detailed survey of the basic properties of κ^κ and 2^κ.

2.2. The club filter

Sets that will play a crucial role in this paper are those related to the *club filter*. As usual, we may identify 2^κ with $\mathscr{P}(\kappa)$ via characteristic functions.

Fact 2.4. *The set* $C := \{a \subseteq \kappa \mid a \text{ contains a club}\}$ *is* $\boldsymbol{\Sigma}^1_1$.

Proof. For every $c \subseteq \kappa$, note that c is closed (in the "club"-sense) if and only if for every $\alpha < \kappa$, $c \cap \alpha$ is closed in α. Therefore, "being closed" is a (topologically) closed property. Being unbounded, on the other hand, is a G_δ property, so "being club" is G_δ. Then for all $a \subseteq \kappa$ we have $a \in C$ iff $\exists c$ (c is club and $c \subseteq a$), which is $\boldsymbol{\Sigma}^1_1$. □

In [13] it was first noticed that the club filter provides a counterexample to the Baire property.

Theorem 2.5 *(Halko–Shelah). The club filter C does not satisfy the Baire property.*

We will prove a generalization of the above, see Theorem 3.10. An immediate corollary of Theorem 2.5 is that in the generalized setting, analytic sets do not satisfy the Baire property. Although the club filter clearly cannot be Borel (Borel sets *do* satisfy the Baire property, in any topological space satisfying the Baire category theorem), it can consistently be $\boldsymbol{\Delta}^1_1$ for successors κ.

Theorem 2.6 *(Mekler–Shelah; Friedman–Wu–Zdomskyy). For any successor cardinal κ, it is consistent that the club filter on κ is $\boldsymbol{\Delta}^1_1$.*

Proof. For $\kappa = \omega_1$, this was first prove in [23]. The argument contained a flaw, which was corrected in [14]. For arbitrary successor cardinals κ, this was proved using different methods in [12]. □

It is also consistent that the club filter is not $\boldsymbol{\Delta}^1_1$—this will follow from Theorem 3.13.

2.3. Absoluteness

Two fundamental results in descriptive set theory are *analytic* (*Mostowski*) *absoluteness* and *Shoenfield absoluteness*. In general, this type of absoluteness does not hold for uncountable κ. For example, let $\kappa = \lambda^+$ for regular λ, pick $S \subseteq \kappa \cap \mathrm{Cof}(\lambda)$ such that both S and $(\kappa \cap \mathrm{Cof}(\lambda)) \setminus S$ are stationary. Let $\mathbb{P}$ be a forcing for adding a club to $S \cup \mathrm{Cof}(<\lambda)$. Then, if Φ is the Σ^1_1 formula defining the club filter $C \subseteq \mathscr{P}(\kappa)$ from Fact 2.4, we have that $V \models \neg\Phi(S \cup \mathrm{Cof}(<\lambda))$ while $V^{\mathbb{P}} \models \Phi(S \cup \mathrm{Cof}(<\lambda))$, so Σ^1_1-absoluteness fails even for κ^+-preserving forcing extensions. On the other hand, Σ^1_1-absoluteness does hold for generic extensions via $<\kappa$-closed forcings.

Lemma 2.7. *Let $\mathbb{P}$ be a $<\kappa$-closed forcing. Then Σ^1_1 formulas are absolute between V and $V^{\mathbb{P}}$.*

Proof. Let $\phi(x)$ be a Σ^1_1 formula with parameters in V. Let $x \in \kappa^\kappa$ and assume $V^{\mathbb{P}} \models \phi(x)$. Let T (in V) be a two-dimensional tree such that $\{x \mid \phi(x)\} = p[T]$, i.e., the projection of T to the first coordinate. Let $h \in \kappa^\kappa \cap V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models (x, h) \in [T]$ and let $\dot{h}$ be a $\mathbb{P}$-name for h.

By induction, build an increasing sequence $\{p_i \mid i < \kappa\}$ of $\mathbb{P}$-conditions, and an increasing sequence $\{t_i \in \kappa^{<\kappa} \mid i < \kappa\}$, such that each $p_i \Vdash t_i \subseteq \dot{h}$. This can be done since at limit stages $\lambda < \kappa$, we can define $t_\lambda := \bigcup_{i<\lambda} t_i$ and pick p_λ below p_i for all $i < \lambda$. Since every p_i forces $(\check{x}, \dot{h}) \in [T]$, it follows that for every i we have $(x{\restriction}\mathrm{len}(t_i), t_i) \in T$. But then (in V) let $g := \bigcup_{i<\kappa} t_i$, so $(x, g) \in [T]$ and therefore $\phi(x)$ holds. □

2.4. Well-order of the reals

In the classical setting, it is well-known that in L there exists a $\mathbf{\Sigma}^1_2$ well-order of the reals. In fact, the well-order is "$\mathbf{\Sigma}^1_2$-good", meaning that both the relation $<_L$ on the reals, and the binary relation defined by

$$\Psi(x, y) \equiv \text{“}x \text{ codes the set of } <_L\text{-predecessors of } y\text{”}$$

is $\mathbf{\Sigma}^1_2$. The proof uses absoluteness of $<_L$ and Ψ between L and initial segments L_δ for countable δ, and the fact that "$E \subseteq \omega \times \omega$ is well-founded" is a $\mathbf{\Pi}^1_1$-predicate on E. In the generalized setting, however, the predicate "$E \subseteq \kappa \times \kappa$ is well-founded" is *closed*, leading to the following result:

Lemma 2.8. *In L, there is a $\mathbf{\Sigma}^1_1$-good well-order of κ^κ.*

Proof. As usual, we have that for $x, y \in \kappa^\kappa$, $x <_L y$ iff $\exists\delta < \kappa^+$ such that $x, y \in L_\delta$ and $L_\delta \models x <_L y$. Using standard tricks, this can be re-written as "$\exists E \subseteq \kappa \times \kappa$ (E is well-founded, $x, y \in \mathrm{ran}(\pi_E)$ and $(\omega, E) \models ZFC^* + V = L + x <_L y$)", where π_E refers to the transitive collapse of (ω, E) onto some $(L_\delta, \in)$ and ZFC^* is a sufficiently large fragment of ZFC. The statement "E is well-founded" is closed because E is well-founded iff $\forall\alpha < \kappa\ E \cap (\alpha \times \alpha)$ is well-founded. Thus we obtain a $\mathbf{\Sigma}^1_1$ statement. A similar argument works with $<_L$ replaced by $\Psi(x, y)$, showing that the well-order is $\mathbf{\Sigma}^1_1$-good. □

2.5. Proper forcing

A ubiquitous tool in the study of the classical Baire and Cantor spaces is Shelah's theory of *proper forcing*. It is a technical requirement on a forcing notion which is just sufficient to imply preservation of ω_1, while itself being preserved by countable support iterations, and moreover having a multitude of natural examples. Over the years, there have been various attempts at generalizing this theory to higher cardinals (see e.g. [28,26,10] for some recent contributions). Of course, we can use the following straightforward generalization:

Definition 2.9. A forcing $\mathbb{P}$ is *κ-proper* if for every sufficiently large θ (e.g. $\theta > 2^{|\mathbb{P}|}$), and for all elementary submodels $M \prec H_\theta$ such that $|M| = \kappa$ and M is closed under $<\kappa$-sequences, for every $p \in \mathbb{P} \cap M$ there exists $q \leq p$ such that for every dense $D \in M$, $D \cap M$ is predense below q.

The above property follows both from the κ^+-c.c. and a κ-version of Axiom A, and implies that κ^+ is preserved, but the property itself is in general not preserved by iterations, see [25, Example 2.4]. Nevertheless, it is a useful formulation that we will need on some occasions.

While a uniform theory for κ-properness is lacking so far, preservation theorems are usually proved either using the κ^+-c.c. or on a case-by-case basis.

Fact 2.10 *(Baumgartner). A forcing $\mathbb{Q}$ is* κ-linked *iff $\mathbb{Q} = \bigcup_{\alpha<\kappa} \mathbb{Q}_\alpha$ where each $\mathbb{Q}_\alpha$ consists of pairwise compatible conditions. A forcing $\mathbb{Q}$ is* well-met *iff for every two compatible conditions $q_1, q_2 \in \mathbb{Q}$ there is a greatest lower bound $q \in \mathbb{Q}$.*

If $\mathbb{P}_\alpha$ is an iteration of length $\alpha > \kappa$ with supports of size $<\kappa$, and every iterand is forced to be κ-linked, $<\kappa$-closed and well-met, then $\mathbb{P}_\alpha$ has the κ^+-c.c.

This was originally proved by Baumgartner in [2], and a modern treatment can be found e.g. in [20, Section V.5] (both expositions deal with $\kappa = \omega_1$ but the proof works for any regular uncountable κ satisfying $\kappa^{<\kappa} = \kappa$).

Fact 2.11.

1. *κ-Sacks forcing $\mathbb{S}_\kappa$ (see Example 3.2) was studied by Kanamori [17], where the following facts were proved:*
   (a) *$\mathbb{S}_\kappa$ satisfies a generalized version of Axiom A (see Definition 3.6 (2)).*
   (b) *Assuming $\diamondsuit_\kappa$, iterations of $\mathbb{S}_\kappa$ with $\leq\kappa$-sized supports also satisfy a version of Axiom A.*
   (c) *If κ is inaccessible, then $\mathbb{S}_\kappa$ is κ^κ-*bounding *(meaning that for every $x \in \kappa^\kappa \cap V^{\mathbb{S}_\kappa}$ there exists $y \in \kappa^\kappa \cap V$ such that $x(i) < y(i)$ for sufficiently large $i < \kappa$), and so are arbitrary iterations of $\mathbb{S}_\kappa$ with $\leq\kappa$-size supports.*
2. *κ-Miller forcing $\mathbb{M}_\kappa$ (see Example 3.2) was studied by Friedman and Zdomskyy [9], where the following facts were proved:*
   (a) *$\mathbb{M}_\kappa$ satisfies a generalized version of Axiom A.*
   (b) *Assuming κ is inaccessible, iterations of $\mathbb{M}_\kappa$ with $\leq\kappa$-sized supports satisfy a version of Axiom A.*

In particular, $\mathbb{S}_\kappa$, $\mathbb{M}_\kappa$ and their iterations are κ-proper in the sense of Definition 2.9 and thus preserve κ^+.

3. Regularity properties

The regularity properties we will consider in this paper are those derived from definable tree-like forcing notions. In this section we give an abstract treatment following the framework introduced by Ikegami in [16], providing sufficient conditions so that the following facts can be proved uniformly:

1. Regularity for Borel sets is true.
2. Regularity for arbitrary $\mathbf{\Sigma}^1_1$ sets is false.
3. Regularity for arbitrary $\mathbf{\Delta}^1_1$ sets is independent.

3.1. Tree-like forcings on κ^κ

Definition 3.1. A forcing notion $\mathbb{P}$ is called *κ-tree-like* iff

1. the conditions of $\mathbb{P}$ are pruned and $<\kappa$-closed trees on κ^κ or 2^κ ordered by $q \leq p$ iff $q \subseteq p$,
2. the full tree ($\kappa^{<\kappa}$ or $2^{<\kappa}$) is an element of $\mathbb{P}$,
3. for all $T \in \mathbb{P}$ and all $s \in T$ the restriction $T{\uparrow}s := \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}$ is also a member of $\mathbb{P}$,
4. the statement "T is a $\mathbb{P}$-tree" is absolute between models of ZFC, and
5. if $\langle T_\alpha \mid \alpha < \lambda\rangle$ is a decreasing sequence of conditions, with $\lambda < \kappa$, then $\bigcap_{\alpha<\lambda} T_\alpha \in \mathbb{P}$.

The first three items are standard, and the fourth one is to make sure that the forcing notion has the same interpretation in all models (in particular in further forcing extensions). Item 5 is a strong form of $<\kappa$-closure of the forcing which is needed for technical reasons. Below are a few examples of κ-tree-like forcings that have either been considered in the literature or are natural generalizations of classical notions.

Example 3.2.

1. κ-*Cohen forcing* $\mathbb{C}_\kappa$. Conditions are the trees corresponding to the basic open sets $[s]$, for $s \in 2^{<\kappa}$ or $\kappa^{<\kappa}$, ordered by inclusion.
2. κ-*Sacks forcing* $\mathbb{S}_\kappa$. A tree T on 2^κ is called a κ-*Sacks tree* if it is pruned, $<\kappa$-closed and
   (a) every node $t \in T$ has a splitting extension in T, and
   (b) for every increasing sequence $\langle s_i \mid i < \lambda\rangle$, $\lambda < \kappa$, of splitting nodes in T, $s := \bigcup_{\alpha<\lambda} s_\alpha$ is a *splitting* node of T.

   $\mathbb{S}_\kappa$ is the partial order of κ-Sacks trees ordered by inclusion.
3. κ-*Miller forcing* $\mathbb{M}_\kappa$. A tree T on $\kappa_\uparrow^{<\kappa}$ is called a κ-*Miller tree* if it is pruned, $<\kappa$-closed and
   (a) every node $t \in T$ has a club-splitting extension in T,
   (b) for every increasing sequence $\langle s_i \mid i < \lambda\rangle$, $\lambda < \kappa$, of club-splitting nodes in T, $s := \bigcup_{i<\lambda} s_i$ is a club-splitting node of T. Moreover, *continuous club-splitting* is required, which is the following property: for every club-splitting limit node $s \in T$, if $\{s_i \mid i < \lambda\}$ is the set of all club-splitting initial segments of s and $C_i := \{\alpha \mid s_i^\frown\langle\alpha\rangle \in T\}$ is the club witnessing club-splitting of s_i for every i, then $C := \{\alpha \mid s^\frown\langle\alpha\rangle \in T\} = \bigcap_{i<\lambda} C_i$ is the club witnessing club-splitting of s.

   $\mathbb{M}_\kappa$ is the partial order of κ-Miller trees ordered by inclusion.
4. κ-*Laver forcing* $\mathbb{L}_\kappa$. A tree T on $\kappa_\uparrow^{<\kappa}$ is a κ-*Laver tree* if all nodes $s \in T$ extending the stem of T are club-splitting. $\mathbb{L}_\kappa$ is the partial order of κ-Laver trees ordered by inclusion.
5. κ-*Mathias forcing* $\mathbb{R}_\kappa$. A κ-Mathias condition is a pair (s, C), where $s \subseteq \kappa$, $\mathrm{len}(s) < \kappa$, $C \subseteq \kappa$ is a club, and $\max(s) < \min(C)$. The conditions are ordered by $(t, D) \leq (s, C)$ iff $t \leq s$, $D \subseteq C$ and $t \setminus s \subseteq C$. Formally, this does not follow Definition 3.1, but we can easily identify conditions (s, C) with trees $T_{(s,C)}$ on $\kappa_\uparrow^{<\kappa}$ defined by $t \in T_{(s,C)}$ iff $\mathrm{ran}(t) \subseteq s \cup C$.
6. κ-*Silver forcing* $\mathbb{V}_\kappa$. If κ is inaccessible, let $\mathbb{V}_\kappa$ consist of κ-Sacks-trees T on $2^{<\kappa}$ which are *uniform*, i.e., for $s, t \in T$, if $\mathrm{len}(s) = \mathrm{len}(t)$ then $s^\frown\langle i\rangle \in T$ iff $t^\frown\langle i\rangle \in T$. Alternatively, we can view conditions of $\mathbb{V}_\kappa$ as functions $f : \kappa \to \{0, 1, \{0, 1\}\}$, such that $f(i) = \{0, 1\}$ holds for all $i \in C$ for some club $C \subseteq \kappa$, ordered by $g \leq f$ iff $\forall i\, (f(i) \in \{0, 1\} \to g(i) = f(i))$.

The generalized κ-Sacks forcing was introduced and studied by Kanamori in [17], and the κ-Miller forcing is its natural variant, studied e.g. by Friedman and Zdomskyy in [9]. The requirement on the trees to be "closed under splitting-nodes" (2(b) and 3(b)) ensure that item 5 of Definition 3.1 is satisfied, and thus that the forcings are $<\kappa$-closed. The property called "continuous club-splitting" was introduced in [9] to facilitate the preservation of measurability. We should note that other generalizations of Miller forcing have also been considered, see e.g. [5].

κ-Silver is a natural generalization of Silver forcing, but the standard proof of Axiom A only works for inaccessible κ.

κ-Laver and κ-Mathias are, again, natural generalizations of their classical counterparts; however, since we require the trees to split into club-many successors at all branches above the stem, any two κ-Laver and κ-Mathias conditions with the same stem are compatible, so both $\mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ are κ^+-centered and hence satisfy the κ^+-c.c. Therefore they are perhaps more reminiscent of the classical *Laver-with-filter* and *Mathias-with-filter* forcings on ω^ω, rather than the actual Laver and Mathias forcing posets. Note that if we would drop club-splitting from the definition and only require stationary or κ-sized splitting instead, we would lose $<\kappa$-closure of the forcing.

Remark 3.3. One notion conspicuous by its absence from Example 3.2 is *random forcing*. To date, it is not entirely clear how random forcing should properly be generalized to uncountable κ. Recently Shelah proposed a definition for κ weakly compact, and a different approach was given by the first author and

Laguzzi in [7]. However, a consensus on the correct definition for arbitrary κ has not been reached so far, so in this work we choose to avoid random forcing, as well as the concept *null ideal* and *Lebesgue measurability*.

The following definition is based on [16, Definition 2.6 and Definition 2.8]. Let $\mathbb{P}$ be a fixed κ-tree-like forcing.

Definition 3.4. Let A be a subset of κ^κ or 2^κ. Then

1. A is $\mathbb{P}$-*null* iff $\forall T \in \mathbb{P}\, \exists S \leq T$ such that $[S] \cap A = \varnothing$. We denote the ideal of $\mathbb{P}$-null sets by $\mathcal{N}_\mathbb{P}$
2. A is $\mathbb{P}$-*meager* iff it is a κ-union of $\mathbb{P}$-null sets. We denote the κ-ideal of $\mathbb{P}$-meager sets by $\mathcal{I}_\mathbb{P}$.
3. A is $\mathbb{P}$-*measurable* iff $\forall T \in \mathbb{P}\, \exists S \leq T$ such that $[S] \subseteq^* A$ or $[S] \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ refers to "modulo $\mathcal{I}_\mathbb{P}$".

For a wide class of tree-like forcing notions, the clause "modulo $\mathcal{I}_\mathbb{P}$" can be eliminated from the above definition: see Lemma 3.8 (2).

3.2. Regularity of Borel sets

In ω^ω, it is not hard to prove that if $\mathbb{P}$ is proper then all analytic sets are $\mathbb{P}$-measurable, using forcing-theoretic arguments and absoluteness techniques (see e.g. [19, Proposition 2.2.3]). These methods are generally not available in the generalized setting. However, we would still like to know that, at least, all Borel subsets of κ^κ are $\mathbb{P}$-measurable for all reasonable examples of $\mathbb{P}$.

Remark 3.5. Closed sets are $\mathbb{P}$-measurable for all $\mathbb{P}$. To see this, let $[U]$ be an arbitrary closed set and let $T \in \mathbb{P}$. If $T \subseteq U$ then we are done, otherwise pick $s \in T \setminus U$, then by Definition 3.1 $T{\uparrow}s \in \mathbb{P}$ and $[T{\uparrow}s] \cap [U] = \varnothing$. It is also easy to see that being $\mathbb{P}$-measurable is closed under complements and $<\kappa$-sized unions and intersections.

It remains to verify closure under κ-sized unions and intersections. For that we introduce some definitions that help to simplify the notion of $\mathbb{P}$-measurability, and moreover will play a crucial role for the rest of this paper.

Definition 3.6. Let $\mathbb{P}$ be a κ-tree-like forcing notion on κ^κ or 2^κ. Then we say that:

1. $\mathbb{P}$ is *topological* if $\{[T] \mid T \in \mathbb{P}\}$ forms a topology base for κ^κ (i.e., for all $S, T \in \mathbb{P}$, $[S] \cap [T]$ is either empty or contains $[R]$ for some $R \in \mathbb{P}$).
2. $\mathbb{P}$ satisfies *Axiom A* iff there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$, with $\leq_0 = \leq$, satisfying:
   (a) $T \leq_\beta S$ implies $T \leq_\alpha S$, for all $\alpha \leq \beta$.
   (b) If $\langle T_\alpha \mid \alpha < \lambda\rangle$ is a sequence of conditions, with $\lambda \leq \kappa$ (in particular $\lambda = \kappa$) satisfying
   $$T_\beta \leq_\alpha T_\alpha \text{ for all } \alpha \leq \beta,$$
   then there exists $T \in \mathbb{P}$ such that $T \leq_\alpha T_\alpha$ for all $\alpha < \lambda$.
   (c) For all $T \in \mathbb{P}$, D dense below T, and $\alpha < \kappa$, there exists an $E \subseteq D$ and $S \leq_\alpha T$ such that $|E| \leq \kappa$ and E is predense below S.
3. $\mathbb{P}$ satisfies *Axiom A** if 2 above holds, but in 2 (c) we additionally require that "$[S] \subseteq \bigcup\{[T] \mid T \in E\}$".

Example 3.7. In Example 3.2, κ-Cohen, κ-Laver and κ-Mathias are topological. By Fact 2.11, κ-Miller and κ-Sacks satisfy Axiom A, and it is not hard to see that in fact they satisfy Axiom A* as well (a direct consequence of the construction). Assuming κ is inaccessible, a generalization of the classical proof shows that κ-Silver also satisfies Axiom A*.

Lemma 3.8.

1. *If $\mathbb{P}$ is topological then a set A is $\mathbb{P}$-measurable iff it satisfies the property of Baire in the topology generated by $\mathbb{P}$. In particular, all Borel sets are $\mathbb{P}$-measurable.*
2. *If $\mathbb{P}$ satisfies Axiom A* then $\mathcal{N}_\mathbb{P} = \mathcal{I}_\mathbb{P}$, and consequently a set A is $\mathbb{P}$-measurable iff $\forall T \in \mathbb{P} \exists S \leq T$ ($[S] \subseteq A$ or $[S] \cap A = \varnothing$) (i.e., we can forget about "modulo $\mathcal{I}_\mathbb{P}$"). Moreover, the collection of $\mathbb{P}$-measurable sets is closed under κ-unions and κ-intersections.*

The proofs are essentially analogous to the classical situation, but let us present them anyway since they are not widely known.

Proof. 1. First of all, notice that if $\mathbb{P}$ is topological then $\mathcal{N}_\mathbb{P}$ is exactly the collection of nowhere dense sets in the $\mathbb{P}$-topology and $\mathcal{I}_\mathbb{P}$ is exactly the ideal of meager sets in the $\mathbb{P}$-topology.

First assume A satisfies the $\mathbb{P}$-Baire property, then let O be an open set in the $\mathbb{P}$-topology such that $A \triangle O$ is $\mathbb{P}$-meager. Given any $T \in \mathbb{P}$, we have two cases: if $[T] \cap O = \varnothing$ then we are done since $[T] \cap A =^* \varnothing$. If $[T] \cap O$ is not empty then there exists a $S \leq T$ such that $[S] \subseteq [T] \cap O$. Then $[S] \subseteq^* A$ holds, so again we are done.

The converse direction is somewhat more involved (cf. [18, Theorem 8.29]). Assume A is $\mathbb{P}$-measurable. Let

- D_1 be a maximal mutually disjoint subfamily of $\{T \in \mathbb{P} \mid [T] \subseteq^* A\}$,
- D_2 be a maximal mutually disjoint subfamily of $\{T \in \mathbb{P} \mid [T] \cap A =^* \varnothing\}$, and
- $D := D_1 \cup D_2$.

Also write $O_1 := \bigcup\{[T] \mid T \in D_1\}$, $O_2 := \bigcup\{[T] \mid T \in D_2\}$ and $O := O_1 \cup O_2$. We will show that $A \triangle O_1$ is $\mathbb{P}$-meager.

Claim 1. *O is $\mathbb{P}$-open dense.*

Proof of Claim. Start with any T. By assumption there exists $S \leq T$ such that $[S] \subseteq^* A$ or $[S] \cap A =^* \varnothing$. In the former case, note that by maximality, there must be some $S' \in D_1$ such that $[S] \cap [S'] \neq \varnothing$. Then find S'' such that $[S''] \subseteq [S] \cap [S']$. Then $[S''] \subseteq O_1$. Likewise, in the case $[S] \cap A =^* \varnothing$ we find a stronger S'' with $[S''] \subseteq O_2$. □ (Claim 1).

Claim 2. *$A \cap O_2$ and $O_1 \setminus A$ are $\mathbb{P}$-meager.*

Proof of Claim. Since the proof of both statements is analogous, we only do the first.

Enumerate $D_2 := \{T_\alpha \mid \alpha < |\kappa^\kappa|\}$. For each α, let $\{X_i^\alpha \mid i < \kappa\}$ be a collection of $\mathbb{P}$-nowhere dense sets, such that $[T_\alpha] \cap A = \bigcup_{i<\kappa} X_i^\alpha$. Now, for every $i < \kappa$, let $Y_i := \bigcup_{\alpha<|\kappa^\kappa|} X_i^\alpha$. We will show that each Y_i is $\mathbb{P}$-nowhere dense. So fix i and pick any $T \in \mathbb{P}$: if $[T]$ is disjoint from all $[T_\alpha]$'s then clearly also $[T] \cap Y_i = \varnothing$. Else, let T_α be such that $[T] \cap [T_\alpha] \neq \varnothing$. Then there exists $S \leq T$ such that $[S] \subseteq [T] \cap [T_\alpha]$. By assumption, $[T_\alpha]$ is disjoint from all $[T_\beta]$'s, and hence from all X_i^β's, for all $\beta \neq \alpha$. Next, since X_i^α is $\mathbb{P}$-nowhere dense, we can find $S' \leq S$ such that $[S'] \cap X_i^\alpha = \varnothing$. But then $[S'] \cap Y_i = \varnothing$, proving that Y_i is indeed $\mathbb{P}$-nowhere dense.

Now clearly $O_2 \cap A$ is completely covered by the collection $\{Y_i \mid i < \kappa\}$, therefore it is meager. □ (Claim 2).

Now it follows from Claim 1 and Claim 2 that $A \triangle O_1 = (O_1 \setminus A) \cup (A \cap O_2) \cup (A \setminus O)$ is a union of three meager sets, hence it is meager.

This proves that the set A has the property of Baire in the topology generated by $\mathbb{P}$.

2. Assume $\mathbb{P}$ satisfies Axiom A*, and let $\{A_i \mid i < \kappa\}$ be a collection of $\mathbb{P}$-null sets. We want to show that $A := \bigcup_{i<\kappa} A_i$ is also $\mathbb{P}$-null. For each i let $D_i := \{T \mid [T] \cap A_i = \varnothing\}$. By assumption, each D_i is dense. Now let $T_0 \in \mathbb{P}$ be given. Using Axiom A* find, inductively, a sequence $\{T_i \mid i < \kappa\}$ as well as a sequence $\{E_i \subseteq D_i \mid i < \kappa\}$ such that

- $T_j \leq_i T_i$ for all $i \leq j$ and
- $[T_i] \subseteq \bigcup\{[T] \mid T \in E_i\}$ for all i.

This can always be done by condition (c) of Axiom A*. Then, by condition (b) there is a T such that $T \leq T_i$ for all i, and hence, $[T] \subseteq \bigcup\{[S] \mid S \in D_i\}$ for all i. In particular, $[T] \cap A_i = \varnothing$ for all $i < \kappa$, proving that $\bigcap A_i$ is $\mathbb{P}$-null.

For the second claim, it suffices to show closure under κ-unions. Consider a collection $\{A_i \mid i < \kappa\}$ of $\mathbb{P}$-measurable sets, and let $T \in \mathbb{P}$. We must find $S \leq T$ such that $[S] \subseteq \bigcup_{i<\kappa} A_i$ or $[S] \cap \bigcup_{i<\kappa} A_i = \varnothing$. If for at least one $i < \kappa$, we can find $S \leq T$ such that $[S] \subseteq A_i$, we are done, so assume that's not the case. Then we have $A_i \cap [T] \in \mathcal{N}_\mathbb{P}$ for all i, because for every $S \in \mathbb{P}$, either $S \nleq T$ in which case we are done, or $S \leq T$ in which case, by $\mathbb{P}$-measurability of A_i and the fact that $\mathcal{I}_\mathbb{P} = \mathcal{N}_\mathbb{P}$, there exists $S' \leq S$ with $[S'] \subseteq A_i$ or $[S'] \cap A_i = \varnothing$—but by our assumption the former is impossible and so the latter must hold. Therefore each $A_i \cap [T]$ is in $\mathcal{N}_\mathbb{P}$ and again by the above we obtain $\bigcup_{i<\kappa}(A_i \cap [T]) \in \mathcal{N}_\mathbb{P}$, so we can find $S \leq T$ with $[S] \cap \bigcup_{i<\kappa} A_i = \varnothing$. □

Note that to prove point 2 above, we do not in fact need the full strength of Axiom A*, but only need that for all $T \in \mathbb{P}$, D dense below T, and $\alpha < \kappa$, there exists $S \leq_\alpha T$ such that $[S] \subseteq \bigcup\{[T] \mid T \in D\}$.

Corollary 3.9. *If $\mathbb{P}$ is either topological or satisfies Axiom A* then all Borel sets are $\mathbb{P}$-measurable.*

3.3. Regularity of $\mathbf{\Sigma}^1_1$ sets

Let us abbreviate "all sets of complexity $\mathbf{\Gamma}$ are $\mathbb{P}$-measurable" by "$\mathbf{\Gamma}(\mathbb{P})$". In the ω^ω case, ZFC proves $\mathbf{\Sigma}^1_1(\mathbb{P})$, and by symmetry $\mathbf{\Pi}^1_1(\mathbb{P})$, but $\mathbf{\Sigma}^1_2(\mathbb{P})$ and $\mathbf{\Delta}^1_2(\mathbb{P})$ are independent of ZFC. But in the case that $\kappa > \omega$ things are dramatically different since by the Halko–Shelah result (Theorem 2.5) $\mathbf{\Sigma}^1_1(\mathbb{C}_\kappa)$ is false, i.e., the Baire property fails for analytic sets. We attempt to find the essential requirements on $\mathbb{P}$ which would allow us to generalize this proof and show, in ZFC, that $\mathbf{\Sigma}^1_1(\mathbb{P})$ fails, i.e., that there is an analytic set which is not $\mathbb{P}$-measurable. It is most convenient to formulate this requirement in terms of the κ-Sacks and κ-Miller forcing notions, see Example 3.2.

Theorem 3.10. *Let $\mathbb{P}$ be a tree-like forcing notion on 2^κ whose conditions are κ-Sacks trees, or a tree-like forcing notion on κ^κ whose conditions are κ-Miller trees. Then $\mathbf{\Sigma}^1_1(\mathbb{P})$ fails.*

Proof. Let us start with the first case. Recall the club-filter C from Fact 2.4, considered as a subset of 2^κ. If C were $\mathbb{P}$-measurable then, in particular, we would have a $T \in \mathbb{P}$ such that $[T] \subseteq^* C$ or $[T] \cap C =^* \varnothing$. First deal with the former case: let $\{X_i \mid i < \kappa\}$ be $\mathbb{P}$-null sets such that $[T] \setminus C = \bigcup_{i<\kappa} X_i$. Inductively, construct a decreasing sequence $\{T_i \mid i < \kappa\}$ of conditions:

- $T_0 = T$.
- Given T_i, first let $T'_i \leq T_i$ be any condition with strictly longer stem, and then let $T_{i+1} \leq T'_i$ be such that $[T_{i+1}] \cap X_i = \varnothing$.

- At limit stages λ, first let $T'_\lambda := \bigcap_{i<\lambda} T_\alpha$, which is a $\mathbb{P}$-condition by item 5 of Definition 3.1. Notice also that $\mathrm{stem}(T'_\lambda) = \bigcup_{i<\lambda} \mathrm{stem}(T_i)$. That is because for every $i < \lambda$, $\bigcup_{i<\lambda} \mathrm{stem}(T_i)$ is in T_i and is the limit of an increasing sequence $\langle \mathrm{stem}(T_j) \mid i \leq j < \lambda \rangle$ of splitnodes of T_i, hence it is also a splitnode in T_i by condition 2(b) of Example 3.2. Therefore it is a splitnode in T'_λ and so is the stem of T'_λ.
  Let $T_\lambda \leq T'_\lambda$ be such that $\mathrm{stem}(T_\lambda) \supseteq \mathrm{stem}(T'_\lambda)^\frown\langle 0\rangle$.

Now let $x := \bigcup_{i<\kappa} \mathrm{stem}(T_i)$. Then x is a branch through T, $x \notin X_i$ for all i, and moreover, there exists a club $c \subseteq \kappa$ such that $x(i) = 0$ for all $i \in c$. In particular, $x \notin C$—contradiction.

To deal with the second case that $[T] \cap C =^* \varnothing$, proceed analogously except that at limit stages, pick $T_\lambda \leq T'_\lambda$ such that $\mathrm{stem}(T_\lambda) \supseteq \mathrm{stem}(T'_\lambda)^\frown\langle 1\rangle$; then it will follow that $x \in C$.

When $\mathbb{P}$ is a tree-like forcing on κ^κ whose conditions are κ-Miller trees, we apply the same argument, but using the following variant of the club-filter: let S be a stationary, co-stationary subset of κ and define

$$C_S := \{a \in \kappa^\kappa \mid \exists c \subseteq \kappa \text{ club such that } \forall i \in c\, (x(i) \in S)\}.$$

Clearly this set is $\mathbf{\Sigma}^1_1$ by the same argument as in Fact 2.4. Proceed exactly as before, choosing members from S or from $\kappa \setminus S$ at limit stages, as desired, which can be achieved using the club-splitting of the trees. □

In the above result, an essential property of the trees T was that $\forall x \in [T]$, the set $\{i < \kappa \mid x{\restriction}i$ is a splitting node of $T\}$ formed a club on κ. Recent work of Philipp Schlicht [27] and Giorgio Laguzzi [22] suggests that this property is directly related to the existence of $\mathbf{\Sigma}^1_1$-counterexamples, since for a version of Sacks-, Miller- and Silver-measurability where the trees are *not* required to have this property, it is consistent that all projective sets are measurable.

3.4. Regularity of $\mathbf{\Delta}^1_1$ sets

With $\mathsf{Borel}(\mathbb{P})$ being provable in ZFC and $\mathbf{\Sigma}^1_1(\mathbb{P})$ inconsistent, we are left with the $\mathbf{\Delta}^1_1$-level.

Lemma 3.11 *(Folklore). If $V = L$ then $\mathbf{\Delta}^1_1(\mathbb{P})$ is false for all tree-like $\mathbb{P}$.*

Proof. Use the $\mathbf{\Sigma}^1_1$-good wellorder of the reals of L from Lemma 2.8, and proceed as in the ω^ω-case, obtaining a $\mathbf{\Delta}^1_1$-counterexample as opposed to a $\mathbf{\Delta}^1_2$ one. □

This is not the only method to produce $\mathbf{\Delta}^1_1$-counterexamples to $\mathbb{P}$-measurability. A completely different method, innate to the generalized setting, is to produce models in which the club filter itself is $\mathbf{\Delta}^1_1$, see Lemma 2.6.

It is known that the Baire property on κ^κ holds for $\mathbf{\Delta}^1_1$ sets in κ^+-product/iterations of κ-Cohen forcing, see e.g. [11, Theorem 49 (7)]. We would like to generalize this to other κ-tree-like forcings. First, we need the following technical result, a strengthening of the concept of *κ-proper* (Definition 2.9). This is again similar to the classical case.

Lemma 3.12. *Let $\mathbb{P}$ be κ-tree-like, and assume that $\mathbb{P}$ either has the κ^+-c.c. or satisfies Axiom A^*. Then for every elementary submodel $M \prec \mathcal{H}_\theta$ of a sufficiently large $\mathcal{H}_\theta$, with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$, and for every $T \in \mathbb{P} \cap M$, there is $T' \leq T$ such that*

$$[T'] \subseteq^* \{x \in \kappa^\kappa \mid x \text{ is } \mathbb{P}\text{-generic over } M\}$$

where $\subseteq^*$ means "modulo $\mathcal{I}_\mathbb{P}$" and a κ-real x is $\mathbb{P}$-generic over M if $\{S \in \mathbb{P} \cap M \mid x \in [S]\}$ is a $\mathbb{P}$-generic filter over M.

Proof. First assume that $\mathbb{P}$ has the κ^+-c.c. Let M be an elementary submodel with $|M| = \kappa$.

Claim. *A real x is $\mathbb{P}$-generic over M if and only if $x \notin B$ for every Borel $\mathbb{P}$-null set B coded in M.*

Proof. Suppose x is $\mathbb{P}$-generic over M, and let B be a $\mathbb{P}$-null set coded in M. Then by elementarity $M \models$ "B is $\mathbb{P}$-null", and $D := \{S \in \mathbb{P} \cap M \mid [S] \cap B = \varnothing\}$ is in M and $M \models$ "D is dense". Since x is $\mathbb{P}$-generic, there exists $S \in D$ such that $x \in [S]$, and therefore, $x \notin B$.

Conversely, suppose $x \notin B$ for every Borel $\mathbb{P}$-null set coded in M. Let $D \subseteq \mathbb{P}$ be a dense set in M, and let A be a maximal antichain inside D. Let $B := \kappa^\kappa \setminus \bigcup\{[S] \mid S \in (A \cap M)\}$ which is a Borel set since $|A| = \kappa$ and has a code in M. Moreover $B \in \mathcal{N}_\mathbb{P}$ since A is a maximal antichain. Therefore, by assumption, $x \notin B$, and hence $x \in [S]$ for some $S \in A \cap M$, i.e., x is $\mathbb{P}$-generic over M. □ (Claim).

Now it is easy to see that $X := \bigcup\{B \mid B$ is a Borel set in $\mathcal{N}_\mathbb{P}$ with code in $M\}$ is a κ-union of $\mathbb{P}$-null sets, hence it is itself in $\mathcal{I}_\mathbb{P}$. In particular, there exists $T' \leq T$ such that $[T'] \subseteq^* \{x \mid x$ is $\mathbb{P}$-generic over $M\} = \kappa^\kappa \setminus X$.

Next, assume instead that $\mathbb{P}$ satisfies Axiom A*. Let $\{D_i \mid i < \kappa\}$ enumerate the dense sets in M, and let $T \in \mathbb{P} \cap M$. As usual, we can apply Axiom A* to inductively find a fusion sequence $\{T_i \mid i < \kappa\}$ and a sequence $\{E_i \subseteq D_i \mid i < \kappa\}$ such that each $E_i \in M$ and $|E_i| \leq \kappa$, and hence $E_i \subseteq M$, and moreover $[T_i] \subseteq \bigcup\{[S] \mid S \in E_i\}$. Let T' be such that $T' \leq T_i$ for all i. Then for every i, $[T'] \subseteq \bigcup\{[S] \mid S \in E_i\}$, so, in particular, every x in $[T']$ is $\mathbb{P}$-generic over M, so we are done. □

Using this strengthening of κ-properness, we are almost in a position to prove that a κ^+-iteration of $\mathbb{P}$ satisfying either the κ^+-c.c. or Axiom A* yields a model of for $\mathbf{\Delta}^1_1(\mathbb{P})$. However, we still have an obstacle, and that is the lack of an abstract preservation theorem for κ-properness, mentioned in Section 2.5. This obstacle makes it impossible to prove the next theorem in an abstract setting including the non-κ^+-c.c. cases. We could formulate it under the assumption that κ-properness is preserved; but in fact we only need several consequences of κ-properness, namely, that κ^+ is preserved and that all new κ-reals appear at some initial stage of the iteration.

Theorem 3.13. *Let $\mathbb{P}$ be a tree-like forcing.*

1. *Suppose $\mathbb{P}$ is κ-linked and well-met (see Fact 2.10), and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $<\kappa$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P})$.*
2. *Suppose $\mathbb{P}$ satisfies Axiom A*, and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $\leq\kappa$. Moreover, assume that $\mathbb{P}_{\kappa^+}$ preserve κ^+ and, moreover, for every $x \in \kappa^\kappa \cap V^{\mathbb{P}_{\kappa^+}}$, there is $\alpha < \kappa^+$ such that $x \in \kappa^\kappa \cap V^{\mathbb{P}_\alpha}$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P})$.*

Proof. The proof works uniformly for both cases. In case 1 we use Fact 2.10 to conclude that $\mathbb{P}_{\kappa^+}$ has the κ^+-c.c., hence preserves κ^+ and has the well-known property that κ-reals in the final extension are caught at an initial stage of the iteration. Note that by Definition 3.1 (5), all tree-like forcings are $<\kappa$-closed.

In $V[G_{\kappa^+}]$, let A be $\mathbf{\Delta}^1_1$, defined by Σ^1_1-formulas ϕ and ψ. Let $S \in \mathbb{P}$ be arbitrary. By the assumption, there exists an $\alpha < \kappa^+$ such that all parameters of ϕ and ψ, as well as S, belong to $V[G_\alpha]$. Moreover, there is a $\beta > \alpha$ such that S belongs to $G(\beta + 1)$ (the $(\beta + 1)$-st component of the generic filter), since it is dense to force this for some $\beta > \alpha$. Let $x_{\beta+1}$ be the real corresponding to $G(\beta + 1)$, i.e., the next $\mathbb{P}$-generic real over $V[G_\beta]$.

We know that in the final model $V[G_{\kappa^+}]$, either $\phi(x_{\beta+1})$ or $\psi(x_{\beta+1})$ holds. As ϕ and ψ are both Σ^1_1 the situation is clearly symmetrical so without loss of generality assume the former. Since $\mathbb{P}$ is $<\kappa$-closed, any iteration of it is also $<\kappa$-closed, so by Lemma 2.7 we have Σ^1_1-absoluteness between $V[G_{\kappa^+}]$ and $V[G_{\beta+1}]$.

In particular, $V[G_{\beta+1}] = V[G_\beta][x_{\beta+1}] \models \phi(x_{\beta+1})$. By the forcing theorem, and since we have assumed $S \in G(\beta+1)$, there exists a $T \in V[G_\beta]$ such that $T \leq S$ and $T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\mathrm{gen}})$.

Now, still in $V[G_\beta]$, take an elementary submodel M of a sufficiently large structure, of size κ, containing T. By elementarity, $M \models$ "$T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\mathrm{gen}})$". Going back to $V[G_{\kappa^+}]$, use Lemma 3.12 to find a $T' \leq T$ such that $[T'] \subseteq^* \{x \mid x \text{ is } \mathbb{P}\text{-generic over } M\}$. Now note that if x is $\mathbb{P}$-generic over M and $x \in [T]$, then $M[x] \models \phi(x)$. By upwards-Σ^1_1-absoluteness between M and $V[G_{\kappa^+}]$, we conclude that $\phi(x)$ really holds. Since this was true for arbitrary $x \in [T']$, we obtain $[T'] \subseteq^* \{x \mid \phi(x)\} = A$. □

The above theorem can be applied to many forcing partial orders $\mathbb{P}$, in particular those from Example 3.2.

Corollary 3.14. *$\mathbf{\Delta}^1_1(\mathbb{P})$ is consistent for $\mathbb{P} \in \{\mathbb{C}_\kappa, \mathbb{S}_\kappa, \mathbb{M}_\kappa, \mathbb{L}_\kappa, \mathbb{R}_\kappa\}$, and if κ is inaccessible, also for $\mathbb{P} = \mathbb{V}_\kappa$.*

Proof. The forcings $\mathbb{C}_\kappa, \mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ have the following two properties: any two conditions with the same stem are compatible, and if S, T are two compatible conditions, then $S \cap T$ is a condition. This implies that all three forcings are κ-linked and well-met.

By Fact 2.11 (1), iterations of $\mathbb{S}_\kappa$ with $\leq\kappa$-sized supports satisfy κ-properness assuming that $\diamondsuit_\kappa$ holds in the ground model, so $\mathbf{\Delta}^1_1(\mathbb{S}_\kappa)$ holds in $L^{\mathbb{S}_{\kappa^+}}$. By Fact 2.11 (2), iterations of $\mathbb{M}_\kappa$ with $\leq\kappa$-sized supports satisfy κ-properness for inaccessible κ. It seems very plausible that by an analogous argument to [17], the same holds for arbitrary κ assuming $\diamondsuit_\kappa$. However, we will leave out the verification of this (potentially very technical) proof because $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$ also follows by a much easier argument, namely Theorem 4.9 (3). Finally, if κ is inaccessible then a straightforward modification of [17, Theorem 6.1] shows that iterations of κ-Silver with $\leq\kappa$-sized supports satisfies κ-properness (the only change in the argument involves the definition of the fusion sequence [17, Definition 1.7] and the amalgamation defined in [17, Page 103]). We leave the details to the reader. □

Remark 3.15. It is clear that in Theorem 3.13 it is enough to add $\mathbb{P}$-generic reals cofinally often, provided that the iteration is $<\kappa$-closed and satisfies the other requirements. For example, we can obtain $\mathbf{\Delta}^1_1(\mathbb{C}_\kappa) + \mathbf{\Delta}^1_1(\mathbb{L}_\kappa) + \mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$ simultaneously by employing a κ^+-iteration of $(\mathbb{C}_\kappa * \mathbb{L}_\kappa * \mathbb{R}_\kappa)$ with supports of size $<\kappa$.

Recall that in the classical setting we had Solovay-style characterization theorems for $\mathbf{\Delta}^1_2$ sets, such as Theorem 1.2 and related results (see [3,16]). In light of Theorem 3.13, one might expect that in the generalized setting, analogous characterization theorems exist for statements concerning $\mathbf{\Delta}^1_1$ sets. However, the following observation shows that this is not the case.

Observation 3.16. Suppose κ is successor. There exists a generic extension of L in which the statement "$\forall r \in 2^\kappa\ \exists x\ (x$ is κ-Cohen over $L[r])$" holds, yet there exists a $\mathbf{\Delta}^1_1$ subset of 2^κ without the Baire property.

Proof. Recall that by Theorem 2.6, it is consistent for the club filter C (Definition 2.4) to be $\mathbf{\Delta}^1_1$-definable. The idea is to adapt the proof of [12, Theorem 1.1] due to Friedman, Wu and Zdomskyy. Since that proof is long and technical, we cannot afford to go into details here, so we only provide a sketch of the argument and leave the details to the reader. In that proof, a model where C is $\mathbf{\Delta}^1_1$ is obtained by a forcing iteration, starting from L, in which cofinally many iterands have the κ^+-c.c. One can then verify that the proof remains correct if, additionally, κ-Cohen reals are added cofinally often to this iteration (in fact, κ-Cohen reals are added naturally in the original proof). Thus we obtain a model in which the club filter is $\mathbf{\Delta}^1_1$ and hence fails to have the Baire property, while clearly the statement "$\forall r \in 2^\kappa\ \exists x\ (x$ is κ-Cohen over $L[r])$" is true. □

A similar argument can be applied to any κ-tree-like forcing $\mathbb{P}$ which satisfies the κ^+-c.c., provided it also satisfies Theorem 3.10 (i.e., whose trees are κ-Sacks or κ-Miller trees).

Table 1
Properties of forcings.

κ-Cohen κ-Laver κ-Mathias	Category 1: topological, κ^+-c.c., ideal $\mathcal{I}_{\mathbb{P}}$ cannot be neglected; $\mathbb{P}$-measurability equivalent to Baire property in $\mathbb{P}$-topology.
κ-Sacks κ-Miller κ-Silver	Category 2: non-topological, Axiom A*, $\mathcal{I}_{\mathbb{P}} = \mathcal{N}_{\mathbb{P}}$ can be neglected.

4. Regularity properties for $\mathbf{\Delta}^1_1$ sets

In the classical setting, regularity properties related to well-known forcing notions on ω^ω or 2^ω have been investigated, and the exact relationship between statements $\mathbf{\Delta}^1_2(\mathbb{P})$ and $\mathbf{\Sigma}^1_2(\mathbb{P})$ has been studied for various forcing notions $\mathbb{P}$. As we saw in the previous section, for generalized reals the $\mathbf{\Delta}^1_1$-level reflects some of these results. We will focus on the forcing notions from Example 3.2, i.e., κ-Cohen, κ-Sacks, κ-Miller, κ-Laver, κ-Mathias and κ-Silver.

Before proceeding, we make a further comment regarding κ-Laver and κ-Mathias, showing that the ideal $\mathcal{I}_{\mathbb{L}_\kappa}$ of $\mathbb{L}_\kappa$-meager sets and the ideal $\mathcal{I}_{\mathbb{R}_\kappa}$ of $\mathbb{R}_\kappa$-meager sets cannot be neglected when discussing the regularity property generated by them.

Lemma 4.1. *The ideal $\mathcal{N}_{\mathbb{L}_\kappa}$ of $\mathbb{L}_\kappa$-null sets is not equal to the ideal $\mathcal{I}_{\mathbb{L}_\kappa}$ of $\mathbb{L}_\kappa$-meager sets. Also, there is an F_σ set A such that no κ-Laver tree is completely contained or completely disjoint from A. The same holds for $\mathbb{R}_\kappa$.*

Proof. Fix a stationary, co-stationary $S \subseteq \kappa$. For each $i < \kappa$ define $A_i := \{x \in \kappa^\kappa_\uparrow \mid \forall j > i(x(j) \in S)\}$ and $A = \bigcup_{i<\kappa} A_i$. Then each A_i is $\mathbb{L}_\kappa$-null, because any κ-Laver tree T can be extended to some $T' \leq T$ with stem s, such that $\mathrm{len}(s) > i$ and for some $j > i$ we have $s(j) \notin S$, so that clearly $[T'] \cap A_i = \varnothing$. On the other hand, A itself cannot be $\mathbb{L}_\kappa$-null, because every κ-Laver tree T contains a branch $x \in [T]$ such that for all j longer then the stem of T we have $x(j) \in S$, and therefore $x \in A$. It is also clear that the set A is F_σ but every κ-Laver tree T contains a branch x which is in A and another branch y which is not in A. The argument for κ-Mathias is analogous. □

Summarizing, the forcings we have introduced can be neatly divided into two categories as presented in Table 1.

4.1. Solovay-style characterizations

By Observation 3.16, we know that a Solovay-style characterization for $\mathbf{\Delta}^1_1(\mathbb{P})$ cannot be achieved in the generalized setting. However, in some cases we can obtain one half of such a characterization.

Lemma 4.2. *$\mathbf{\Delta}^1_1(\mathbb{C}_\kappa)$ implies that for every $r \in \kappa^\kappa$ there exists a κ-Cohen real over $L[r]$.*

Proof. The proof is completely analogous to the classical case, see e.g. [1, Theorem 9.2.1], except that we obtain a $\mathbf{\Delta}^1_1$-counterexample as opposed to a $\mathbf{\Delta}^1_2$ one, using the $\mathbf{\Sigma}^1_1$-good wellorder of L (Lemma 2.8). A central ingredient of the classical proof is the Kuratowski–Ulam (Fubini for Category) theorem, which, as we mentioned, is valid on the generalized Baire space. A detailed argument has also been worked out in the PhD Thesis of Laguzzi, see [21, Theorem 75]. □

Lemma 4.3. *$\mathbf{\Delta}^1_1(\mathbb{S}_\kappa)$ implies that for every $r \in \kappa^\kappa$ there is an $x \in 2^\kappa \setminus L[r]$.*

Proof. This follows directly from Lemma 3.11. □

Let us define, for $x, y \in \kappa^\kappa$, the *eventual domination* relation: $x <^* y$ iff $\exists\alpha\forall\beta > \alpha\,(x(\beta) < y(\beta))$. We will simply say "$y$ dominates x" for $x <^* y$ and if $X \subseteq \kappa^\kappa$ we will say "y dominates X" iff $\forall x \in X\,(x <^* y)$. We will also say "$y$ is unbounded over x" iff $x \not>^* y$ and "y is unbounded over X" iff $\forall x \in X\,(x \not>^* y)$. Note that for the next lemma, it is not relevant whether we talk about domination in the space of all elements of κ^κ or only the strictly increasing ones.

In [3, Theorem 6.1] it is proved that $\mathbf{\Delta}^1_2(\mathbb{M})$ implies the existence of unbounded reals over $L[r]$ for every real r. This generalizes to the κ^κ-context assuming κ is an inaccessible.

Lemma 4.4. *Suppose κ is inaccessible. Then $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$ implies that for every $r \in \kappa^\kappa$ there is an $x \in \kappa^\kappa_\uparrow$ which is unbounded over $\kappa^\kappa_\uparrow \cap L[r]$.*

Proof. The proof is based on the proof of [3, Theorem 6.1]. Assuming that there are no unbounded reals over $\kappa^\kappa_\uparrow \cap L[r]$ we will construct a $\mathbf{\Sigma}^1_1$-definable sequence $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ of reals in $L[r]$ which is dominating, well-ordered by $<^*$, and satisfies some additional technical properties. This will yield two non-κ-Miller-measurable sets A and B defined by $A := \{x \in \kappa^\kappa_\uparrow \mid$ the least α such that $x \leq^* f_\alpha$ is even$\}$ and $B := \{x \in \kappa^\kappa_\uparrow \mid$ the least α such that $x \leq^* f_\alpha$ is odd$\}$, where, by convention, limit ordinals are considered even.

To begin with, we fix an enumeration $\langle \boldsymbol{\sigma}_i \mid i < \kappa\rangle$ of $\kappa^{<\kappa}_\uparrow \setminus \{\varnothing\}$. Let $\ulcorner\sigma\urcorner$ denote i such that $\sigma = \boldsymbol{\sigma}_i$, and also well-order $\kappa^{<\kappa}_\uparrow \setminus \{\varnothing\}$ by $\preceq$, defined by $\sigma \preceq \tau$ iff $\ulcorner\sigma\urcorner \leq \ulcorner\tau\urcorner$. We also use the following notation: for all $\sigma \in \kappa^{<\kappa}_\uparrow$ of successor length, let $\sigma(\text{last})$ denote the last digit of σ, i.e., $\sigma(\text{len}(\sigma) - 1)$.

Next, let $\mathcal{C}$ denote the set $\{\sigma \in \kappa^{<\kappa}_\uparrow \mid \text{len}(\sigma)$ is a successor$\}$. Define a fixed function $\varphi_0 : \mathcal{C} \to \kappa$ by letting $\varphi_0(\sigma)$ be the least $i < \kappa$ such that $\boldsymbol{\sigma}_i(0) > \sigma(\text{last})$. The function φ_0 should be understood as a "lower bound" on potential other functions $\varphi : \mathcal{C} \to \kappa$ satisfying $\boldsymbol{\sigma}_{\varphi(\sigma)}(0) > \sigma(\text{last})$.

Let T be a given κ-Miller tree T, and assume, without loss of generality, that every splitting node of T is club-splitting. We recursively define a collection $\langle \tilde{\tau}^T_\sigma \mid \sigma \in \kappa^{<\kappa}_\uparrow\rangle$ of split-nodes of T, and another collection $\langle \tau^T_\sigma \mid \sigma \in \mathcal{C}\rangle$, as follows:

- $\tilde{\tau}^T_\varnothing = \text{stem}(T)$.
- Assuming $\tilde{\tau}^T_\sigma$ is defined, and given a $\beta < \kappa$, let $\tau^T_{\sigma^\frown\langle\beta\rangle}$ be $\boldsymbol{\sigma}_i$ for the least i such that
  - $\tilde{\tau}^T_\sigma {}^\frown \boldsymbol{\sigma}_i \in \text{Split}(T)$, and
  - $\boldsymbol{\sigma}_i(0) > \beta$.

  Then let $\tilde{\tau}^T_{\sigma^\frown\langle\beta\rangle} := \tilde{\tau}^T_\sigma {}^\frown \tau^T_{\sigma^\frown\langle\beta\rangle}$.
- For σ with $\text{len}(\sigma) = \lambda$ limit, let $\tilde{\tau}^T_\sigma := \bigcup_{\alpha<\lambda} \tilde{\tau}^T_{\sigma\restriction\alpha}$. Note that $\tilde{\tau}^T_\sigma \in \text{Split}(T)$ by the assumption that limits of splitting nodes in T are splitting.

Intuitively, each τ^T_σ, for σ of successor length, gives us a $\preceq$-minimal extension within the tree T, whose first digit is strictly higher then the a-priori-prescribed value $\sigma(\text{last})$. Define a function $\varphi_T : \mathcal{C} \to \kappa$ by $\varphi_T(\sigma) := \ulcorner\tau^T_\sigma\urcorner$. This function will be used as a lower bound later. Notice that for any κ-Miller tree T we have $\varphi_0 \leq \varphi_T$, and in fact $\varphi_0 = \varphi_{(\kappa^{<\kappa}_\uparrow)}$ (i.e., the φ_T for $T = \kappa^{<\kappa}_\uparrow =$ the trivial $\mathbb{M}_\kappa$-condition).

It is worth noting that since the values of $\varphi_T(\sigma)$ and $\varphi_0(\sigma)$ only depend on $\sigma(\text{last})$, these functions could also be construed as functions from κ to κ. However, for technical reasons, it is necessary to consider them as functions from $\mathcal{C}$ to κ.

Next, for a fixed function $f : \kappa \to \kappa$, another function $\varphi : \kappa^{<\kappa}_\uparrow \to \kappa$ satisfying $\varphi_0 \leq \varphi$, and an ordinal $\beta < \kappa$, we define a special set $S = S(\varphi, f, \beta)$ of κ-reals. This set will be defined by specifying "fronts" S_α, for $\alpha < \kappa$. Each S_α will be a subset of $\kappa^{<\kappa}_\uparrow$, satisfying the following two requirements:

1. $|S_\alpha| < \kappa$, and
2. $\forall\rho \in S_\alpha\,(\text{len}(\rho) \geq \alpha)$.

Moreover, every $\rho \in S_{\alpha+1}$ will be a proper extension of a $\rho' \in S_\alpha$. We construct the S_α recursively as follows:

- $S_0 := \{\boldsymbol{\sigma}_i \mid i \leq \beta\}$.
- $S_1 := \{\rho^\frown\boldsymbol{\sigma}_i \mid \rho \in S_0, i \leq \varphi(\langle\beta\rangle) \text{ and } \boldsymbol{\sigma}_i(0) > \beta\}$.
  Notice that since $\varphi_0(\langle\beta\rangle) \leq \varphi(\langle\beta\rangle)$ there is at least one $\boldsymbol{\sigma}_i$ satisfying the above requirement. In particular, all elements of S_1 have length ≥ 1. It is also clear that $|S_1| < \kappa$.
- Let $\text{height}(S_1) := \sup\{\text{len}(\rho) \mid \rho \in S_1\}$ and let $f^*(1) := \sup(\{\beta\} \cup \{f(\xi) \mid \xi < \text{height}(S_1)\})$. Now let

  $$S_2 := \{\rho^\frown\boldsymbol{\sigma}_i \mid \rho \in S_1, i \leq \varphi(\langle\beta, f^*(1)\rangle) \text{ and } \boldsymbol{\sigma}_i(0) > f^*(1)\}.$$

  Again notice that since $\varphi_0(\langle\beta, f^*(1)\rangle) \leq \varphi(\langle\beta, f^*(1)\rangle)$, there exists at least one $\boldsymbol{\sigma}_i$ as above, so all element of S_2 have length ≥ 2. Also it is clear that $|S_2| < \kappa$.
- Generally, assume S_α is defined as well as $f^*(\xi)$ for all $\xi < \alpha$. Let $\text{height}(S_\alpha) := \sup\{\text{len}(\rho) \mid \rho \in S_\alpha\}$, which is an ordinal $< \kappa$ by the inductive assumption that $|S_\alpha| < \kappa$. Let $f^*(\alpha) := \sup(\{\beta\} \cup \{f(\xi) \mid \xi < \text{height}(S_\alpha)\})$. Then let

  $$S_{\alpha+1} := \{\rho^\frown\boldsymbol{\sigma}_i \mid \rho \in S_\alpha, i \leq \varphi(\langle\beta, f^*(1), \dots, f^*(\alpha)\rangle) \text{ and } \boldsymbol{\sigma}_i(0) > f^*(\alpha)\}.$$

  As before, $\varphi_0(\langle\beta, f^*(1), \dots, f^*(\alpha)\rangle) \leq \varphi(\langle\beta, f^*(1), \dots, f^*(\alpha)\rangle)$ implies that all members of $S_{\alpha+1}$ have length $\geq \alpha + 1$. Also $|S_{\alpha+1}| < \kappa$ is clear.
- Suppose λ is limit. Let S_λ be the collection of $\rho \in \kappa^{<\kappa}_{\uparrow}$ such that $\rho = \bigcup_{\alpha<\lambda} \rho_\alpha$ for some strictly $\subseteq$-increasing sequence $\{\rho_\alpha \mid \alpha < \lambda\}$ with $\rho_\alpha \in S_\alpha$. Clearly all such ρ have length $\geq \lambda$. By the inductive assumption that $|S_\alpha| < \kappa$ for all $\alpha < \lambda$, and the fact that κ is inaccessible, it follows that $|S_\lambda| < \kappa$.

Finally we let $S = S(\varphi, f, \beta)$ to be the set of all κ-reals x such that $x = \bigcup_{\alpha<\kappa} \rho_\alpha$ for some strictly $\subseteq$-increasing sequence $\{\rho_\alpha \mid \alpha < \kappa\}$ with $\rho_\alpha \in S_\alpha$. The essential properties of $S(\varphi, f, \beta)$ are summarized in the next sublemma:

Sublemma 4.5.

1. *For every $S(\varphi, f, \beta)$, there exists a function $g \in \kappa^\kappa$ which bounds $S(\varphi, f, \beta)$ (i.e., $\forall x \in S(\varphi, f, \beta)\ \forall i < \kappa\ ((x(i) < g(i)))$).*
2. *Every $x \in S(\varphi, f, \beta)$ is cofinally often above f (i.e., $x \not\leq^* f$).*
3. *For every κ-Miller tree T, f and φ satisfying $\varphi_T <^* \varphi$, there exists $\beta < \kappa$ such that $[T] \cap S(\varphi, f, \beta) \neq \varnothing$.*

Proof.

1. By construction, if ρ is any initial segment of any $x \in S(\varphi, f, \beta)$ with $\text{len}(\rho) = \alpha$, then ρ must be an initial segment of some sequence from S_α. We can thus define g by stipulating that $g(\alpha)$ be above $\rho(\alpha)$ for all $\rho \in S_{\alpha+1}$, which can always be done since $|S_{\alpha+1}| < \kappa$. Now it is clear that for every $x \in S(\varphi, f, \beta)$, for every α we have $x(\alpha) < g(\alpha)$ (another way to explain this is: the tree generated by $\bigcup_{\alpha<\kappa} S_\alpha$ is $<\kappa$-branching).
2. By construction, each $S_{\alpha+1}$ contains only those $\rho^\frown\boldsymbol{\sigma}_i$ where $\boldsymbol{\sigma}_i(0) > f^*(\alpha)$. In particular $\boldsymbol{\sigma}_i(0) > f(\text{len}(\rho))$. Therefore $x(\xi) > f(\xi)$ happens cofinally often for every $x \in S(\varphi, f, \beta)$.
3. This is the main point of the proof. First, note that since $\varphi_T <^* \varphi$, there are only $<\kappa$-many σ satisfying $\varphi_T(\sigma) \geq \varphi(\sigma)$. In particular, we can pick $\beta < \kappa$ such that
   (a) $\beta > \ulcorner\text{stem}(T)\urcorner$, and

(b) $\varphi_T(\langle\beta\rangle^\frown\sigma) < \varphi(\langle\beta\rangle^\frown\sigma)$ holds for *all* σ.

After β has been fixed, the set $S(\varphi, f, \beta)$ is also fixed. In particular, f^* can be computed from f as it was done in the construction of the S_α's. Let

$$\vec{f} := \langle\beta\rangle^\frown\langle f^*(\alpha) \mid 1 \leq \alpha < \kappa\rangle$$

and for all $\alpha < \kappa$ use the abbreviation:

$$\rho_\alpha := \tilde{\tau}^T_{\vec{f}\upharpoonright\alpha}.$$

Then $x := \bigcup_{\alpha<\kappa} \rho_\alpha = \bigcup_{\alpha<\kappa} \tilde{\tau}^T_{\vec{f}\upharpoonright\alpha}$ is a branch through $[T]$. On the other hand, we claim that $\rho_\alpha \in S_\alpha$ for all α:

- Since $\ulcorner\text{stem}(T)\urcorner < \beta$ and $\rho_0 = \tilde{\tau}^T_\varnothing = \text{stem}(T)$, by construction $\rho_0 \in S_0$.
- Since $\varphi_T(\langle\beta\rangle) < \varphi(\langle\beta\rangle)$, $\ulcorner\tau^T_{\langle\beta\rangle}\urcorner = \varphi_T(\langle\beta\rangle)$, $\tau^T_{\langle\beta\rangle}(0) > \beta$, and

$$\rho_1 = \tilde{\tau}^T_{\langle\beta\rangle} = \tilde{\tau}^T_\varnothing {}^\frown \tau^T_{\langle\beta\rangle} = \rho_0^\frown\tau^T_{\langle\beta\rangle},$$

  by construction $\rho_1 \in S_1$.
- Assume $\rho_\alpha \in S_\alpha$. Since $\varphi_T(\vec{f}\upharpoonright(\alpha+1)) < \varphi(\vec{f}\upharpoonright(\alpha+1))$, $\ulcorner\tau^T_{\vec{f}\upharpoonright(\alpha+1)}\urcorner = \varphi_T(\vec{f}\upharpoonright(\alpha+1))$, $\tau^T_{\vec{f}\upharpoonright(\alpha+1)}(0) > f^*(\alpha)$ and

$$\rho_{\alpha+1} = \tilde{\tau}^T_{\vec{f}\upharpoonright(\alpha+1)} = \tilde{\tau}^T_{\vec{f}\upharpoonright\alpha}{}^\frown\tau^T_{\vec{f}\upharpoonright(\alpha+1)} = \rho_\alpha^\frown\tau^T_{\vec{f}\upharpoonright(\alpha+1)},$$

  by construction $\rho_{\alpha+1} \in S_{\alpha+1}$.
- For limits λ we have $\rho_\lambda = \tilde{\tau}^T_{\vec{f}\upharpoonright\lambda} = \bigcup_{\alpha<\lambda} \tilde{\tau}^T_{\vec{f}\upharpoonright\alpha} = \bigcup_{\alpha<\lambda} \rho_\alpha$. Since inductively $\rho_\alpha \in S_\alpha$, by definition we have $\rho_\lambda \in S_\lambda$.

Since $\rho_\alpha \in S_\alpha$ for all $\alpha < \kappa$ we obtain $x = \bigcup_{\alpha<\kappa} \rho_\alpha \in S(\varphi, f, \beta)$, as had to be shown. □ (Sublemma).

To complete the proof of the main lemma, assume, towards contradiction, that $\kappa^\kappa_\uparrow \cap L[r]$ is a dominating set, for some r. Construct a sequence $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ of elements of $\kappa^\kappa_\uparrow \cap L[r]$, and an auxiliary sequence $\langle\varphi_\alpha \mid \alpha < \kappa^+\rangle$ of elements of $\kappa^{\mathcal{C}} \cap L[r]$, in such a way that:

1. $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ and $\langle\varphi_\alpha \mid \alpha < \kappa^+\rangle$ are well-ordered by $<^*$,
2. $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ is a dominating subset of $\kappa^\kappa_\uparrow \cap L[r]$ and $\langle\varphi_\alpha \mid \alpha < \kappa^+\rangle$ is a dominating subset of $\kappa^{\mathcal{C}} \cap L[r]$,
3. all φ_α are pointwise strictly above φ_0,
4. $f_{\alpha+1}$ dominates $S(\varphi_\alpha, f_\alpha, \beta)$ for all β, and
5. both sequences have $\mathbf{\Sigma}^1_1$-definitions.

To see that this can be done, at each step α inductively pick the $<_{L[a]}$-least f_α and φ_α dominating all the previous functions; to satisfy point 4 above, use Sublemma (1) to dominate each $S(\varphi_\alpha, f_\alpha, \beta)$ by a corresponding function g_β, and then dominate $\{g_\beta \mid \beta < \kappa\}$ by another g.

Now, as suggested earlier, define $A := \{x \in \kappa^\kappa_\uparrow \mid$ the least f_α which dominates x is even$\}$ and $B := \{x \in \kappa^\kappa_\uparrow \mid$ the least f_α which dominates x is odd$\}$. Clearly $A \cap B = \varnothing$, and by assumption $A \cup B = \kappa^\kappa_\uparrow$. Since the sequence of f_α's was $\mathbf{\Sigma}^1_1$-definable, the sets A and B are also $\mathbf{\Sigma}^1_1$-definable, hence they are both $\mathbf{\Delta}^1_1$. To reach a contradiction, let T be a κ-Miller tree, and we will show that $[T]$ contains an element in A and an element in B. Since the sequence $\langle\varphi_\alpha \mid \alpha < \kappa^+\rangle$ is dominating, there exists an α such that for all $\xi \geq \alpha$ we have $\varphi_T <^* \varphi_\xi$. In particular $\varphi_T <^* \varphi_\alpha$ and $\varphi_T <^* \varphi_{\alpha+1}$. By point 3 of the Sublemma, we can find β and β' such that

$$[T] \cap S(\varphi_\alpha, f_\alpha, \beta) \neq \varnothing, \text{ and}$$
$$[T] \cap S(\varphi_{\alpha+1}, f_{\alpha+1}, \beta') \neq \varnothing.$$

Without loss of generality α is even. Let y be an element of the first set. By point 2 of the Sublemma, $y \not<^* f_\alpha$, and by construction, $y <^* f_{\alpha+1}$. Hence $y \in B$. Likewise, let y' be an element of the second set. Then by an analogous argument $y' \not<^* f_{\alpha+1}$ but $y' <^* f_{\alpha+2}$. Hence $y' \in A$. This completes the proof. □

Question 4.6. *Can Lemma 4.4 be proved without assuming that κ is inaccessible?*

So far, these are the only generalizations of classical Solovay-style characterizations known to us. The other result due to Brendle and Löwe linked Laver-measurability with dominating reals. However, that proof does not seem to generalize to the κ^κ-setting because κ-Laver-measurability differs from classical Laver-measurability in the sense that the ideal $\mathcal{I}_\mathbb{L}$ cannot be neglected (see Lemma 4.1). Therefore the following is still open:

Question 4.7. *Does $\mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ imply that for every $r \in \kappa^\kappa$, there is an x which is dominating over $L[r]$?*

Likewise, currently we do not have suitable Solovay-style consequences of the assumptions $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$. In the classical setting, there is a connection between these properties and splitting/unsplit reals.

Question 4.8. *Can the hypotheses $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$ be linked to the existence of (a suitable generalization of) splitting/unsplit reals?*

4.2. Comparing $\mathbf{\Delta}^1_1(\mathbb{P})$

The next questions we want to ask are: for which $\mathbb{P}$ and $\mathbb{Q}$ does $\mathbf{\Delta}^1_1(\mathbb{P})$ imply $\mathbf{\Delta}^1_1(\mathbb{Q})$, and for which $\mathbb{P}$ and $\mathbb{Q}$ can we construct models where $\mathbf{\Delta}^1_1(\mathbb{P}) + \neg\mathbf{\Delta}^1_1(\mathbb{Q})$ holds? We will prove several implications for arbitrary pointclasses $\mathbf{\Gamma}$ in Lemma 4.9. Classical counterparts of such implications are well-known but generally much easier to prove, as the uncountable context provides combinatorial challenges not present when $\kappa = \omega$.

Separating regularity properties is currently very difficult for the following two reasons:

1. We do not have good Solovay-style characterizations, and
2. We do not have good preservation theorems for forcing iterations.

We will finish this section with the only example of such a separation result currently known to us.

Lemma 4.9. *Let $\mathbf{\Gamma}$ be a class of subsets of κ^κ or 2^κ closed under continuous preimages (in particular $\mathbf{\Gamma} = \mathbf{\Delta}^1_1$). Then*

1. $\mathbf{\Gamma}(\mathbb{M}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{S}_\kappa)$.
2. $\mathbf{\Gamma}(\mathbb{V}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{S}_\kappa)$.
3. $\mathbf{\Gamma}(\mathbb{C}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{M}_\kappa)$.
4. $\mathbf{\Gamma}(\mathbb{L}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{M}_\kappa)$.
5. $\mathbf{\Gamma}(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{M}_\kappa)$.
6. *If κ is inaccessible, then $\mathbf{\Gamma}(\mathbb{C}_\kappa) \Rightarrow \mathbf{\Gamma}(\mathbb{V}_\kappa)$.*

Proof.

1. Let $A \subseteq 2^\kappa$ be a set in $\mathbf{\Gamma}$ and let T be a κ-Sacks tree. We must find a κ-Sacks tree below T whose branches are completely contained in or disjoint from A. Let φ be the natural order-preserving bijection identifying $2^{<\kappa}$ with $\mathrm{Split}(T)$, and φ^* the induced homeomorphism between 2^κ and $[T]$. Further, fix a stationary, co-stationary set $S \subseteq \kappa$ and enumerate $S := \{\xi_\alpha \mid \alpha < \kappa\}$ and $\kappa \setminus S := \{\eta_\alpha \mid \alpha < \kappa\}$. Let ψ be a map from $\kappa_\uparrow^{<\kappa}$ to $2^{<\kappa}$ defined by:
   - $\psi(\varnothing) = \varnothing$.
   - $\psi(s^\frown\langle\alpha\rangle) := \begin{cases} \psi(s)^\frown\langle 1\rangle^\frown 0^\beta{}^\frown\langle 1\rangle & \text{if } \alpha \in S \text{ and } \alpha = \xi_\beta \\ \psi(s)^\frown\langle 0\rangle^\frown 0^\beta{}^\frown\langle 1\rangle & \text{if } \alpha \notin S \text{ and } \alpha = \eta_\beta \end{cases}$
     where 0^β denotes a β-sequence of 0's.
   - $\psi(s) := \bigcup_{\alpha<\lambda} \psi(s\restriction\alpha)$, if $\mathrm{len}(s) = \lambda$ for a limit ordinal.

   The function ψ is different from a standard encoding of ordinals by binary sequences, but it is clear that ψ is bijective, since there is an obvious algorithm to compute $\psi^{-1}(s)$ for any $s \in 2^{<\kappa}$. The reason for using this specific function is that we want $\psi(s)$ to be a splitting node whenever s is a club-splitting node. Clearly, ψ induces a homeomorphism ψ^* between $\kappa_\uparrow^\kappa$ and $2^\kappa \setminus \mathbb{Q}$, where we use $\mathbb{Q}$ to denote the *generalized rationals*, i.e., $\mathbb{Q} := \{x \in 2^\kappa \mid |\{i \mid x(i) = 1\}| < \kappa\}$.
   Let $A' := (\varphi^* \circ \psi^*)^{-1}[A]$, which is in $\mathbf{\Gamma}$ by assumption. By $\mathbf{\Gamma}(\mathbb{M}_\kappa)$ we can find a κ-Miller tree R such that $[R] \subseteq A'$ or $[R] \cap A' = \varnothing$, w.l.o.g. the former. Let $R' := \{\psi(s) \mid s \in R\}$. First, note that R' is a κ-Sacks tree: this follows because for any $s \in \mathrm{Split}(R)$ there are $\alpha \in S$ and $\beta \notin S$ such that both $s^\frown\langle\alpha\rangle$ and $s^\frown\langle\beta\rangle$ are in R, which implies that both $\psi(s)^\frown\langle 1\rangle$ and $\psi(s)^\frown\langle 0\rangle$ are in R', so $\psi(s) \in \mathrm{Split}(R')$. Moreover, since ψ^* is a homeomorphism, we know that $[R'] \setminus \mathbb{Q} = (\psi^*)``[R] \subseteq (\varphi^*)^{-1}[A]$. But since $\mathbb{Q}$ is a set of size $\kappa$ we can easily find a refinement $R'' \subseteq R'$, which is still a $\kappa$-Sacks tree and moreover $[R''] \subseteq (\psi^*)``[R] \subseteq (\varphi^*)^{-1}[A]$. Then $(\varphi^*)``[R'']$ generates a κ-Sacks tree which is completely contained in $[T] \cap A$.
2. Let $A \in \mathbf{\Gamma}$ and $T \in \mathbb{S}_\kappa$ and φ and φ^* be as above. Then $A' := (\varphi^*)^{-1}[A]$ is in $\mathbf{\Gamma}$ so there exists a κ-Silver tree S such that $[S] \subseteq A$ or $[S] \cap A = \varnothing$. As S is a κ-Sacks tree, clearly $\varphi``S$ generates a κ-Sacks tree below T whose branches are completely contained in or completely disjoint from A.
3. Now let $A \subseteq \kappa_\uparrow^\kappa$ be in $\mathbf{\Gamma}$ and let T be a κ-Miller tree. By shrinking if necessary, we may assume T to have the property that all splitting nodes are club-splitting. Let φ be the natural order-preserving bijection between $\kappa_\uparrow^{<\kappa}$ and $\mathrm{Split}(T)$, and φ^* the induced homeomorphism between $\kappa_\uparrow^\kappa$ and $[T]$. Let $A' := (\varphi^*)^{-1}[A]$. As A' has the Baire property by $\mathbf{\Gamma}(\mathbb{C}_\kappa)$, let $[s]$ be a basic open set such that $[s] \subseteq^* A'$ or $[s] \cap A' =^* \varnothing$, and without loss of generality assume the former. Let $\{X_i \mid i < \kappa\}$ be nowhere dense sets such that $[s] \setminus A' = \bigcup_{i<\kappa} X_i$. We will inductively construct a κ-Miller tree S such that $[S] \subseteq A'$ and $[S] \cap X_i = \varnothing$ for all $i < \kappa$.
   - Let S_0 be the tree generated by $\{s\}$.
   - Suppose S_i has been defined for $i < \kappa$. Let $\mathrm{Term}(S_i)$ be the collection of terminal branches of S_i (i.e., those $\sigma \in S_i$ such that $\mathrm{Succ}_{S_i}(\sigma) = \varnothing$), and for each $\sigma \in \mathrm{Term}(S_i)$ and $\alpha < \kappa$, let $\tau_{\sigma,\alpha}$ be an extension of $\sigma^\frown\langle\alpha\rangle$ such that $[\tau_{\sigma,\alpha}] \cap X_i = \varnothing$. Now let S_{i+1} be the tree generated by $\{\tau_{\sigma,\alpha} \mid \sigma \in \mathrm{Term}(S_i)$ and $\alpha < \kappa\}$.
   - For limits $\lambda < \kappa$, let S_λ be the tree generated by cofinal branches through $\bigcup_{\alpha<\lambda} S_\alpha$.

   By construction, $S := \bigcup_{i<\kappa} S_i$ is a κ-Miller tree (all splitting nodes of S are in fact fully splitting). Moreover $[S] \subseteq [s]$ and $[S] \cap X_i = \varnothing$ for all $i < \kappa$. In particular, $[S] \subseteq A'$. But now it follows easily that $\varphi``S$ generates a κ-Miller tree below T, whose branches are completely contained in A.
4. This follows a similar strategy as above, but using the topology generated by $\mathbb{L}_\kappa$ instead of the standard topology. Let $A \in \kappa_\uparrow^\kappa$ be in $\mathbf{\Gamma}$, $T \in \mathbb{M}_\kappa$, φ and φ^* be as above, and let $A' := (\varphi^*)^{-1}[A]$. As A' is $\mathbb{L}_\kappa$-measurable, there is a κ-Laver tree R such that $[R] \subseteq^* A'$ or $[R] \cap A' =^* \varnothing$, where $\subseteq^*$ and $=^*$ means

"modulo $\mathcal{I}_{\mathbb{L}_\kappa}$". Without loss of generality assume the former and let $\{X_i \mid i < \kappa\}$ be in $\mathcal{N}_{\mathbb{L}_\kappa}$ such that $[R] \setminus A' = \bigcup_{i<\kappa} X_i$. Again we will construct a κ-Miller tree S such that $[S] \subseteq A'$ and $[S] \cap X_i = \varnothing$ for all $i < \kappa$.

We will need to perform a fusion argument on $\mathbb{M}_\kappa$, so we introduce some terminology. For a κ-Miller tree S, a node $s \in S$ is called an *i-th splitting node* iff $s \in \mathrm{Split}(S)$ and the set $\{j < i \mid s{\restriction}j \in \mathrm{Split}(S)\}$ has order-type i. $\mathrm{Split}_i(S)$ denotes the set of i-th splitting nodes of S. The standard fusion for $\mathbb{M}_\kappa$ (cf. Fact 2.11 (2)) is defined by $S' \leq_i S$ iff $S' \leq S$ and $\mathrm{Split}_i(S') = \mathrm{Split}_i(S)$. We will build a fusion sequence $\{S_i \mid i < \kappa\}$ of κ-Miller trees, but with the following additional property

$$(*) \qquad \forall i\, \forall s \in \mathrm{Split}_i(S_i)\ (S_i{\uparrow}s \text{ is a } \kappa\text{-Laver tree with stem } s).$$

Note that if s is as above, then every $t \in S_i$ extending s also has the property that $S_i{\uparrow}t$ is a κ-Laver tree with stem t.

- Let $S_0 := R$.
- Suppose S_i has been defined for $i < \kappa$. Pick $\sigma \in \bigcup\{\mathrm{Succ}_{S_i}(\rho) \mid \rho \in \mathrm{Split}_i(S_i)\}$. By $(*)$ we know that $S_i{\uparrow}\rho$, and therefore also $S_i{\uparrow}\sigma$, is a κ-Laver tree. So let $S_\sigma \leq S_i{\uparrow}\sigma$ be a κ-Laver tree such that $[S_\sigma] \cap X_i = \varnothing$. Then let

  $$S_{i+1} := \bigcup\{S_\sigma \mid \sigma \in \bigcup\{\mathrm{Succ}_{S_i}(\rho) \mid \rho \in \mathrm{Split}_i(S_i)\}\}.$$

  By construction S_{i+1} is a κ-Miller tree, $S_{i+1} \leq_i S_i$, and condition $(*)$ is satisfied.
- For limits $\lambda < \kappa$, let $S_\lambda := \bigcap_{i<\lambda} S_i$. By a standard fusion argument, S_λ is a κ-Miller tree and $S_\lambda \leq_i S_i$ for all $i < \lambda$. Moreover, any $\sigma \in \mathrm{Split}_\lambda(S_\lambda)$ is the extension of a λ-splitting node of S_i for every i, so by condition $(*)$, $S_i{\uparrow}\sigma$ is a κ-Laver tree with stem σ, for every $i < \lambda$. By $<\kappa$-closure of $\mathbb{L}_\kappa$, it follows that $S_\lambda{\uparrow}\sigma = \bigcap_{i<\lambda}(S_i{\uparrow}\sigma)$ is a κ-Laver tree with stem σ, hence S_λ satisfies condition $(*)$.

By construction, $S := \bigcap_{i<\kappa} S_i$ is a κ-Miller tree, $[S] \subseteq [R]$, and $[S] \cap X_i = \varnothing$ for all $i < \kappa$. In particular, $[S] \subseteq A'$. Now it follows that φ"S generates a κ-Miller tree below T, whose branches are completely contained in A.

5. This part is completely analogous to 4. Note that κ-Mathias conditions are special kinds of κ-Laver trees, and $\mathbb{R}_\kappa$ is also $<\kappa$-closed.
6. Here it is easier to consider $\mathbb{C}_\kappa$ on 2^κ as opposed to κ^κ. It is not hard to see that the two properties are equivalent for $\mathbf{\Gamma}$. Let $A \subseteq 2^\kappa$ be in $\mathbf{\Gamma}$, let $T \in \mathbb{V}_\kappa$, let φ be the natural order-preserving bijection between 2^κ and the splitnodes of T, and let φ^* be the induced homeomorphism between 2^κ and $[T]$. Let $A' := (\varphi^*)^{-1}[A]$, and using $\mathbf{\Gamma}(\mathbb{C}_\kappa)$ let $s \in 2^{<\kappa}$ be such that $[s] \subseteq^* A'$ or $[s] \cap A' =^* \varnothing$, without loss of generality the former. Let X_i be nowhere dense such that $[s] \setminus A' = \bigcup_{i<\kappa} X_i$. As before, we will inductively construct a κ-Silver tree S such that $[S] \subseteq [s]$ and $[S] \cap X_i = \varnothing$ for all i.

   In this construction, it will be easier to view κ-Silver conditions as functions from κ to $\{0, 1, \{0, 1\}\}$. We will use the following notation: for $f : \alpha \to \{0, 1, \{0, 1\}\}$ let

   $$[f] := \{x \in 2^\alpha \mid \forall i\, (f(i) \in \{0, 1\} \to x(i) = f(i))\}.$$

   Notice that if $f : \kappa \to \{0, 1, \{0, 1\}\}$ and $f(i) = \{0, 1\}$ for club-many i, then the corresponding κ-Silver tree can be defined as $S_f := \{\sigma \in 2^{<\kappa} \mid \sigma \in [f{\restriction}\mathrm{len}(\sigma)]\}$, and we have $[S_f] = [f]$. We will construct a function f as the limit of f_α's, defined as follows:

   - $f_0 := s$.
   - Since X_0 is nowhere dense, let τ_1 be such that $[s^\frown\langle 0\rangle^\frown\tau_1] \cap X_0 = \varnothing$. Then let $\tau_2 \supseteq \tau_1$ be such that $[s^\frown\langle 1\rangle^\frown\tau_2] \cap X_0 = \varnothing$. Now set

     $$f_1 := s^\frown\langle\{0, 1\}\rangle^\frown\tau_2.$$

Notice that for any $x \in 2^\kappa$ extending any $\sigma \in [f_1]$ we have $x \notin X_0$.

- Suppose f_i is defined for $i < \kappa$. Let $\{\sigma_\alpha \mid \alpha < 2^i\}$ enumerate all sequences in $[f_i^\frown\langle\{0,1\}\rangle]$ and define $\{\tau_\alpha \mid \alpha < 2^i\}$ by induction as follows:
  - $\tau_0 = \varnothing$.
  - If τ_α is defined let $\tau_{\alpha+1} \supseteq \tau_\alpha$ be such that $[\sigma_\alpha^\frown\tau_{\alpha+1}] \cap X_i = \varnothing$.
  - For limits λ let $\tau_\lambda := \bigcup_{\alpha<\lambda} \tau_\alpha$.

  Then define $\tau_{2^i} := \bigcup_{\alpha<2^i} \tau_\alpha$ and notice that $\tau_{2^i} \in 2^\delta$ for $\delta < \kappa$ since κ was inaccessible. Now let

$$f_{i+1} := f_i^\frown\langle\{0,1\}\rangle^\frown\tau_{2^i}.$$

  It is clear that any $x \in 2^\kappa$ extending any $\sigma \in [f_{i+1}]$ is not in X_i.
- For γ limit, let $f_\gamma := \bigcup_{i<\gamma} f_i$.

Finally, we let $f := \bigcup_{i<\kappa} f_i$. By construction $f(i) = \{0,1\}$ for club-many $i < \kappa$, and clearly every $x \in [f]$ is not in X_i for any $i < \kappa$. Hence $S_f := \{\sigma \in 2^{<\kappa} \mid \sigma \in [f{\restriction}\mathrm{len}(\sigma)]\}$ is a κ-Silver tree with $[S_f] \subseteq A'$. Then φ“S_f generates a κ-Silver subtree of T which is completely contained in A, as had to be shown. □

Focusing on $\mathbf{\Gamma} = \mathbf{\Delta}^1_1$, we can summarize the contents of the above results in Fig. 1.[4] Of particular interest are two implications which are present in the classical setting but still seem open in the general setting:

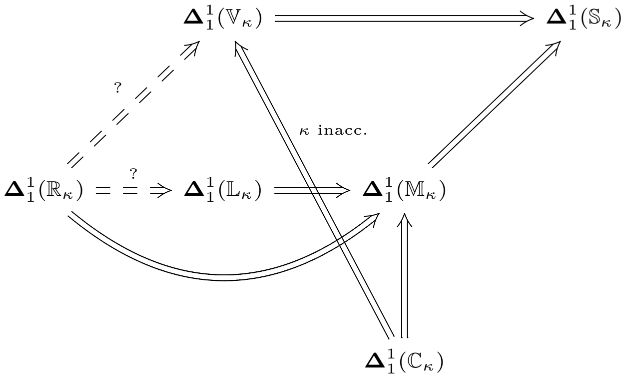

Fig. 1. Diagram of implications for $\mathbf{\Delta}^1_1$.

Question 4.10. *Is $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ true? Is $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ (at least for κ inaccessible) true?*

As mentioned, currently we can prove only the following separation theorem.

Theorem 4.11. *Suppose κ is inaccessible. Then it is consistent that $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{S}_\kappa)$ hold whereas $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$, $\mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$, $\mathbf{\Delta}^1_1(\mathbb{C}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$ fail.*

Proof. It is sufficient to establish $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa) + \neg\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$. Perform a κ^+-iteration of κ-Silver forcing, starting in L, with supports of size κ. An argument completely analogous to [17, Theorem 6.1] shows that this iteration of κ-Silver forcing is κ-proper (so the conditions necessary to apply Theorem 3.13 are satisfied, i.e., κ^+ is preserved and κ-reals in the final extension are captured by an initial segment), and moreover, is κ^κ-*bounding*, i.e., every function $f \in \kappa^\kappa$ in the extension is dominated by a $g \in \kappa^\kappa$ in the ground model. By Theorem 3.13 the generic extension satisfies $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$, while the statement “$\forall r\ \exists x$ (x is unbounded over $\kappa^\kappa \cap L[r]$)” is false, so by Lemma 4.4 $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa)$ fails. □

[4] We arrange the diagram in this particular way in order to be consistent with previous presentations of similar diagrams, e.g. in [6].

Notice that by Remark 3.15 and Lemma 4.9 we can obtain $\mathbf{\Delta}^1_1(\mathbb{P})$ for all $\mathbb{P} \in \{\mathbb{C}_\kappa, \mathbb{S}_\kappa, \mathbb{M}_\kappa, \mathbb{L}_\kappa, \mathbb{R}_\kappa\}$, and also for $\mathbb{P} = \mathbb{V}_\kappa$ if κ is inaccessible, simultaneously in one model, namely $L^{(\mathbb{C}_\kappa * \mathbb{L}_\kappa * \mathbb{R}_\kappa)_{\omega_1}}$.

5. Open questions

We have carried out an initial study of regularity properties related to forcing notions on the generalized reals; but many questions remain open, particularly with regard to the specific examples presented in Section 4.

Question 5.1.

1. *Can Lemma 4.4 be proved without assuming that κ is inaccessible?*
2. *Does $\mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ imply that for every $r \in \kappa^\kappa$, there is an x which is dominating over $L[r]$?*
3. *Can the hypotheses $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa)$ be linked to the existence of (a suitable generalization of) splitting/unsplit reals?*

A more long-term goal would be to find a complete diagram of implications for generalized $\mathbf{\Delta}^1_1$ sets.

Question 5.2. *Which additional implications from Fig. 1 can be proved in ZFC? Which are consistently false? Specifically, does $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Delta}^1_1(\mathbb{L}_\kappa)$ and $\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \Rightarrow \mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$ (at least for κ inaccessible) hold?*

In a more conceptual direction, one should try to better understand the exact role of the club filter, which provides counterexamples for $\mathbf{\Sigma}^1_1$-regularity. For example, perhaps one could prove that the club filter, up to some adequate notion of equivalence, is the *only* $\mathbf{\Sigma}^1_1$-counterexample. Alternatively, one could try to focus on regularity properties such as the ones considered in [27,22], and try to gain a better understanding why the club filter is a counterexample for some regularity properties but not for others. For example, by recent results of Laguzzi and the first author, projective measurability is consistent for a version of Silver forcing in which the splitting levels occur on a normal measure on κ as opposed to the club filter.

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3.6 Full-splitting Miller trees and infinitely often equal reals

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This project started during a research visit to Hamburg in February 2014 (where Laguzzi was a postdoc at the time), after which we continued collaborating via email. The motivation was due to me as well as most of the proofs with some contributions by Laguzzi.

- ***Lemma 2.1***: *Khomskii*
- ***Theorem 2.3***: *Khomskii*
- ***Lemma 2.10***: *Khomskii*
- ***Theorem 3.4***: *Khomskii, with contributions by Laguzzi*
- ***Lemma 4.2***: *Khomskii*
- ***Theorem 4.3***: *Khomskii*
- ***Theorem 4.4***: *Khomskii*
- ***Definition 4.5 and Theorem 4.6***: *Collaboration*
- ***Theorem 5.3***: *Khomskii*
- ***Lemma 5.7***: *Khomskii*

The paper was written by Khomskii.

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Full-splitting Miller trees and infinitely often equal reals

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ABSTRACT

We investigate two closely related partial orders of trees on ω^ω: the *full-splitting Miller trees* and the *infinitely often equal trees*, as well as their corresponding σ-ideals. The former notion was considered by Newelski and Rosłanowski while the latter involves a correction of a result of Spinas. We consider some Marczewski-style regularity properties based on these trees, which turn out to be closely related to the property of Baire, and look at the dichotomies of Newelski–Rosłanowski and Spinas for higher projective pointclasses. We also provide some insight concerning a question of Fremlin whether one can add an infinitely often equal real without adding a Cohen real, which was recently solved by Zapletal.

1. Introduction

A common theme in descriptive set theory and forcing theory on the reals are perfect-set-style *dichotomy theorems*—statements asserting that all Borel (or analytic) sets are either in a σ-ideal $\mathfrak{I}$ on ω^ω, or else contain the branches of a certain kind of tree. When $\mathbb{P}$ denotes the partial order of these trees ordered by inclusion, such a theorem guarantees that there is a dense embedding

$$\mathbb{P} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$$

from $\mathbb{P}$ to the partial order of Borel sets positive with respect to $\mathfrak{I}$ (also ordered by inclusion), and hence that the two posets are forcing-equivalent. The most famous result of this kind is the original *perfect set theorem*, showing that the Sacks partial order (perfect trees ordered by inclusion) densely embeds into the partial order of uncountable Borel sets. Jindřich Zapletal [18,19] developed an extensive theory of *idealized forcing*,

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i.e., forcing with $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ for various σ-ideals $\mathfrak{I}$ on the reals. In Zapletal's framework, properties of the forcing can be studied directly using properties of the σ-ideal. On the other hand, there is a long-established tradition of studying forcing properties using combinatorics on trees. A dichotomy theorem provides the best of both worlds, since it allows us to freely switch back and forth between the "idealized" and the "tree" framework, depending on which suits the situation better.

In this paper we consider two closely related dichotomies. The following two definitions are due to Newelski and Rosłanowski [14].

Definition 1.1. A tree $T \subseteq \omega^{<\omega}$ is called a *full-splitting Miller tree* iff every $t \in T$ has an extension $s \in T$ such that s is *full-splitting*, i.e., $s^\frown \langle n \rangle \in T$ for every n. Let $\mathbb{FM}$ denote the partial order of full-splitting Miller trees ordered by inclusion.

Definition 1.2. For $f : \omega^{<\omega} \to \omega$, let

$$D_f := \{x \in \omega^\omega \mid \forall^\infty n\, (x(n) \neq f(x \restriction n))\}.$$

Then $\mathfrak{D}_\omega := \{A \subseteq \omega^\omega \mid A \subseteq D_f \text{ for some } f\}$.

The original motivation of [14] was the connection to infinite games of the same kind as used by Morton Davis in [4] in the proof of the perfect set theorem from determinacy, but played on ω^ω instead of 2^ω. Let $G^*(A)$ be the game in which Player I chooses $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and Player II chooses $n_i \in \omega$, and I wins iff $s_0{}^\frown \langle n_0 \rangle{}^\frown s_1{}^\frown \langle n_1 \rangle{}^\frown \cdots \in A$. It is easy to see (cf. [15]) that Player I wins $G^*(A)$ if and only if there exists a tree $T \in \mathbb{FM}$ such that $[T] \subseteq A$, and Player II wins $G^*(A)$ if and only if $A \in \mathfrak{D}_\omega$. General properties of so-called *Mycielski ideals* (i.e., ideals of sets for which II wins a corresponding game) imply that $\mathfrak{D}_\omega$ is a σ-ideal on ω^ω. Using Solovay's "unfolding" method (see e.g. [9, Exercise 27.14]) it follows from the determinacy of closed games that analytic sets are either $\mathfrak{D}_\omega$-small or contain $[T]$ for some $T \in \mathbb{FM}$.

The next concept is due to Spinas [16].

Definition 1.3. For every $x \in \omega^\omega$ let $K_x := \{y \in \omega^\omega \mid \forall^\infty n\ (x(n) \neq y(n))\}$, and let $\mathfrak{I}_{\text{ioe}}$ be the σ-ideal generated by K_x, for $x \in \omega^\omega$.

In [16], $\mathfrak{I}_{\text{ioe}}$-positive sets were called "countably infinitely often equal families", since a set A is $\mathfrak{I}_{\text{ioe}}$-positive if and only if for every countable sequence of reals $\{x_i \mid i < \omega\}$ there exists $a \in A$ which hits every x_i infinitely often. The following result was claimed in [16, Theorem 3.3]: "every analytic set is either $\mathfrak{I}_{\text{ioe}}$-small or contains $[T]$ for some $T \in \mathbb{FM}$". This dichotomy is clearly in error, as the simple example below shows:

Example 1.4. Let T be the tree on $\omega^{<\omega}$ defined as follows:

- If $|s|$ is even then $\text{succ}_T(s) = \{0, 1\}$.
- If $|s|$ is odd then $\text{succ}_T(s) = \begin{cases} 2\mathbb{N} & \text{if } s(|s|-1) = 0 \\ 2\mathbb{N}+1 & \text{if } s(|s|-1) = 1 \end{cases}$

where $\text{succ}_T(s) := \{n \mid s^\frown \langle n \rangle \in T\}$. Clearly T is $\mathfrak{I}_{\text{ioe}}$-positive but cannot contain a full-splitting subtree.

The correct dichotomy for the ideal $\mathfrak{I}_{\text{ioe}}$ involves a subtle modification of the concept of a full-splitting tree, suggested by Spinas in private communication.

Definition 1.5 *(Spinas).* A tree $T \subseteq \omega^\omega$ is called an *infinitely often equal tree*, or simply *ioe-tree*, if for each $t \in T$ there exists $N > |t|$, such that for every $k \in \omega$ there exists $s \in T$ extending t such that $s(N) = k$. Let $\mathbb{IE}$ denote the partial order of ioe-trees ordered by inclusion.

Clearly $\mathbb{FM} \subseteq \mathbb{IE}$ while the converse is false by Example 1.4. It is not hard to see that the proof of [16, Theorem 3.3] does yield the following correct dichotomy theorem: "every analytic set is either $\mathfrak{I}_{\text{ioe}}$-small or contains $[T]$ for some $T \in \mathbb{IE}$." This dichotomy, like the one of Newelski–Rosłanowski, also allows for an easy analysis in terms of infinite games, see Definition 4.5 and Theorem 4.6. Moreover, an argument as in Theorem 4.2 provides an alternative, arguably more elementary, proof of the dichotomy. The partial order $\mathbb{IE}$ has been considered in unpublished work of Goldstern and Shelah [7], but hasn't been studied elsewhere to our knowledge.

Summarizing the situation, we have two closely related perfect-set-style dichotomy theorems leading to the following dense embeddings:

$$\mathbb{FM} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{D}_\omega$$
$$\mathbb{IE} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{I}_{\text{ioe}}$$

We will study these objects from various points of view. In Section 2 we look at some general properties of these two forcings/ideals, relating them to one another as well as to Cohen forcing and the meager ideal. In Section 3 we consider regularity properties generated by these forcings/ideals which are closely related to the property of Baire, and in Section 4 we focus on the dichotomies themselves, but for projective classes above analytic. Section 5 is devoted to an interesting problem concerning the forcing $\mathbb{IE}$ and Cohen reals.

We use standard set-theoretic notation; for a tree $T \subseteq \omega^{<\omega}$ and $t \in T$, we write $\text{succ}_T(t) = \{n \mid t^\frown \langle n \rangle \in T\}$ and $T{\uparrow}t$ to denote $\{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$. We will frequently use the notation D_f and K_x to refer to the generators of the σ-ideals $\mathfrak{D}_\omega$ and $\mathfrak{I}_{\text{ioe}}$, as in Definitions 1.2 and 1.3. Also, we will say that two reals $x, y \in \omega^\omega$ are *infinitely often equal* (*ioe*) if $\exists^\infty n\, (x(n) = y(n))$ and *eventually different* (*evd*) if $\forall^\infty n\, (x(n) \neq y(n))$.

We would like to thank Martin Goldstern, Otmar Spinas and Wolfgang Wohofsky for some useful discussion and advice.

2. Some general properties

The first easy observations involve the relationships between $\mathfrak{D}_\omega$, $\mathfrak{I}_{\text{ioe}}$ and the ideal $\mathcal{M}$ of meager subsets of ω^ω.

Lemma 2.1. *$\mathfrak{I}_{\text{ioe}} \subsetneq \mathfrak{D}_\omega \subsetneq \mathcal{M}$.*

Proof. For every $x \in \omega^\omega$ let $f_x : \omega^{<\omega} \to \omega$ be defined by $f_x(s) := x(|s|)$. Then it is easy to see that $K_x \subseteq D_{f_x}$, and since $\mathfrak{D}_\omega$ is a σ-ideal $\mathfrak{I}_{\text{ioe}} \subseteq \mathfrak{D}_\omega$ follows. On the other hand, if T is the tree from Example 1.4 then $[T] \in \mathfrak{D}_\omega$ (because $[T]$ does not contain an $\mathbb{FM}$-subtree) but $[T] \notin \mathfrak{I}_{\text{ioe}}$.

To see $\mathfrak{D}_\omega \subsetneq \mathcal{M}$ notice that for $f : \omega^{<\omega} \to \omega$ the sets $D_{f,N} := \{y \mid \forall n > N\, (y(n) \neq f(y{\restriction}n))\}$ are nowhere dense, so $D_f = \bigcup_{N\in\omega} D_{f,N}$ is meager. On the other hand $\{x \mid \forall n\, (x(2n) = 0)\}$ is meager but contains a full-splitting Miller tree, hence it is not in $\mathfrak{D}_\omega$. □

Lemma 2.2 *(Newelski–Rosłanowski). There exists a continuous function $\varphi : \omega^\omega \to \omega^\omega$ such that for all meager A, $\varphi^{-1}[A] \in \mathfrak{D}_\omega$.*

Proof. Given a fixed enumeration $\{s_i \mid i < \omega\}$ of $\omega^{<\omega}$, let φ be defined by

$$\varphi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown s_{x(2)}{}^\frown \ldots$$

Also let $\varphi' : \omega^{<\omega} \to \omega^{<\omega}$ be a function on initial segments such that $\varphi(x) = \bigcup_{n\in\omega} \varphi'(x{\restriction}n)$. Then, given a nowhere dense set $X \subseteq \omega^\omega$, define $f : \omega^{<\omega} \to \omega$ as follows: given $s \in \omega^{<\omega}$ find t such that $[\varphi'(s)^\frown t] \cap X = \varnothing$, and let i be such that $t = s_i$. Then set $f(s) = i$. One can easily verify that $\varphi^{-1}[X] \subseteq D_f$, which is sufficient since $\mathfrak{D}_\omega$ is a σ-ideal.

An alternative way to view this is as follows: given an arbitrary $\mathbb{FM}$-tree T, $\varphi``[T]$ is non-meager. $\square$

The relationship between $\mathfrak{I}_{\text{ioe}}$, $\mathfrak{D}_\omega$ and $\mathcal{M}$ is also apparent by considering cardinal invariants. Recall the definitions of $\text{cov}(\mathfrak{I}), \text{add}(\mathfrak{I}), \text{cof}(\mathfrak{I})$ and $\text{non}(\mathfrak{I})$ for σ-ideals on ω^ω (see e.g. [1, Section 1.3]). The following result is an easy generalization of [14, Theorem 3.1 and Corollary 3.3].

Theorem 2.3.

1. $\text{cov}(\mathfrak{I}_{\text{ioe}}) = \text{cov}(\mathfrak{D}_\omega) = \text{cov}(\mathcal{M})$ *and* $\text{non}(\mathfrak{I}_{\text{ioe}}) = \text{non}(\mathfrak{D}_\omega) = \text{non}(\mathcal{M})$.
2. $\text{add}(\mathfrak{I}_{\text{ioe}}) = \text{add}(\mathfrak{D}_\omega) = \omega_1$ *and* $\text{cof}(\mathfrak{I}_{\text{ioe}}) = \text{cof}(\mathfrak{D}_\omega) = 2^{\aleph_0}$.

Proof.

1. Since $\mathfrak{I}_{\text{ioe}} \subseteq \mathfrak{D}_\omega \subseteq \mathcal{M}$ it immediately follows that $\text{cov}(\mathcal{M}) \leq \text{cov}(\mathfrak{D}_\omega) \leq \text{cov}(\mathfrak{I}_{\text{ioe}})$. For the other direction, we recall Bartoszyński's characterization [1, Theorem 2.4.1] saying that $\text{cov}(\mathcal{M})$ is the least size of an *eventually different family*, i.e., a family $F \subseteq \omega^\omega$ such that for every $x \in \omega^\omega$ there exists $y \in F$ which is eventually different from x. From this it easily follows that $\text{cov}(\mathfrak{I}_{\text{ioe}}) \leq \text{cov}(\mathcal{M})$. The proof for non is dual.
2. This follows from the following claim, proved by Newelski and Rosłanowski in [14, Theorem 3.2].

   Claim 2.4 *(Newelski–Rosłanowski). Let $\{x_\alpha \mid \alpha < 2^{\aleph_0}\}$ be a collection of reals such that $\forall\alpha \neq \beta\ \exists^\infty n\ (x_\alpha(n) \neq x_\beta(n))$, and for each α put $X_\alpha := \{x \mid \forall n\ (x(n) \neq x_\alpha(n))\}$. Then for every uncountable $F \subseteq 2^{\aleph_0}$, $\bigcup_{\alpha\in F} X_\alpha \notin \mathfrak{D}_\omega$.*

   Since each $A_\alpha \in \mathfrak{I}_{\text{ioe}} \subseteq \mathfrak{D}_\omega$ while $\bigcup_{\alpha\in F} X_\alpha \notin \mathfrak{D}_\omega \supseteq \mathfrak{I}_{\text{ioe}}$, the above claim implies the result for both $\mathfrak{D}_\omega$ and $\mathfrak{I}_{\text{ioe}}$. $\square$

Turning to forcing properties, let us recall some results of Zapletal.

Definition 2.5. A σ-ideal $\mathfrak{I}$ on ω^ω is *σ-generated by closed sets* if every set in $\mathfrak{I}$ is contained in an F_σ-set in $\mathfrak{I}$.

Theorem 2.6 *(Zapletal). If $\mathfrak{I}$ is a σ-ideal on ω^ω σ-generated by closed sets then the forcing $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ is proper and preserves Baire category (non-meager ground-model sets remain non-meager in the extension).*

Proof. See [19, Theorem 4.1.2]. $\square$

Corollary 2.7. $\mathbb{FM}$ *and* $\mathbb{IE}$ *are proper and preserve Baire category. In particular, they do not add dominating or random reals.*

Proof. The generators D_f and K_x are clearly F_σ sets, so the results follows by Zapletal's theorem. It is not too hard to provide direct Axiom A-style proofs for this, in fact for $\mathbb{FM}$ it was already done in [14, Section 2]. $\square$

The following concept is very practical when dealing with idealized forcing notions, and was first explicitly defined in [3,8].

Definition 2.8. Let $\mathfrak{I}$ be a σ-ideal on the reals, and assume that membership of Borel sets in the ideal is a $\boldsymbol{\Sigma}^1_2$ predicate (on Borel codes). Let M be a model of set theory containing all countable ordinals and the defining parameter of $\mathfrak{I}$. Then a real x is called $\mathfrak{I}$-*quasigeneric* over M if and only if for every Borel set $B \in \mathfrak{I}$ with Borel code in M, $x \notin B$.

The importance of $\boldsymbol{\Sigma}^1_2$-definability is that the statement $B \in \mathfrak{I}$ should be absolute between M and V. In general, being an $\mathfrak{I}$-quasigeneric real is much weaker then being a $(B(\omega^\omega) \setminus \mathfrak{I})$-generic real. For example, a real is Sacks-quasigeneric (i.e., $\mathfrak{I}_{\mathsf{ctbl}}$-quasigeneric, where $\mathfrak{I}_{\mathsf{ctbl}}$ is the ideal of countable subsets of ω^ω) over M if and only if $x \notin M$; and it is Miller-quasigeneric (i.e., K_σ-quasigeneric, where K_σ is the ideal of σ-*compact* subsets of ω^ω) over M if and only if it is unbounded over $\omega^\omega \cap M$. However, a $(\mathcal{B}(\omega^\omega) \setminus \mathfrak{I})$-generic real is always $\mathfrak{I}$-quasigeneric. When $\mathfrak{I}$ is a ccc ideal then the two notions are equivalent.

Definition 2.9. A real x is called *infinitely often equal* (*ioe*) over a model M, iff $\forall y \in \omega^\omega \cap M\ \exists^\infty n\ (x(n) = y(n))$. A real x is called *infinitely often following* (*iof*) over a model M, iff $\forall f \in \omega^{(\omega^{<\omega})} \cap M\ \exists^\infty n\ (x(n) = f(x{\restriction}n))$.

Lemma 2.10. *Let M be a model of set theory with $\omega_1 \subseteq M$ and x a real. Then:*

1. *x is $\mathfrak{I}_{\text{ioe}}$-quasigeneric over M iff it is ioe over M, and*
2. *x is $\mathfrak{D}_\omega$-quasigeneric over M iff it is iof over M.*

Proof. The proofs of both statements are analogous so let us only show the first. If x avoids $\mathfrak{I}_{\text{ioe}}$-small Borel sets coded in M, then for any $y \in \omega^\omega \cap M$, K_y is a Borel $\mathfrak{I}_{\text{ioe}}$-small set coded in M, so $x \notin K_y$, so x is ioe to y. Conversely, suppose x is ioe over M and $B \in \mathfrak{I}_{\text{ioe}}$ is a Borel set coded in M. Since "$B \in \mathfrak{I}_{\text{ioe}}$" is a Σ^1_2 statement on the code of B, by absoluteness $M \models B \in \mathfrak{I}_{\text{ioe}}$. Therefore there are $x_i \in M$ such that $B \subseteq \bigcup_{i<\omega} K_{x_i}$ (this statement is Π^1_1, hence absolute). But x is ioe to all x_i, so by definition $x \notin K_{x_i}$ for all i, hence $x \notin B$. $\square$

Therefore, $\mathbb{IE}$ canonically adds an ioe real, whereas $\mathbb{FM}$ canonically adds an iof real. From Lemma 2.1 it immediately follows that a Cohen real is an iof real, and an iof real is an ioe real. Also, from Lemma 2.2 it follows that if x is an iof real then $\varphi(x)$ is a Cohen real (so $\mathbb{FM}$ adds a Cohen real). Moreover, the following is well-known:

Fact 2.11 *(Bartoszyński/Folklore). If $V_0 \subseteq V_1 \subseteq V_2$ are models of set theory, in V_1 there is an ioe real over V_0 and in V_2 there is an ioe real over V_1, then in V_2 there is a Cohen real over V_0.*

Corollary 2.12. $\mathbb{IE} * \mathbb{IE}$ *adds a Cohen real.*

So $\mathbb{C}$, $\mathbb{FM}$ and $\mathbb{IE}$ all have a very similar effect on the structure of the real line. For example, an ω_2-iteration of $\mathbb{FM}$ or $\mathbb{IE}$ with countable supports yields the same values for the cardinal invariants in Cichoń's diagram as an ω_2-iteration/product of Cohen forcing, namely $\omega_1 = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \omega_2$. Newelski and Rosłanowski observed that "it seems that forcing $\mathbb{FM}$ is the best one for adding Cohen reals in countable support iterations." In fact, we do not know the answer to the following basic question:

Question 2.13. *What is a "natural" forcing property (e.g. adding or not adding certain types of reals) which distinguishes* $\mathbb{FM}$ *from Cohen forcing?*

The situation with $\mathbb{IE}$ and adding Cohen reals is much more subtle, and is closely related to issues of homogeneity, which we will now describe.

Definition 2.14 *(Zapletal).* A σ-ideal $\mathfrak{I}$ on ω^ω is *homogeneous* if for every $\mathfrak{I}$-positive Borel set B, there exists a Borel function $f : \omega^\omega \to B$ such that f-preimages of $\mathfrak{I}$-small sets are $\mathfrak{I}$-small.

Remark 2.15. The meager ideal is well-known to be homogeneous, and for $\mathfrak{D}_\omega$, observe that for every full-splitting Miller tree T there exists a natural homeomorphism ψ between ω^ω and $[T]$ (generated by the identification of $\omega^{<\omega}$ with the split-nodes of T), with the additional property that for every full-splitting tree S, $\psi``[S]$ generates a full-splitting sub-tree of T. It follows that $\mathfrak{D}_\omega$ is homogeneous.

On the other hand, $\mathfrak{I}_{\text{ioe}}$ fails to be homogeneous—this will indirectly follow both from Corollary 3.6 and from Lemma 5.2. The crucial point is the following idea from unpublished work of Goldstern and Shelah [7]:

Definition 2.16. A tree $T \subseteq \omega^{<\omega}$ is called an *almost full-splitting Miller tree* iff every $t \in T$ has an extension $s \in T$ such that $\forall^\infty n\, (s^\frown \langle n \rangle \in T)$.

Lemma 2.17 *(Goldstern–Shelah). There exists a $T^{\text{GS}} \in \mathbb{IE}$ such that* every *$\mathbb{IE}$-subtree of T^{GS} is an almost full-splitting Miller tree.*

Proof. Construct T^{GS} in such a way that:

1. Every splitting note $t \in T^{\text{GS}}$ is full-splitting.
2. If $s \neq t$ are splitting nodes of T^{GS} then $|s| \neq |t|$.
3. If $t \in T^{\text{GS}}$ is a non-splitting node of T then $t(|t| - 1) = 0$.

Such a tree can easily be constructed inductively after fixing some bijection $f : \omega^{<\omega} \cong \omega$. It is not hard to see that if S is any sub-tree of T^{GS} which is an ioe-tree, then it has to be an almost full-splitting, since the only way that a node of S can be extended to "hit" an arbitrary $k > 0$ on some fixed level, is to extend that node to a $t \in S$ such that $t^\frown \langle k \rangle \in S$ for all $k > 0$. □

An argument just as in the proof of Lemma 2.2 easily extends to show that if T is an almost full-splitting Miller tree then $\varphi``[T]$ is non-meager, implying that $T^{\text{GS}} \Vdash_{\mathbb{IE}}$ "$\varphi(\dot{x}_G)$ is a Cohen real". Nevertheless, since $\mathfrak{I}_{\text{ioe}}$ is not homogeneous, there is no a priori reason why there could not be some other $\mathbb{IE}$-condition forcing that no Cohen reals are added. We shall return to this question in Section 5.

3. Marczewski-type regularity properties

A vast array of "Marczewski-type" regularity properties have been considered in the literature, where a set $A \subseteq \omega^\omega$ is considered "measurable" if every set in a certain partial order can be shrunk to a smaller set in the same partial order, which is completely contained in, or disjoint from, the given set A, possibly modulo a suitable ideal. Polish mathematicians had a strong interest in such properties for a long time, e.g. [17]. More modern treatments include [2,3,5], while [8,12,11] provide more abstract treatments in the setting of *forcing with trees* or *idealized forcing*. See also [13] for a related treatment in terms of *category bases*. The Baire property, Lebesgue measurability, the Ramsey property and many other properties can be formulated as Marczewski-type properties. Following this setting we define:

Definition 3.1. A set $A \subseteq \omega^\omega$ is called

- $\mathbb{FM}$-*measurable* iff $\forall T \in \mathbb{FM}\, \exists S \in \mathbb{FM}\, (S \leq T$ and $[S] \subseteq A$ or $[S] \cap A = \varnothing)$.
- $\mathbb{IE}$-*measurable* iff $\forall T \in \mathbb{IE}\, \exists S \in \mathbb{IE}\, (S \leq T$ and $[S] \subseteq A$ or $[S] \cap A = \varnothing)$.

We also define *weak* (local) versions of the above.

Definition 3.2. A set $A \subseteq \omega^\omega$ is

- *weakly* $\mathbb{FM}$*-measurable* iff $\exists S \in \mathbb{FM}$ ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).
- *weakly* $\mathbb{IE}$*-measurable* iff $\exists S \in \mathbb{IE}$ ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).

If $\mathbf{\Gamma}$ is some pointclass of sets (e.g. Borel, projective etc.) we follow standard practice and use the notation $\mathbf{\Gamma}(\mathbb{FM}), \mathbf{\Gamma}(\mathbb{IE})$, $\mathbf{\Gamma}(\text{w}\mathbb{FM})$ and $\mathbf{\Gamma}(\text{w}\mathbb{IE})$ to refer to the statements "all sets in $\mathbf{\Gamma}$ are $\mathbb{FM}$-measurable", "...are $\mathbb{IE}$-measurable", "...are weakly $\mathbb{FM}$-measurable" and "...are weakly $\mathbb{IE}$-measurable", respectively. $\mathbf{\Gamma}$(Baire) refers to "all sets in $\mathbf{\Gamma}$ have the property of Baire".

Usually, the homogeneity of the ideal/partial order of trees (in the sense of Definition 2.14) ensures that for sufficiently nice pointclasses $\mathbf{\Gamma}$, the "weak" notion of measurability is equivalent to the strong one.

Observation 3.3. Let $\mathbf{\Gamma}$ be a pointclass closed under continuous pre-images. Since $\mathfrak{D}_\omega$ is homogeneous (in fact witnessed by a continuous reduction) it is easy to see that $\mathbf{\Gamma}(\mathbb{FM}) \Leftrightarrow \mathbf{\Gamma}(\text{w}\mathbb{FM})$.

Theorem 3.4. *Let* $\mathbf{\Gamma}$ *be a pointclass closed under continuous pre-images. Then the following are equivalent:*

1. $\mathbf{\Gamma}$(*Baire*)
2. $\mathbf{\Gamma}(\mathbb{FM})$
3. $\mathbf{\Gamma}(\mathbb{IE})$

Proof.

- 1 ⇒ 2. Let $A \subseteq \omega^\omega$ be a set in $\mathbf{\Gamma}$. By $\mathbf{\Gamma}$(Baire) we can find a basic open set $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ stand for "modulo a meager set". Without loss of generality, assume the former. Then there is a G_δ set $B \subseteq A$ which is co-meager in $[s]$. Since $\mathfrak{D}_\omega \subseteq \mathcal{M}$, B cannot be $\mathfrak{D}_\omega$-small, hence it contains an $\mathbb{FM}$-tree. By Observation 3.3 this is sufficient.
- 2 ⇒ 3. We say that an ioe-tree T is *in strict form* if it can be written as follows:
  - for every $\sigma \in \omega^{<\omega}$, there exists $N_\sigma \subseteq \omega^n$, for some $n \geq 1$, such that
    - $\forall k\, \exists! s \in N_\sigma\ (s(n-1) = k)$, and
    - for $m < (n-1)$, there is some k such that $s(m) \neq k$ for all $s \in N_\sigma$.

    We use $\text{len}(N_\sigma) = n$ to denote the length of N_σ, and we canonically enumerate N_σ as $\{s^\sigma_k \mid k < \omega\}$, in such a way that $s^\sigma_k(n-1) = k$.
  - T is the tree generated by sequences of the form

$$s^{\varnothing}_{n_0} {}^\frown s^{\langle n_0\rangle}_{n_1} {}^\frown s^{\langle n_0,n_1\rangle}_{n_2} {}^\frown \dots {}^\frown s^{\langle n_0,n_1,n_2,\dots,n_{\ell-1}\rangle}_{n_\ell}$$

    for some sequence $\langle n_0, n_1, \dots, n_\ell\rangle$.

  Ioe-trees in strict form are somewhat easier to visualize and deal with. If T is in strict form, $t \in T$, and we need to find the first N such that $\forall k\, \exists s \supseteq t$ with $s(N) = k$, we only need to find the (unique) shortest sequence σ such that $t \subseteq s^{\varnothing}_{\sigma(0)} {}^\frown s^{\sigma\restriction 1}_{\sigma(1)} {}^\frown \dots {}^\frown s^{\sigma\restriction|\sigma|-1}_{\sigma(|\sigma|-1)}$, and then we have $N = \sum_{i=0}^{|\sigma|} N_{\sigma\restriction i}$. Every ioe-tree T can be pruned to an ioe-subtree $S \leq T$ in strict form.

  So, let $A \in \mathbf{\Gamma}$ and let T be an ioe-tree, assuming, without loss of generality, that T is in strict form. Define a function $\psi' : \omega^{<\omega} \to T$ inductively by $\psi'(\varnothing) = \varnothing$ and $\psi'(\sigma^\frown \langle n\rangle) := \psi'(\sigma)^\frown s^\sigma_n$. This gives rise to a natural homeomorphism $\psi : \omega^\omega \cong [T]$. Since $\psi^{-1}[A]$ is also in $\mathbf{\Gamma}$ we can find a full-splitting tree S such that $[S] \subseteq \psi^{-1}[A]$ or $[S] \cap \psi^{-1}[A] = \varnothing$. So we will be done if we can show that ψ"$[S]$ generates an ioe-subtree of T. But this follows from the definition of ψ' and the fact that the last digit of every s_n^σ is n.

- $3 \Rightarrow 1$. Recall the function φ from Lemma 2.2. Let $A \in \mathbf{\Gamma}$ and let $A' := \varphi^{-1}[A]$, also in $\mathbf{\Gamma}$. Now recall the Goldstern–Shelah tree T^{GS} from Lemma 2.17. Since A' is $\mathbb{IE}$-measurable, there exists $S \leq T^{\mathrm{GS}}$ such that $[S] \subseteq A$ or $[S] \cap A = \varnothing$, without loss of generality the former. But since S is an almost full-splitting tree, $\varphi``[S]$ is not meager, but it is analytic, so it is comeager in some basic open $[s]$. Then $[s] \subseteq^* A$. This is sufficient because $\mathbf{\Gamma}(\text{Baire})$ is equivalent to the assertion that for all $A \in \mathbf{\Gamma}$ there exists a basic open $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$. □

In the "$3 \Rightarrow 1$"-direction of the above proof, the Golstern–Shelah tree was used in a quintessential way; this suggests that the property called *weak* $\mathbb{IE}$*-measurability* behaves substantially different. Indeed, the following theorem is the most surprising result of this section.

Theorem 3.5. $\mathbf{\Delta}^1_2$(*Baire*) $\Rightarrow \mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$.

Corollary 3.6. *It is consistent that* $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ *is true while* $\mathbf{\Sigma}^1_2(\mathbb{IE})$ *is false; in particular* $\mathbb{IE}$*- and weak* $\mathbb{IE}$*-measurability are not classwise equivalent.*

Proof. We know that the ω_1-Cohen model $L^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_2(\text{Baire}) + \neg\mathbf{\Sigma}^1_2(\text{Baire})$ ([1, Sections 9.3 and 9.3]). Therefore, by Theorems 3.4 and 3.5 we have $L^{\mathbb{C}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE}) + \neg\mathbf{\Sigma}^1_2(\mathbb{IE})$. □

This is the first instance we know of where such a situation occurs in the context of a very naturally defined ideal.

Proof of Theorem 3.5. Assume $\mathbf{\Delta}^1_2$(Baire). Let $A \subseteq \omega^\omega$ be a $\mathbf{\Sigma}^1_2$ set. We have to find an $\mathbb{IE}$-tree T such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

We may assume that for some r, $\omega_1^{L[r]} = \omega_1$, since otherwise $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ follows easily (for example from $\mathbf{\Sigma}^1_2$(Baire)). We may also assume, without loss of generality, that the parameters in the definition of A are in $L[r]$. Using the Borel decomposition of $\mathbf{\Sigma}^1_2$ sets we can write $A = \bigcup_{\alpha<\omega_1} B_\alpha$, where B_α are Borel sets coded in $L[r]$. If there exists at least one α such that $B_\alpha \notin \mathfrak{I}_{\mathrm{ioe}}$, then there is an $\mathbb{IE}$-tree T with $[T] \subseteq B_\alpha \subseteq A$ and we are done. So suppose that all B_α are $\mathfrak{I}_{\mathrm{ioe}}$-small. For each α, since $L[r] \models B_\alpha \in \mathfrak{I}_{\mathrm{ioe}}$, we can fix a sequence $\langle x_i^\alpha \mid i < \omega\rangle$ of reals in $L[r]$ such that $B_\alpha \subseteq \bigcup_{i<\omega} K_{x_i^\alpha}$.

Let $\rho : \omega^\omega \to \omega^\omega$ be defined by $\rho(x) := \langle x(0), x(2), x(4), \dots\rangle$. By $\mathbf{\Delta}^1_2$(Baire), we know that in V there is a Cohen real c over $L[r]$ ([1, Theorem 9.2.1]). Then c is infinitely often equal over $L[r]$, and in particular, infinitely often equal to $\rho(x_i^\alpha)$ for all $\alpha < \omega_1, i < \omega$. Let T_c be the $\mathbb{FM}$-tree such that

$$[T_c] = \{y \mid \rho(y) = c\}.$$

We claim that $[T_c] \cap A = \varnothing$. Let $a \in A$, then there is some $\alpha < \omega_1$ such that $a \in B_\alpha$. By absoluteness of "$B_\alpha \subseteq \bigcup_{i<\omega} K_{x_i^\alpha}$", there is some $i < \omega$ such that a is eventually different from x_i^α. Let $N \in \omega$ be such that $\forall n > N\ (a(n) \neq x_i^\alpha(n))$. But since c is ioe to $\rho(x_i^\alpha)$, we can easily find $n > N$ such that $c(n) = x_i^\alpha(2n) \neq a(2n)$. By definition this implies that $a \notin [T_c]$. □

4. The dichotomy for higher projective sets

Definition 4.1. Let us say that a set $A \subseteq \omega^\omega$ satisfies the $\mathbb{FM}$-*dichotomy* if A is either $\mathfrak{D}_\omega$-small or contains $[T]$ for an $\mathbb{FM}$-tree T, and that is satisfies the $\mathbb{IE}$-*dichotomy* if A is either $\mathfrak{I}_{\mathrm{ioe}}$-small or contains $[T]$ for some $\mathbb{IE}$-tree T. We use $\mathbf{\Gamma}(\mathbb{FM}\text{-dich})$ and $\mathbf{\Gamma}(\mathbb{IE}\text{-dich})$ to abbreviate "all sets in $\mathbf{\Gamma}$ have the $\mathbb{FM}$-dichotomy" and "all sets in $\mathbf{\Gamma}$ have the $\mathbb{IE}$-dichotomy", respectively.

So we know that $\mathbf{\Sigma}^1_1(\mathbb{FM}\text{-dich})$ and $\mathbf{\Sigma}^1_1(\mathbb{IE}\text{-dich})$ are true, but we can also ask to which higher projective levels the dichotomies can be extended. Note that these properties are stronger then their Marczewski-counterparts, i.e., $\mathbf{\Gamma}(\mathbb{FM}\text{-dich}) \Rightarrow \mathbf{\Gamma}(\mathbb{FM})$ and $\mathbf{\Gamma}(\mathbb{IE}\text{-dich}) \Rightarrow \mathbf{\Gamma}(\mathbb{IE})$ for all projective pointclasses $\mathbf{\Gamma}$. The main result of this section concerns the consistency strength of $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ and $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$. In general, statements of this kind have a rather unpredictable behavior: for example, K_σ-regularity (see [10]) for $\mathbf{\Sigma}^1_2$ sets is equiconsistent with ZFC, while the perfect set property for $\mathbf{\Sigma}^1_2$ (even $\mathbf{\Pi}^1_1$) sets has the strength of an inaccessible. In yet other cases, it is actually inconsistent (Zapletal, see [11, Proposition 2.4.4]). The properties considered here will fall into the second category.

First we prove a "Mansfield–Solovay-style" theorem for $\mathbb{FM}$ and $\mathbb{IE}$. Its proof uses a completely classical Cantor-Bendixson analysis and it is worth noting that an analogous argument replacing trees on $\omega \times \omega_1$ by trees on $\omega \times \omega$ provides alternative (arguably more elementary) proofs of the Newelski–Rosłanowski and the Spinas dichotomy theorems, i.e., $\mathbf{\Sigma}^1_1(\mathbb{FM}\text{-dich})$ and $\mathbf{\Sigma}^1_1(\mathbb{IE}\text{-dich})$.

Lemma 4.2.

1. *For any $\Sigma^1_2(r)$ set A, either there exists an $\mathbb{FM}$-tree $U \in L[r]$, such that $[U] \subseteq A$, or A can be covered by $\mathfrak{D}_\omega$-small Borel sets coded in $L[r]$.*
2. *For any $\Sigma^1_2(r)$ set A, either there exists an $\mathbb{IE}$-tree $U \in L[r]$, such that $[U] \subseteq A$, or A can be covered by $\mathfrak{I}_{\mathrm{ioe}}$-small Borel sets coded in $L[r]$.*

Proof. 1. Let $T \in L[r]$ be a tree on $\omega \times \omega_1$ such that $A = p[T]$ (where p denote the projection to the first coordinate). For any tree S on $\omega \times \omega_1$ define

$$S' := \{(s,h) \in S \mid \exists s' \supseteq s \, \forall n \, \exists h' \supseteq h \, ((s'^\frown \langle n \rangle, h') \in S)\}.$$

Next define inductively $T^{(0)} := T$, $T^{(\alpha+1)} := (T^{(\alpha)})'$ and $T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}$. As the above construction is absolute, it follows that all $T^{(\alpha)}$ are in $L[r]$.

Let α be least such that $T^{(\alpha)} = T^{(\alpha+1)}$, and consider two cases:

- $T^{(\alpha)} \neq \varnothing$. Then every $(s,h) \in T^{(\alpha)}$ has the property that there exists s' such that for every n there is h' extending h with $(s'^\frown \langle n \rangle, h') \in T^{(\alpha)}$. Using this it is easy to inductively construct a tree $U \subseteq \omega^{<\omega}$, such that every $s \in U$ has a full-splitting extension $s' \in U$, i.e., U is an $\mathbb{FM}$-tree, and moreover, such that given any branch $x \in [U]$, there is a corresponding branch $g \in {\omega_1}^\omega$ such that $(x,g) \in [T^{(\alpha)}] \subseteq [T]$. Therefore $x \in p[T] = A$.
- $T^{(\alpha)} = \varnothing$. In this case, for every $\gamma < \alpha$ and every $h \in (\omega_1)^{<\omega}$ we define a function $f_{\gamma,h} : \omega^{<\omega} \to \omega$ by:

$$f_{\gamma,h}(s) = n \;:\Longleftrightarrow\; \exists h' \supseteq h \, ((s,h') \in T^{(\gamma)}) \text{ and}$$
$$n \text{ is least such that } \forall h'' \supseteq h \, ((s^\frown \langle n \rangle, h'') \notin T^{(\gamma)})$$

  if such an n exists, and $f_{\gamma,h}(s) = 0$ otherwise.
  Since each $f_{\gamma,h}$ is explicitly constructed from $T^{(\gamma)}$ and h, clearly it is in $L[r]$. Also let $D_{f_{\gamma,h}} := \{x \mid \forall^\infty n \, (x(n) \neq f_{\gamma,h}(x{\restriction}n))\}$ be the Borel $\mathfrak{D}_\omega$-small sets corresponding to $f_{\gamma,h}$, clearly also coded in $L[r]$. We will finish the proof by concluding that $A \subseteq \bigcup\{D_{f_{\gamma,h}} \mid \gamma < \alpha, h \in (\omega_1)^{<\omega}\}$.
  Take any $x \in A$, and let $g \in {\omega_1}^\omega$ be such that $(x,g) \in [T]$. Let $\gamma < \alpha$ be least such that $(x,g) \in [T^{(\gamma)}] \setminus [T^{(\gamma+1)}]$, and let $s \subseteq x$ and $h \subseteq g$ be such that $(s,h) \in T^{(\gamma)} \setminus T^{(\gamma+1)}$. By definition of $T^{(\gamma+1)} := (T^{(\gamma)})'$ we know that for any s' extending s there exists n such that $(s'^\frown \langle n \rangle, h') \notin T^{(\gamma)}$ for any h' extending h. Take any $k > |s|$. Then $(x{\restriction}k, g{\restriction}k) \in T^{(\gamma)}$, and let n be least such that $(x{\restriction}k^\frown \langle n \rangle, h') \notin T^{(\gamma)}$ for any h' extending h. But then, the definition of $f_{\gamma,h}$ implies that $f_{\gamma,h}(x{\restriction}k) = n$. On the other hand,

$(x\restriction(k+1), g\restriction(k+1))$ is also in $T^{(\gamma)}$, and so $x(k)$ cannot have value n. In particular $x(k) \neq f_{\gamma,h}(x\restriction k)$. Since this argument applies for all $k > |s|$, this proves $x \in D_{f_{\gamma,h}}$.

2. The argument is completely analogous, so we only mention the changes that need to be made. Here, the pruning operation for a tree S on $\omega \times \omega_1$ is defined as follows:

$$S' := \{(s,h) \in S \mid \exists N > |s|\, \forall k\, \exists s' \supseteq s\, \exists h' \supseteq h \\ ((s',h') \in S, |s'| = N+1 \text{ and } s'(N) = k)\}.$$

Then, in the case that $T^{(\alpha)} = \varnothing$ proceed as follows: for every $\gamma < \alpha$, $s \in \omega^{<\omega}$ and $h \in (\omega_1)^{<\omega}$ define a real $x_{\gamma,s,h} \in \omega^\omega$ by:

$$x_{\gamma,s,h}(N) = k \;:\Longleftrightarrow\; (s,h) \in T^{(\gamma)} \text{ and } N > |s| \text{ and} \\ k \text{ is least s.t. } \forall s' \supseteq s\, \forall h' \supseteq h\, ((s',h') \in T^{(\gamma)} \wedge |s'| = N+1 \to s'(N) \neq k)$$

if such a k exists, and 0 otherwise. The proof is completed by showing that $A \subseteq \bigcup\{K_{(x_{\gamma,s,h})} \mid \gamma < \alpha, s \in \omega^{<\omega}, h \in (\omega_1)^{<\omega}\}$; details are left to the reader. □

We are now ready to prove the main theorem of this section. Recall that by Lemma 2.10 iof reals are $\mathfrak{D}_\omega$-quasigeneric and ioe reals are $\mathfrak{I}_{\text{ioe}}$-quasigeneric, which will be frequently used in the proof.

Theorem 4.3. *The following are equivalent:*

1. $\boldsymbol{\Sigma}^1_2(\mathbb{FM}\text{-dich})$
2. $\boldsymbol{\Sigma}^1_2(\mathbb{IE}\text{-dich})$
3. $\forall r \in \omega^\omega\ \{x \mid x \textit{ is not iof over } L[r]\} \in \mathfrak{D}_\omega$
4. $\forall r \in \omega^\omega\ \{x \mid x \textit{ is not ioe over } L[r]\} \in \mathfrak{I}_{\text{ioe}}$
5. $\forall r \in \omega^\omega\ (\omega_1^{L[r]} < \omega_1)$

Proof. First we prove $1 \Leftrightarrow 3 \Leftrightarrow 5$.

- $1 \Rightarrow 3$. Fix an arbitrary r and let $X := \{x \mid x \text{ is not iof over } L[r]\}$. It is not hard to see that X is a $\Sigma^1_2(r)$ set, so by assumption either $X \in \mathfrak{D}_\omega$ or there is some $T \in \mathbb{FM}$ such that $[T] \subseteq X$. We will show that the second option is impossible.
From $\boldsymbol{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ we have $\boldsymbol{\Sigma}^1_2(\mathbb{FM})$, and by Theorem 3.4 also $\boldsymbol{\Sigma}^1_2(\text{Baire})$. In particular, there is a Cohen real c, which is an iof real, over $L[r,T]$. Let $T \in \mathbb{FM}$ and recall that there is a homeomorphism $\psi : \omega^\omega \cong [T]$ such that ψ-preimages of $\mathfrak{D}_\omega$-small sets are $\mathfrak{D}_\omega$-small (Remark 2.15). Since being an iof real is the same as being $\mathfrak{D}_\omega$-quasigeneric, it easily follows that $\psi(c)$ is an iof real in $[T]$. This contradicts $[T] \subseteq X$.
- $3 \Rightarrow 1$. Notice that Lemma 4.2(1) actually says: every $\boldsymbol{\Sigma}^1_2$ set A either contains $[T]$ for $T \in \mathbb{FM}$ or $A \subseteq \{x \mid x \text{ is not } \mathfrak{D}_\omega\text{-quasigeneric over } L[r]\} = \{x \mid x \text{ is not iof over } L[r]\}$, from which the result follows.
- $5 \Rightarrow 3$. If $\omega_1^{L[r]} < \omega_1$ then $\{x \mid x \text{ is not iof over } L[r]\} = \bigcup\{B \mid B \text{ is a Borel } \mathfrak{D}_\omega\text{-small set coded in } L[r]\}$ is a countable union of $\mathfrak{D}_\omega$-small sets.
- $3 \Rightarrow 5$. Recall Claim 2.4 used in the proof of $\text{add}(\mathfrak{D}_\omega) = \omega_1$, which, in particular, implies that for any family $F = \{x_\alpha \mid \alpha < \omega_1\}$ of reals satisfying $\forall \alpha \neq \beta\ \exists^\infty n\ (x_\alpha(n) \neq x_\beta(n))$, and letting $X_\alpha := \{x \mid \forall n\ (x(n) \neq x_\alpha(n))\}$, we have
  - $X_\alpha \in \mathfrak{I}_{\text{ioe}} \subseteq \mathfrak{D}_\omega$ for all $\alpha < \omega_1$, and
  - $\bigcup_{\alpha<\omega_1} X_\alpha \notin \mathfrak{D}_\omega$.

If $\omega_1^{L[r]} = \omega_1$ for some r, then we have an F as above satisfying $F \subseteq \omega^\omega \cap L[r]$. But then $\{x \mid x$ is not iof over $L[r]\} = \bigcup\{B \mid B$ is a Borel $\mathfrak{D}_\omega$-small coded in $L[r]\} \supseteq \bigcup\{X_\alpha \mid \alpha < \omega_1\}$ cannot be $\mathfrak{D}_\omega$-small.

The proofs of $2 \Leftarrow 4 \Leftrightarrow 5$ are analogous. For $2 \Rightarrow 4$ we must be more careful since the ideal $\mathfrak{I}_{\text{ioe}}$ is not homogeneous, so we cannot conclude that there is an ioe real inside $[T]$ for every $T \in \mathbb{IE}$, just from the existence of ioe reals. However, using the same trick as in the "$2 \Rightarrow 3$"-direction of the proof of Theorem 3.4 we can argue as follows: given a $T \in \mathbb{IE}$ in strict form, find a homeomorphism ψ between ω^ω and $[T]$ such that ψ-preimages of $\mathfrak{I}_{\text{ioe}}$-small sets are $\mathfrak{D}_\omega$-small. Then, if c is an iof real, $\psi(c)$ is an ioe real. □

The consistency of the $\mathbb{FM}$- and $\mathbb{IE}$-dichotomies beyond $\mathbf{\Sigma}^1_2$ sets can be established in the Solovay model and from determinacy hypotheses. Here we should note that, while the remaining results of this section are not particularly surprising, they are nevertheless not "trivial" results, since, as we already mentioned, there are dichotomies which are true for analytic sets but are inconsistent for $\mathbf{\Sigma}^1_2$ sets, so there is no a priori reason to believe that our dichotomies for higher projective sets are consistent.

In the next theorem we will assume familiarity with the Solovay model (see e.g. [9, Section 11] for details). $\text{Col}(\omega, {<}\kappa)$ will denote the standard Lévy partial order for collapsing an inaccessible κ to ω_1.

Theorem 4.4. *Let κ be inaccessible and let G be $\text{Col}(\omega, {<}\kappa)$-generic over V. Then in $V[G]$ all sets definable from countable sequences of ordinals satisfy the $\mathbb{FM}$- and the $\mathbb{IE}$-dichotomy, and in $L(\mathbb{R})^{V[G]}$ all sets of reals satisfy the $\mathbb{FM}$- and $\mathbb{IE}$-dichotomy.*

Proof. The proofs of both dichotomies are similar; we prove the $\mathbb{FM}$-case in detail and leave the $\mathbb{IE}$-case to the reader.

Let $A \subseteq \omega^\omega$ be a set in $V[G]$, defined by ϕ and a countable sequence of ordinals $\vec{a}$. By well-known properties of the Lévy collapse, there is a formula $\tilde{\phi}$ such that for all x:

$$V[G] \models \phi(\vec{a}, x) \iff V[\vec{a}][x] \models \tilde{\phi}(\vec{a}, x).$$

Assume that $A \notin \mathfrak{D}_\omega$. In particular, A cannot be covered by Borel $\mathfrak{D}_\omega$-small sets coded in $V[\vec{a}]$, since $V[\vec{a}]$ only contains countably many reals. So there is an $x \in A$ which is $\mathfrak{D}_\omega$-quasigeneric over $V[\vec{a}]$, i.e., iof over $V[\vec{a}]$. By another standard property of the Lévy collapse, there is a $\text{Col}(\omega, {<}\kappa)$-generic H such that $V[G] = V[\vec{a}][H]$, and moreover, a complete suborder Q of $\text{Col}(\omega, {<}\kappa)$ in $V[\vec{a}]$, such that $|Q| < \kappa$ and $x \in V[\vec{a}][H \cap Q]$. Then in $V[\vec{a}]$, there is a Q-name $\dot{x}$ for x and a condition $p \in Q \cap H$ satisfying:

$$p \Vdash_Q \tilde{\phi}(\vec{a}, \dot{x}) \ \wedge \ \text{“}\dot{x} \text{ is iof over } V[\vec{a}]\text{”}.$$

Since $\omega_1^{V[G]}$ is inaccessible in $V[\vec{a}]$, in $V[G]$ there are only countably many Q-dense sets in $V[\vec{a}]$. Let $\{D_i \mid i < \omega\}$ enumerate all of them.

In $V[G]$, by induction we will construct $U \subseteq \omega^{<\omega}$, and for every $t \in U$ a corresponding Q-condition p_t, such that

1. $s \subseteq t \iff p_t \leq p_s$,
2. for every t, $p_t \Vdash t \subseteq \dot{x}$,
3. for every t, $p_t \in D_i$, where $i = |\{s \in U \mid s \subseteq t\}|$, and
4. the downward-closure of U is an $\mathbb{FM}$-tree.

Let $p_\varnothing \leq p$ be any condition in D_0. Suppose $s \in U$ has been constructed, $p_s \in Q$ satisfies $p_s \Vdash s \subseteq \dot{x}$, and $i := |\{s' \in U \mid s' \subseteq s\}|$. Extend p_s to $p'_s \in D_i$.

Claim. *There exists $t \supseteq s$ such that for all n, there is $q \leq p'_s$ such that $q \Vdash t^\frown \langle n \rangle \subseteq \dot{x}$.*

Proof. Suppose not: so for any $t \supseteq s$ there is n such that no $q \leq p'_s$ forces $t^\frown \langle n \rangle \subseteq \dot{x}$, so that $p'_s \Vdash t^\frown \langle n \rangle \not\subseteq \dot{x}$. Define a function $f : \omega^{<\omega} \to \omega$ in $V[\vec{a}]$ by letting $f(t)$ be such n as above, for all $t \supseteq s$, and $f(t) = 0$ for other t. Then we have

$$p'_s \Vdash \forall t \supseteq s \; (t^\frown \langle f(t) \rangle \not\subseteq \dot{x}).$$

Since also $p'_s \Vdash s \subseteq \dot{x}$, in particular we have

$$p'_s \Vdash \forall n > |s| \; (\dot{x}{\restriction}n^\frown \langle f(\dot{x}{\restriction}n) \rangle \not\subseteq \dot{x})$$

and so

$$p'_s \Vdash \forall n > |s| \; (\dot{x}(n) \neq f(\dot{x}{\restriction}n)),$$

contradicting the fact that $p'_s \Vdash$ "$\dot{x}$ is iof over $V[\vec{a}]$". □ (Claim).

By the claim, we can fix such a t, and for every n, a condition $p_{t^\frown\langle n \rangle} \leq p'_s$ forcing $t^\frown \langle n \rangle \subseteq \dot{x}$. Finally we add all these $t^\frown \langle n \rangle$ to the set U, and this completes the inductive construction.

Let $T(U)$ be the tree generated by U. It is clear that $T(U) \in \mathbb{FM}$, so it only remains to show that $[T(U)] \subseteq A$. In $V[G]$, let y be any real in $[T(U)]$. We have to show that $V[G] \models \phi(\vec{a}, y)$. By construction, y can be viewed as the limit of some $\{t_n \mid n < \omega\}$, where all $t_n \in U$. Let $G_y := \{q \in Q \mid \exists n \, (p_{t_n} \leq q)\}$. Since G_y meets every D_i, it is Q-generic over $V[\vec{a}]$, and since $p_{t_n} \Vdash_Q t_n \subseteq \dot{x}$ for every n, we know that $\dot{x}[G_y] = y$. Also, since $p \Vdash_Q V[\vec{a}][\dot{x}] \models \tilde{\phi}(\vec{a}, \dot{x})$, it follows that $V[\vec{a}][y] \models \tilde{\phi}(\vec{a}, y)$, and therefore $V[G] \models \phi(\vec{a}, y)$. This completes the proof of the $\mathbb{FM}$-dichotomy.

The proof of the $\mathbb{IE}$-dichotomy is analogous, replacing $\mathfrak{D}_\omega$ by $\mathfrak{I}_{\text{ioe}}$, $\mathbb{FM}$ by $\mathbb{IE}$ and "iof reals" by "ioe reals". The corresponding claim must read as follows: "*There is $N > |s|$ such that for all $k \in \omega$ there are t and $q \leq p'_s$ such that $t \supseteq s$, $|t| = N + 1$, $t(N) = k$ and $q \Vdash t \subseteq \dot{x}$.*" The claim is proved by assuming the contrary and producing a real $z \in V[\vec{a}]$ such that $p'_s \Vdash \forall N > |s| \; (\dot{x}(N) \neq z(N))$, contradicting $p'_s \Vdash$ "$\dot{x}$ is ioe over $V[\vec{a}]$". □

Another way to extend the dichotomy beyond $\mathbf{\Sigma}^1_2$ sets is by the use of infinite games; the $\mathbb{FM}$-dichotomy was originally motivated by a Morton Davis-like game. The following game corresponds to the $\mathbb{IE}$-dichotomy.

Definition 4.5. Let $G^{\mathbb{IE}}(A)$ be the game in which players I and II play as follows:

I:	N_0		s_0 N_1		s_1 N_2		…	…	
II:		k_0		k_1		k_2			…

where $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$, $N_i \geq 1$, $k_i \in \omega$, and the following rules must be obeyed for all i:

- $|s_i| = N_i$,
- $s_i(N_i - 1) = k_i$.

Then player I wins iff $z := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$. For technical reasons, we formalize the game as if Player I makes two consecutive moves rather than a pair (s_i, N_{i+1}).

Theorem 4.6.

1. *Player I has a winning strategy in* $G^{\mathbb{IE}}(A)$ *iff there is an* $\mathbb{IE}$*-tree* T *such that* $[T] \subseteq A$*.*
2. *Player II has a winning strategy in* $G^{\mathbb{IE}}(A)$ *iff* $A \in \mathfrak{I}_{\text{ioe}}$*.*

Proof. We will only show the left-to-right direction of 2. Let τ be a winning strategy for player II. Suppose $p = \langle N_0, k_0, s_0, N_1, \dots, k_{\ell-1}, s_{\ell-1}\rangle$ is a *position of the game* of length 3ℓ (and $p = \varnothing$ if $\ell = 0$). Then we define $p^* := s_0{}^\frown \dots {}^\frown s_{\ell-1}$ (and $p^* = \varnothing$ if $p = \varnothing$), and for an $x \in \omega^\omega$ we say

- p is *compatible* with x iff $p^* \subseteq x$, and
- p *rejects* x iff p is compatible with x but for any N_ℓ, $k_\ell := \tau(p^\frown \langle N_\ell\rangle)$ is such that $p^\frown \langle N_\ell, k_\ell, s_\ell\rangle$ is incompatible with x for any s_ℓ satisfying $s_\ell(N_\ell) = k_\ell$—in other words, for any N_ℓ, $x(\sum_{i=0}^{\ell} N_i - 1) \neq k_\ell$.

Let $H_p := \{x \mid p \text{ rejects } x\}$. For any $x \in A$ there must be a p which rejects it, otherwise x would be a play according to τ. So $A \subseteq \bigcup_p H_p$, and we shall be done if we can show that $H_p \in \mathfrak{I}_{\text{ioe}}$ for any p.

Fix p of length 3ℓ as before, let $M := \sum_{i=0}^{\ell-1} N_i$, and define a real z as follows: for $N < M$ let $z(N) = 0$, and for $N \geq M$, let $N_\ell := N - M + 1$ and let $z(N) := \tau(p^\frown \langle N_\ell\rangle)$. Suppose p rejects x, which by definition means that for any N_ℓ, $x(\sum_{i=0}^{\ell} N_i - 1) \neq \tau(p^\frown \langle N_\ell\rangle)$. In particular, for any $N \geq M$ we have

$$x(N) = x(M + N_\ell - 1) = x(\sum_{i=0}^{\ell} N_i - 1) \neq \tau(p^\frown \langle N_\ell\rangle) = z(N).$$

Hence $H_p \subseteq K_z = \{x \mid \forall^\infty n\, (x(n) \neq z(n)\}$ which completes the proof. □

5. Half a Cohen real

Recall Fact 2.11, which says that if we iteratively add two ioe reals to a model of set theory then we add a Cohen real; for that reason, an ioe real has sometimes received the name "half a Cohen real". A natural questions which appeared in Fremlin's list of open problems [6] is:

Question 5.1 *(Fremlin). Is it possible to add an ioe real without adding a Cohen real?*

This question was recently answered in the positive by Zapletal [20] using rather unorthodox methods.

Theorem 5.2 *(Zapletal 2013). Let* X *be a compact metrizable space which is infinite-dimensional, and all of its compact subsets are either infinite-dimensional or zero-dimensional. Let* $\mathfrak{I}$ *be the* σ*-ideal* σ*-generated by the compact zero-dimensional subsets of* X*. Then* $\mathcal{B}(X) \setminus \mathfrak{I}$ *adds an ioe real but not a Cohen real.*

In spite of the beauty of this result, Zapletal himself mentions: "as the usual approach towards forcing problems includes a direct combinatorial construction of a suitable poset, the following question is natural: ... is there a combinatorial description of a forcing satisfying [Theorem 5.2] which does not mention topological dimension?"

As $\mathbb{IE}$ seems, in a sense, to be a "minimal" forcing for adding ioe reals, we may wonder whether $\mathbb{IE}$ does not add Cohen reals below some condition, thus providing an alternative solution to Fremlin's problem. The main purpose of this section is to prove the following property for $\mathbb{IE}$:

Theorem 5.3. *For every continuous function* $f : \omega^\omega \to \omega^\omega$ *there exists a* $T \in \mathbb{IE}$ *such that* $f``[T]$ *is meager.*

The relation between this result and Fremlin's problem is given by the following:

Fact 5.4 *(Zapletal). If $\mathfrak{I}$ is a σ-ideal generated by closed sets then $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ has the* continuous reading of names*: for every $\dot{x}$ and B such that $B \Vdash \dot{x} \in \omega^\omega$, there exists $C \leq B$ and a continuous $f : C \to \omega^\omega$ (in the ground model) such that $C \Vdash \dot{x} = f(\dot{x}_{\text{gen}})$.*

Since $\mathfrak{I}_{\text{ioe}}$ is σ-generated by closed sets, the above fact can be applied to $\mathbb{IE}$. So if we can find an $\mathbb{IE}$-condition T_0 and strengthen Theorem 5.3 to "for every $S \leq T_0$ and every continuous $f : [S] \to \omega^\omega$, there exists $T \leq S$ such that $f``[T]$ is meager", it will follow that $\mathbb{IE}$ does not add Cohen reals below $T_0$: for any $\dot{x}$, find $f$ and $S \leq T_0$ such that $S \Vdash \dot{x} = f(\dot{x}_{\text{gen}})$, then find $T \leq S$ such that $f``[T]$ is meager, implying that $T \Vdash$ "$\dot{x}$ is not Cohen" since $T \Vdash \dot{x} \in f``[T]$.

Proof of Theorem 5.3. The proof is quite unusual in the following sense: first we prove that it holds under the assumption that $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$, and then argue that the assumption can be dropped by absoluteness.

Lemma 5.5. *Assume* $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$*. Then for every continuous function $f : \omega^\omega \to \omega^\omega$ there exists a $T \in \mathbb{IE}$ such that $f``[T]$ is meager.*

Proof. Towards contradiction, assume that the theorem is false and fix an $f : \omega^\omega \to \omega^\omega$ such that $f``[T]$ is non-meager for every $T \in \mathbb{IE}$. This is equivalent to saying that f-preimages of meager sets are $\mathfrak{I}_{\text{ioe}}$-small. Let $\{X_\alpha \mid \alpha < \text{add}(\mathcal{M})\}$ be a collection of meager sets such that $\bigcup_\alpha X_\alpha$ is non-meager. We will derive a contradiction by showing that for every basic open $[s]$ there is a basic open $[t] \subseteq [s]$ such that $[t] \cap \bigcup_\alpha X_\alpha =^* \varnothing$.

Fix $[s]$ and a homeomorphism $\psi : \omega^\omega \cong [s]$. Every $X'_\alpha := \psi^{-1}[X_\alpha]$ is still meager, so every $Y_\alpha := f^{-1}[X'_\alpha]$ is $\mathfrak{I}_{\text{ioe}}$-small. For each α let $\langle x_i^\alpha \mid i < \omega\rangle$ be such that $Y_\alpha \subseteq \bigcup_i K_{x_i^\alpha}$. Now, letting ρ be the function defined by $\rho(x) := \langle x(0), x(2), \dots\rangle$, using $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$ and Bartoszyński's characterization of $\text{cov}(\mathcal{M})$, we find that $\{\rho(x_i^\alpha) \mid i < \omega, \alpha < \text{add}(\mathcal{M})\}$ is not an eventually different family, hence there exists c which is infinitely often equal to all $\rho(x_i^\alpha)$.

Construct T_c such that $[T_c] = \{y \mid \rho(y) = c\}$, and by exactly the same argument as in the proof of Theorem 3.5 we know that $[T_c] \cap Y_\alpha = \varnothing$ for every α. But then, by assumption, $f``[T_c]$ is non-meager, and then also $\psi``f``[T_c]$ is non-meager; but it is analytic, hence comeager in a basic open $[t]$. This completes the proof since $[t]$ avoids $\bigcup_\alpha X_\alpha$ modulo meager. □ (Lemma)

To conclude the theorem from the lemma we use a simple absoluteness argument, i.e., we check the complexity of the statement "for all continuous $f : \omega^\omega \to \omega^\omega$ there is $T \in \mathbb{IE}$ such that $f``[T]$ is meager". Note the following:

1. "$f : \omega^\omega \to \omega^\omega$ is a continuous function" can be expressed as "$f' : \omega^{<\omega} \to \omega^{<\omega}$ is monotone and unbounded along each real", which is $\mathbf{\Pi}^1_1$ on (the code of) f'.
2. "$T \in \mathbb{IE}$" is arithmetic on the code of T.
3. $f``[T]$ is an analytic set whose code is recursive in f' and T.
4. For an analytic set to be meager is $\mathbf{\Pi}^1_1$.

Then the statement in question can be expressed as:

$$\forall f' \, (f' \text{ is continuous } \to \exists T \, (T \in \mathbb{IE} \wedge f``[T] \in \mathcal{M}))$$

which is a $\mathbf{\Pi}^1_3$ sentence, hence downward absolute between forcing extensions of V and V itself. So the proof is completed by going to any forcing extension $V^{\mathbb{P}}$ satisfying $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$ (e.g., add ω_2 Cohen reals), applying the lemma and then applying absoluteness to conclude that the statement was already true in V. □

Unfortunately, we do not know whether the proposed strengthening of Theorem 5.3 is valid below some condition T_0, i.e., whether there is T_0 such that for any $S \leq T_0$ and any continuous $f : [S] \to \omega^\omega$ there is $T \leq S$ such that $f``[T]$ is meager. Certainly, a sufficient condition for this would be a T_0 such that for every $S \leq T_0$, $\mathbb{E}{\restriction}S$ is homogeneous (in the sense of Definition 2.14).

On the other hand, by Lemma 2.17 we know that such a T_0 certainly cannot be the trivial condition, since the tree T^{GS} forces that Cohen reals are added. It is not so hard to find other $\mathbb{E}$-conditions, aside of T^{GS}, which also add Cohen reals, but it is not clear whether such conditions are dense.

Notice that the question of $\mathbb{E}$ adding Cohen reals can also be formulated in the setting of the following (closed) game:

Definition 5.6. Let $G^{\mathbb{E}}_{\neg\mathbb{C}}$ be the game defined as follows:

$$\begin{array}{r||llr} \text{I:} & & S \leq T_0, f : [S] \to \omega^\omega \text{ continuous} & \dots \\ \hline \text{II:} & T_0 \in \mathbb{E} & & T \leq S \end{array}$$

$$\begin{array}{cccc} & (s_0, x(0)) & (s_1, x(1)) & \dots \\ \hline \dots & t_0 & t_1 & \dots \end{array}$$

where $s_i, t_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and $x(i) \in \omega$ are such that $x \in [T]$. Assuming all the rules are followed, Player I wins iff $f(x) = s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$.

Lemma 5.7. *If Player I wins $G^{\mathbb{E}}_{\neg\mathbb{C}}$ then every $\mathbb{E}$-condition forces that Cohen reals are added. If Player II wins $G^{\mathbb{E}}_{\neg\mathbb{C}}$ then, letting T_0 be II's first move, $T_0 \Vdash$ "there are no Cohen reals".*

Proof. After the first three moves have been played and $f : [S] \to \omega^\omega$ and $T \leq S$ have been chosen, the rest of the game is essentially Solovay's unfolded version of the Banach–Mazur game, and by a standard argument (see, e.g., [9, Exercise 27.14]) it follows that if Player I wins that game, then $f``[T]$ is comeager in a basic open set, whereas if Player II wins that game, then $f``[T]$ is meager. The rest is clear. □

6. Questions

The most interesting question seems to be the following:

Question 6.1. *Is there an $\mathbb{E}$-condition forcing that no Cohen reals are added, or does $\mathbb{E}$ always add Cohen reals? This can be formulated as "who wins the game $G^{\mathbb{E}}_{\neg\mathbb{C}}$?".*

In Section 3 we completely solved the question of projective regularity for $\mathbb{FM}$- and $\mathbb{E}$-measurability, but not yet for the (arguably more interesting) *weak* $\mathbb{E}$-measurability. We have the following implications:

$$\boldsymbol{\Delta}^1_2(\mathbb{E}) \Leftrightarrow \boldsymbol{\Delta}^1_2(\text{Baire}) \Rightarrow \boldsymbol{\Sigma}^1_2(\text{w}\mathbb{E}) \Rightarrow \boldsymbol{\Delta}^1_2(\text{w}\mathbb{E}),$$

where the first equivalence is due to Theorem 3.4, the second implication due to Theorem 3.5 and the third one trivial. But we do not know anything about the reverse implications. In particular

Question 6.2. *Are $\boldsymbol{\Delta}^1_2$(Baire) and $\boldsymbol{\Delta}^1_2(\text{w}\mathbb{E})$ equivalent? If not, then are $\boldsymbol{\Delta}^1_2$(Baire) and $\boldsymbol{\Sigma}^1_2(\text{w}\mathbb{E})$ equivalent, or are $\boldsymbol{\Sigma}^1_2(\text{w}\mathbb{E})$ and $\boldsymbol{\Delta}^1_2(\text{w}\mathbb{E})$ equivalent?*

A related question is:

Question 6.3. *Can* $\mathbf{\Delta}^1_2(\mathrm{w}\mathbb{IE})$ *and* $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ *be characterized in terms of the existence of quasi-generic reals over* $L[r]$*?*

Finally, in Theorem 4.3 we characterized $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ and $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$, but did not talk about the $\mathbf{\Delta}^1_2$- and $\mathbf{\Pi}^1_1$-levels.

Question 6.4. *Can* $\mathbf{\Pi}^1_1(\mathbb{FM}\text{-dich})$ *and* $\mathbf{\Pi}^1_1(\mathbb{IE}\text{-dich})$ *be added to the list of equivalent statements in Theorem 4.3?*

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3.7 Almost disjoint refinements and mixing reals

Barnabas Farkas, Yurii Khomskii and Zoltán Vydyánszki, Fundamenta Mathematicae, to appear.

This paper was largely the initiative of Farkas. My contribution began with some descriptive set theoretic ideas which then led to the proof of Theorem 1.6. Following this, the rest of the paper was conducted following informal collaborative meetings, mostly between Farkas and Khomskii at the KGRC in Vienna and, separately, between Farkas and Vydyánszki in Budapest. The contributions are roughly as follows:

- ***Theorem 1.6****: Farkas, with contributions by Khomskii*
- ***Corollary 1.8****: Farkas, with contributions by Khomskii*
- ***Theorem 1.9, cf. Propositions 5.2, Proposition 5.3****: Farkas, with contributions by Khomskii and Vydyánszki,*
- ***Proposition 1.10****: Farkas*
- ***Theorem 1.12, cf. Theorem 6.1 and Fact 6.2****: Farkas and Vydyánszki*
- ***Proposition 1.14, cf. Proposition 7.2 and Proposition 7.3****: Collaboration of all three author*

The paper was written by Farkas.

ALMOST DISJOINT REFINEMENTS AND MIXING REALS

BARNABÁS FARKAS, YURII KHOMSKII, AND ZOLTÁN VIDNYÁNSZKY

Abstract. We investigate families of subsets of ω with almost disjoint refinements in the classical case as well as with respect to given ideals on ω. More precisely, we study the following topics and questions:

1) Examples of projective ideals.

2) We prove the following generalization of a result due to J. Brendle: If $V \subseteq W$ are transitive models, $\omega_1^W \subseteq V$, $\mathcal{P}(\omega) \cap V \neq \mathcal{P}(\omega) \cap W$, and $\mathcal{I}$ is an analytic or coanalytic ideal coded in V, then there is an $\mathcal{I}$-almost disjoint refinement ($\mathcal{I}$-ADR) of $\mathcal{I}^+ \cap V$ in W, that is, a family $\{A_X : X \in \mathcal{I}^+ \cap V\} \in W$ such that (i) $A_X \subseteq X$, $A_X \in \mathcal{I}^+$ for every X and (ii) $A_X \cap A_Y \in \mathcal{I}$ for every distinct X and Y.

3) The existence of perfect $\mathcal{I}$-almost disjoint ($\mathcal{I}$-AD) families; and the existence of a "nice" ideal $\mathcal{I}$ on ω with the property: Every $\mathcal{I}$-AD family is countable but $\mathcal{I}$ is nowhere maximal.

4) The existence of $(\mathcal{I}, \mathrm{Fin})$-almost disjoint refinements of families of $\mathcal{I}$-positive sets in the case of everywhere meager (e.g. analytic or coanalytic) ideals. We show that under Martin's Axiom if $\mathcal{I}$ is an everywhere meager ideal and $\mathcal{H} \subseteq \mathcal{I}^+$ with $|\mathcal{H}| < \mathfrak{c}$, then $\mathcal{H}$ has an $(\mathcal{I}, \mathrm{Fin})$-ADR, that is, a family $\{A_H : H \in \mathcal{H}\}$ such that (i) $A_H \subseteq H$, $A_H \in \mathcal{I}^+$ for every H and (ii) $A_{H_0} \cap A_{H_1}$ is finite for every distinct $H_0, H_1 \in \mathcal{H}$.

5) Connections between classical properties of forcing notions and adding mixing reals (and mixing injections), that is, a (one-to-one) function $f : \omega \to \omega$ such that $|f[X] \cap Y| = \omega$ for every $X, Y \in [\omega]^\omega \cap V$. This property is relevant concerning almost disjoint refinements because it is very easy to find an almost disjoint refinement of $[\omega]^\omega \cap V$ in every extension $V \subseteq W$ containing a mixing injection over V.

1. Introduction

Let us begin with our motivations which led us to work on almost disjoint refinements and their generalizations. First of all, the following easy fact seems to be somewhat surprising (see also Proposition 1.10):

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Key words and phrases. analytic ideal, coanalytic ideal, almost disjoint family, almost disjoint refinement, Mansfield-Solovay Theorem, mixing real, meager ideal.

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Fact 1.1. *If $\mathcal{H} \subseteq [\omega]^\omega (= \{X \subseteq \omega : |X| = \omega\})$ is of size $< \mathfrak{c}$, then $\mathcal{H}$ has an* almost-disjoint refinement *$\{A_H : H \in \mathcal{H}\}$, that is, (i) $A_H \in [H]^\omega$ for every $H \in \mathcal{H}$ and (ii) $|A_H \cap A_K| < \omega$ for every $H \neq K$ from $\mathcal{H}$.*

The following theorem due to B. Balcar and P. Vojtáš is probably the most well-know general result on the existence of almost-disjoint refinements.

Theorem 1.2. (see [BaV80]) *Every ultrafilter on ω has an almost-disjoint refinement.*

B. Balcar and T. Pazák, and independently J. Brendle proved the following theorem:

Theorem 1.3. *(see* [BaP10], [LS08]*) Assume that $V \subseteq W$ are transitive models and $\mathcal{P}(\omega) \cap V \neq \mathcal{P}(\omega) \cap W$. Then $[\omega]^\omega \cap V$ has an almost-disjoint refinement in W (where by* transitive model *we mean a transitive model of a "large enough" finite fragment of* ZFC*).*

One of our main results is a generalization of this theorem in the context of "nice" ideals on ω, that is, we change the notion of *smallness* in the setting above by replacing *finite* with *element of an ideal $\mathcal{I}$*.

In order to formulate our generalization and to give a setting to our other related results, we have to introduce some notations and the appropriate versions of the classical notions.

Let $\mathcal{I}$ be an ideal on a countably infinite set X. We always assume that $[X]^{<\omega} = \{Y \subseteq X : |Y| < \omega\} \subseteq \mathcal{I}$ and $X \notin \mathcal{I}$. Let us denote by $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ the family of $\mathcal{I}$-*positive* sets, and by $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$ the *dual filter* of $\mathcal{I}$. If $Y \in \mathcal{I}^+$ then let $\mathcal{I} \restriction Y = \{A \in \mathcal{I} : A \subseteq Y\} = \{B \cap Y : B \in \mathcal{I}\}$ be the *restriction* of $\mathcal{I}$ to Y (an ideal on Y). If X is clear from the contex, then the ideal of finite subsets of X will be denoted by Fin.

Definition 1.4. We say that a non-empty family $\mathcal{A} \subseteq \mathcal{I}^+$ is $\mathcal{I}$-*almost-disjoint* ($\mathcal{I}$-AD) if $A \cap B \in \mathcal{I}$ for every distinct $A, B \in \mathcal{A}$. A family $\mathcal{A} \subseteq \mathcal{I}^+$ is $(\mathcal{I}, \mathrm{Fin})$-AD if $|A \cap B| < \omega$ for every distinct $A, B \in \mathcal{A}$.

Definition 1.5. Let $\mathcal{H} \subseteq \mathcal{I}^+$. We say that a family $\mathcal{A} = \{A_H : H \in \mathcal{H}\}$ is an $\mathcal{I}$-*AD refinement* ($\mathcal{I}$-ADR) of $\mathcal{H}$ if (i) $A_H \subseteq H$, $A_H \in \mathcal{I}^+$ for every H, and (ii) $A_{H_0} \cap A_{H_1} \in \mathcal{I}$ for every distinct $H_0, H_1 \in \mathcal{H}$ (in paticular, $\mathcal{A}$ is an $\mathcal{I}$-AD family). If $\mathcal{I} = \mathrm{Fin}$ we simply say AD-refinement (ADR).

We say that a family $\mathcal{A} = \{A_H : H \in \mathcal{H}\}$ is an $(\mathcal{I}, \mathrm{Fin})$-*AD refinement* ($(\mathcal{I}, \mathrm{Fin})$-ADR) of $\mathcal{H}$ if (i) holds and (ii)' $|A_{H_0} \cap A_{H_1}| < \omega$ for every distinct $H_0, H_1 \in \mathcal{H}$.

Notice that an ideal on a countably infinite X can be regarded as a subset of the Polish space $2^X \simeq 2^\omega$ using a bijection between X and ω. Thus, it makes sense to talk about Borel, analytic, etc ideals and about certain descriptive properties of ideals, such as the Baire property or meagerness (it is easy to see that these properties do not depend on the choice of the bijection). In the past two decades the study of certain definable (e.g. Borel, analytic, coanalytic, etc.) ideals has become a central topic in set theory. It turned out that they play an important role in combinatorial set theory, and in the theory of cardinal invariants of the continuum as well as the theory of forcing (see e.g. [Ma91], [So99], [F], [Hr11] and many other publications).

Now we can formulate our generalization of Theorem 1.3:

Theorem 1.6. *Assume that $V \subseteq W$ are transitive models, $\omega_1^W \subseteq V$, $\mathcal{P}(\omega) \cap V \neq \mathcal{P}(\omega) \cap W$, and $\mathcal{I}$ is an analytic or coanalytic ideal coded in V. Then there is an $\mathcal{I}$-ADR of $\mathcal{I}^+ \cap V$ in W.*

We say that an ideal $\mathcal{I}$ on X (where $|X| = \omega$) is *everywhere meager* if $\mathcal{I} \restriction Y$ is meager in $\mathcal{P}(Y)$ for every $Y \in \mathcal{I}^+$. In particular, analytic and coanalytic ideals are everywhere meager because their restrictions are also analytic and coanalytic, respectively, hence have the Baire property, and we can apply the following well-known characterisation theorem (due to Sierpiński (1)⟷(2), and Talagrand (2)⟷(3), for the proofs see e.g. [BrJ, Thm 4.1.1-2]).

Theorem 1.7. *Let $\mathcal{I}$ be an ideal on ω. Then the following are equivalent: (1) $\mathcal{I}$ has the Baire property, (2) $\mathcal{I}$ is meager, and (3) there is a partition $\{P_n : n \in \omega\}$ of ω into finite sets such that $\{n \in \omega : P_n \subseteq A\}$ is finite for each $A \in \mathcal{I}$.*

From now on, when working with partitions of a set, we always assume that every element of the partition is nonempty. From this theorem we can also deduce the following important corollary:

Corollary 1.8. *If $\mathcal{I}$ is a meager ideal, then there is a perfect $(\mathcal{I}, \mathrm{Fin})$-AD family. In particular, if $\mathcal{I}$ is everywhere meager, then there are perfect $(\mathcal{I}, \mathrm{Fin})$-AD families on every $X \in \mathcal{I}^+$.*

Proof. It is easy to define a perfect AD family $\mathcal{A}$ on ω (e.g. consider the branches of $2^{<\omega}$ in $\mathcal{P}(2^{<\omega})$). Fix a partition $(P_n)_{n\in\omega}$ of ω into finite sets such that $\{n \in \omega : P_n \subseteq A\}$ is finite for every $A \in \mathcal{I}$. For each $A \in \mathcal{A}$ let $A' = \bigcup\{P_n : n \in A\} \in \mathcal{I}^+$, and let $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$. Then $|A' \cap B'| < \omega$ for every distinct $A, B \in \mathcal{A}$ hence $\mathcal{A}'$ is an $(\mathcal{I}, \mathrm{Fin})$-AD family. The function $\mathcal{P}(\omega) \to \mathcal{P}(\omega)$, $A \mapsto A'$ is injective and continuous hence $\mathcal{A}'$ is perfect. □

Concerning the reverse implications in Corollary 1.8, we prove the following.

Theorem 1.9.

(a) *The existence of a perfect $(\mathcal{I}, \mathrm{Fin})$-AD family does not imply that $\mathcal{I}$ is meager.*
(b) *If $\mathfrak{b} = \mathfrak{c}$ then there is an non-meager ideal $\mathcal{I}$ such that there are perfect $(\mathcal{I}, \mathrm{Fin})$-AD families on every $X \in \mathcal{I}^+$. Here $\mathfrak{c}$ stands for the continuum and $\mathfrak{b}$ for the* bounding number, *that is, $\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega$ is $\leq^*$-unbounded$\}$ where $f \leq^* g$ iff the set $\{n \in \omega : f(n) > g(n)\}$ is finite.*
(c) *There is an ideal $\mathcal{I}$ such that every $\mathcal{I}$-AD family is countable but $\mathcal{I}$ is nowhere maximal, that is, $\mathcal{I} \restriction X$ is not a prime ideal for any $X \in \mathcal{I}^+$ (in particular, there are infinite $\mathcal{I}$-AD families).*
(d) *It is independent from* ZFC *whether the example in (c) can be chosen as $\undertilde{\Sigma}^1_2$.*

Corollary 1.8 has an easy but important application. Clearly, if $\mathcal{I}$ is an ideal on ω then there is a family (e.g. $\mathcal{I}^+$) of size $\mathfrak{c}$ which does not have any $\mathcal{I}$-ADR's. Conversely, we have the following very special case of results from [BgHM84] and [BaSV81]:

Proposition 1.10. *If $\mathcal{I}$ is an everywhere meager ideal and $\mathcal{H} \in [\mathcal{I}^+]^{<\mathfrak{c}}$, then $\mathcal{H}$ has an $\mathcal{I}$-ADR.*

Proof. Let $\mathcal{H} = \{H_\alpha : \alpha < \kappa\}$. Applying Corollary 1.8, we can fix an $\mathcal{I}$-AD family $\mathcal{A} = \{A_\xi : \xi < \kappa^+\}$ on H_0 and for every $\beta < \kappa$ let $T_\beta = \{\xi < \kappa^+ : H_\beta \cap A_\xi \in \mathcal{I}^+\}$, furthermore let $R = \{\beta < \kappa : |T_\beta| \leq \kappa\}$ (we know that $0 \notin R$). By induction on $\alpha \in \kappa \setminus R$ we can pick a

$$\xi_\alpha \in T_\alpha \setminus \Big(\bigcup_{\beta \in R} T_\beta \cup \{\xi_{\alpha'} : \alpha' \in \alpha \setminus R\} \Big)$$

because $|T_\alpha| = \kappa^+$ and $|\bigcup\{T_\beta : \beta \in R\}| \leq \kappa$, and let $E_\alpha = H_\alpha \cap A_{\xi_\alpha} \in \mathcal{I}^+$. Then the family $\{E_\alpha : \alpha \in \kappa \setminus R\}$ is an $\mathcal{I}$-ADR of $\{H_\alpha : \alpha \in \kappa \setminus R\}$. We can continue the procedure on $\{H_\beta : \beta \in R\}$ because $E_\alpha \cap H_\beta \in \mathcal{I}$ for every $\alpha \in \kappa \setminus R$ and $\beta \in R$. □

This proposition motivates the following:

Question 1.11. Let $\mathcal{I}$ be an everywhere meager ideal and $\mathcal{H} \in [\mathcal{I}^+]^{<\mathfrak{c}}$. Does $\mathcal{H}$ have an $(\mathcal{I}, \mathrm{Fin})$-ADR?

We answer this question, at least consistently:

Theorem 1.12. *Assume* MA_κ *and let $\mathcal{I}$ be an everywhere meager ideal, then every $\mathcal{H} \in [\mathcal{I}^+]^{\leq\kappa}$ has an* $(\mathcal{I}, \mathrm{Fin})$*-ADR.*

We also define new notions of mixing and injective mixing reals, and investigate connections between adding (injective) mixing reals and classical properties of forcing notions (such as adding Cohen/random/splitting/dominating reals and the Laver/Sacks-properties).

Definition 1.13. Let $\mathbb{P}$ be a forcing notion. We say that an $f \in \omega^\omega \cap V^{\mathbb{P}}$ is a *mixing real* over V if $|f[X] \cap Y| = \omega$ for every $X, Y \in [\omega]^\omega \cap V$. If f is one-to-one, then we call it an *injective mixing real* or *mixing injection*.

Our results are summarized in the following proposition.

Proposition 1.14. *Let $\mathbb{P}$ be a forcing notion.*

(i) *If $\mathbb{P}$ adds random reals, then it adds mixing reals.*
(ii) *If $\mathbb{P}$ adds dominating reals, then it adds mixing reals.*
(iii) *If $\mathbb{P}$ adds Cohen reals, then it adds mixing injections.*
(iv) *If $\mathbb{P}$ adds mixing injections, then it adds unbounded reals.*
(v) *If $\mathbb{P}$ has the Laver-property, then it does not add injective mixing reals.*

Our paper is organized as follows. In Section 2 we recall some notations and classical results of descriptive set theory we will need later.

The next two sections are focused on descriptive aspects of nice ideals and almost disjoint refinements. In Section 3 we present a plethora of examples of Borel and projective ideals on ω. In Section 4 we prove Theorem 1.6 by modifying Brendle's proof of Theorem 1.3.

The next two sections contain rather combinatorial results. In Section 5 we prove Theorem 1.9, as well as study some problems concerning the possible generalizations of Corollary 1.8 on the second level of the projective hierarchy. In Section 6 we prove Theorem 1.12.

In Section 7 we study the notions of mixing and injective mixing reals. In this section we will heavily use standard facts about forcing notions, for the details see [BrJ].

Finally, in Section 8, we list some open questions concerning our results.

2. Descriptive set theory and ideals

As usual, $\utilde{\Sigma}^0_\alpha, \utilde{\Pi}^0_\alpha$ will stand for the αth level of the Borel hierarchy while we denote by $\utilde{\Sigma}^1_n, \utilde{\Pi}^1_n$ the levels of the projective hierarchy. If r is a real, the appropriate relativised versions are denoted by $\Sigma^0_\alpha(r), \Pi^0_\alpha(r)$, etc. For the ambiguous classes we write $\utilde{\Delta}^i_\alpha$ and $\Delta^i_\alpha(r)$.

Suppose that $\mathcal{I}$ is an ideal on the set X. As mentioned before, if X is countable then we can talk about complexity of ideals: $\mathcal{I}$ is F_σ, $\utilde{\Sigma}^0_\alpha$, $\utilde{\Pi}^1_n$, etc if $\mathcal{I} \subseteq \mathcal{P}(X) \simeq 2^X$ is an F_σ, $\utilde{\Sigma}^0_\alpha$, $\utilde{\Pi}^1_n$, etc set in the usual compact Polish topology on 2^X. If we fix a bijection between ω and X we can define the collection of $\Sigma^0_\alpha(r), \Pi^0_\alpha(r)$, etc subsets of 2^X as well. If $X = \omega^n, \Delta = \{(n,m) \in \omega^2 : m \leq n\}, [\omega]^n, 2^{<\omega}, \omega^{<\omega}, \mathbb{Q}(= \{\text{rational numbers}\})$ then the we will always assume that the bijection is the usual, recursive one.

For example, $\mathrm{Fin} = [\omega]^{<\omega}$ is an F_σ ideal, $\mathcal{Z} = \{A \subseteq \omega : |A \cap n|/n \to 0\}$ is $F_{\sigma\delta}$, and $\mathrm{Conv} = \{A \subseteq \mathbb{Q} \cap [0,1] : A$ has only finitely many accumulation points$\}$ is $F_{\sigma\delta\sigma}$, etc (see more examples in Section 3). Similarly, we can associate descriptive complexity to any $\mathcal{X} \subseteq \mathcal{P}(\omega)$, and we can also talk about the Baire property and measurability of subsets of $\mathcal{P}(\omega)$. Clearly, if $Y \in \mathcal{I}^+$ then $\mathcal{I} \restriction Y$ belongs to the same Borel or projective class in $\mathcal{P}(Y)$ as $\mathcal{I}$ in $\mathcal{P}(\omega)$ (simply because $\mathcal{I} \restriction Y$ is a continuous preimage of $\mathcal{I}$).

For a family $\mathcal{H} \subset 2^X$ we will denote by $\mathrm{id}(\mathcal{H})$ the ideal generated by the sets in $\mathcal{H}$. We say that an ideal $\mathcal{I}$ on a countably infinite set X is

- *tall* if every infinite subset of X contains an infinite element of $\mathcal{I}$;
- a *P-ideal* if for every sequence $A_n \in \mathcal{I}$ $(n \in \omega)$, there is an $A \in \mathcal{I}$ such that $A_n \subseteq^* A$, that is, $|A_n \setminus A| < \omega$ for every n.

We will need the following two fundamental results of descriptive set theory (see e.g. in [J]):

Theorem 2.1. (Shoenfield Absoluteness Theorem) *If $V \subseteq W$ are transitive models, $\omega_1^W \subseteq V$, and $r \in \omega^\omega \cap V$, then $\Sigma^1_2(r)$ formulas are absolute between V and W.*

Corollary 2.2. *If $X \subseteq \mathcal{P}(\omega)$ is an analytic or coanalytic set in the parameter $r \in \omega^\omega$, then the statement "X is an ideal" is absolute for transitive models $V \subseteq W$ with $\omega_1^W \subseteq V$ and $r \in V$.*

Proof. Let $\varphi(x,r)$ be a $\Sigma^1_1(r)$ or $\Pi^1_1(r)$ definition of X ($r \in \omega^\omega$). Then the statement "X is an ideal" is the conjunction of the following formulas (i) $\forall a \in \mathrm{Fin}\ \varphi(a,r)$, (ii) $\forall\ x,y$ ($x \nsubseteq y$ or $\neg\varphi(y,r)$ or $\varphi(x,r)$), and (iii) $\forall\ x,y$ ($\neg\varphi(x,r)$ or $\neg\varphi(y,r)$ or $\varphi(x \cup y, r)$). In particular, "X is an ideal" is $\Pi^1_2(r)$ and hence we can apply the Shoenfield Absoluteness Theorem. □

Theorem 2.3. (Mansfield-Solovay Theorem) *If $A \nsubseteq L[r]$ is a $\Sigma^1_2(r)$ set, then A contains a perfect subset.*

Other than these notions and results above, we will use descriptive set theoretic tools such as Γ-completeness, Γ-hardness, etc which can all be found in [K].

Let $\mathrm{Tree} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree}\}$ be the usual Polish space of all trees on ω (a closed subset on $\mathcal{P}(\omega^{<\omega})$) and as usual, we denote by $[T] = \{x \in \omega^\omega : \forall\ n\ x \restriction n \in T\}$ the *body* of T, i.e. the set of all branches of T.

3. Examples of Borel and projective ideals

There are many classical examples of Borel ideals. Here we present some of those that have easily understandable definitions, and the reader can see that these examples are motivated by a wide variety of backgrounds. For the important role of these ideals, especially in characterisation results, see [Hr11].

Some F_σ ideals:

Summable ideals. Let $h : \omega \to [0,\infty)$ be a function such that $\sum_{n\in\omega} h(n) = \infty$. The *summable ideal associated to h* is

$$\mathcal{I}_h = \Big\{A \subseteq \omega : \sum_{n \in A} h(n) < \infty\Big\}.$$

It is easy to see that a summable ideal $\mathcal{I}_h$ is tall iff $\lim_{n\to\infty} h(n) = 0$, and that summable ideals are F_σ P-ideals. The *classical summable ideal* is $\mathcal{I}_{1/n} = \mathcal{I}_h$ where $h(n) = 1/(n+1)$, or $h(0) = 1$ and $h(n) = 1/n$ if $n > 0$. We know that there are tall F_σ P-ideals which are not summable ideals: Farah's example (see [F, Example 1.11.1]) is the following ideal:

$$\mathcal{I}_F = \Big\{A \subseteq \omega : \sum_{n<\omega} \frac{\min\{n, |A \cap [2^n, 2^{n+1})|\}}{n^2} < \infty\Big\}.$$

The *eventually different ideals.*

$$\mathcal{ED} = \Big\{A \subseteq \omega \times \omega : \limsup_{n\to\infty} |(A)_n| < \infty\Big\}$$

where $(A)_n = \{k \in \omega : (n,k) \in A\}$, and $\mathcal{ED}_{\mathrm{fin}} = \mathcal{ED} \restriction \Delta$ where $\Delta = \{(n,m) \in \omega \times \omega : m \leq n\}$. $\mathcal{ED}$ and $\mathcal{ED}_{\mathrm{fin}}$ are not P-ideals.

The *van der Waerden ideal:*

$$\mathcal{W} = \{A \subseteq \omega : A \text{ does not contain arbitrary long arithmetic progressions}\}.$$

Van der Waerden's well-known theorem says that $\mathcal{W}$ is a proper ideal. $\mathcal{W}$ is not a P-ideal. For some set-theoretic results about this ideal see e.g. [Fl09] and [Fl10].

The *random graph ideal:*

$$\text{Ran} = \text{id}\big(\{\text{homogeneous subsets of the random graph}\}\big)$$

where the *random graph* (ω, E), $E \subseteq [\omega]^2$ is up to isomorphism uniquely determined by the following property: If $A, B \in [\omega]^{<\omega}$ are nonempty and disjoint, then there is an $n \in \omega \setminus (A \cup B)$ such that $\{\{n,a\} : a \in A\} \subseteq E$ and $\{\{n,b\} : b \in B\} \cap E = \emptyset$. A set $H \subseteq \omega$ is (E-)homogeneous iff $[H]^2 \subseteq E$ or $[H]^2 \cap E = \emptyset$. Ran is not a P-ideal.

The *ideal of graphs with finite chromatic number*:

$$\mathcal{G}_{\text{fc}} = \{E \subseteq [\omega]^2 : \chi(\omega, E) < \omega\}.$$

It is not a P-ideal.

Solecki's ideal: Let $\text{CO}(2^\omega)$ be the family of clopen subsets of 2^ω (it is easy to see that $|\text{CO}(2^\omega)| = \omega$), and let $\Omega = \{A \in \text{CO}(2^\omega) : \lambda(A) = 1/2\}$ where λ is the usual product measure on 2^ω. The ideal $\mathcal{S}$ on Ω is generated by $\{I_x : x \in 2^\omega\}$ where $I_x = \{A \in \Omega : x \in A\}$. $\mathcal{S}$ is not a P-ideal.

Some $F_{\sigma\delta}$ ideals:

Density ideals. Let $(P_n)_{n\in\omega}$ be a sequence of pairwise disjoint finite subsets of ω and let $\vec{\mu} = (\mu_n)_{n\in\omega}$ be a sequences of measures, μ_n is concentrated on P_n such that $\limsup_{n\to\infty} \mu_n(\omega) > 0$. The *density ideal generated by* $\vec{\mu}$ is

$$\mathcal{Z}_{\vec{\mu}} = \Big\{A \subseteq \omega : \lim_{n\to\infty} \mu_n(A) = 0\Big\}.$$

A density ideal $\mathcal{Z}_{\vec{\mu}}$ is tall iff $\max\{\mu_n(\{i\}) : i \in P_n\} \xrightarrow{n\to\infty} 0$, and density ideals are $F_{\sigma\delta}$ P-ideals. The *density zero ideal* $\mathcal{Z} = \{A \subseteq \omega : \lim_{n\to\infty} |A \cap n|/n = 0\}$ is a tall density ideal because let $P_n = [2^n, 2^{n+1})$ and $\mu_n(A) = |A \cap P_n|/2^n$. It is easy to see that $\mathcal{I}_{1/n} \subsetneq \mathcal{Z}$, and Szemerédi's famous theorem implies that $\mathcal{W} \subseteq \mathcal{Z}$ (see [Sz75]). The stronger statement $\mathcal{W} \subseteq \mathcal{I}_{1/n}$ is a still open Erdős prize problem.

The *ideal of nowhere dense subsets of the rationals:*

$$\text{Nwd} = \{A \subseteq \mathbb{Q} : \text{int}(\overline{A}) = \emptyset\}$$

where $\text{int}(\cdot)$ stands for the interior operation on subsets of the reals, and $\overline{A}$ is the closure of A in $\mathbb{R}$. Nwd is not a P-ideal.

The *trace ideal of the null ideal*: Let $\mathcal{N}$ be the σ-ideal of subsets of 2^ω with measure zero (with respect to the usual product measure). The G_δ*-closure* of a set $A \subseteq 2^{<\omega}$ is $[A]_\delta = \{x \in 2^\omega : \exists^\infty\, n\ x \restriction n \in A\}$, a G_δ subset of 2^ω. The trace of $\mathcal{N}$ is defined by

$$\text{tr}(\mathcal{N}) = \{A \subseteq 2^{<\omega} : [A]_\delta \in \mathcal{N}\}.$$

It is a tall $F_{\sigma\delta}$ P-ideal.

Some tall $F_{\sigma\delta\sigma}$ (non P-)ideals:

The ideal Conv is generated by those infinite subsets of $\mathbb{Q}\cap[0,1]$ which are convergent in $[0,1]$, in other words

$$\mathrm{Conv}=\big\{A\subseteq\mathbb{Q}\cap[0,1]:|\text{accumulation points of }A\text{ (in }\mathbb{R})|<\omega\big\}.$$

The Fubini product of Fin by itself:

$$\mathrm{Fin}\otimes\mathrm{Fin}=\big\{A\subseteq\omega\times\omega:\forall^\infty\, n\in\omega\ |(A)_n|<\omega\big\}.$$

Some non-tall ideals:

An important F_σ ideal:

$$\mathrm{Fin}\otimes\{\emptyset\}=\big\{A\subseteq\omega\times\omega:\forall^\infty\, n\in\omega\ (A)_n=\emptyset\big\},$$

and its $F_{\sigma\delta}$ brother (a density ideal):

$$\{\emptyset\}\otimes\mathrm{Fin}=\big\{A\subseteq\omega\times\omega:\forall\, n\in\omega\ |(A)_n|<\omega\big\}.$$

Applying the Baire Category Theorem, it is easy to see that there are no G_δ (i.e. $\utilde{\Pi}^0_2$) ideals and we already presented many F_σ (i.e. $\utilde{\Sigma}^0_2$) ideals. In general, we have Borel ideals at arbitrary high levels of the Borel hierarchy:

Theorem 3.1. (see [C85] and [C88]) *There are $\utilde{\Sigma}^0_\alpha$- and $\utilde{\Pi}^0_\alpha$-complete ideals for every $\alpha\geq 3$.*

About ideals on the ambiguous levels of the Borel hierarchy see [E94].

We also present some (co)analytic examples.

Theorem 3.2. (see [Z90, page 321]) *For every $x\in\omega^\omega$ let $I_x=\{s\in\omega^{<\omega}:x\restriction|s|\not\leq s\}$ where $\leq$ is the coordinatewise ordering on every ω^n. Then the ideal on $\omega^{<\omega}$ generated by $\{I_x:x\in\omega^\omega\}$ is $\utilde{\Sigma}^1_1$-complete.*

Theorem 3.3. *The ideal of graphs without infinite complete subgraphs,*

$$\mathcal{G}_c=\big\{E\subseteq[\omega]^2:\forall\, X\in[\omega]^\omega\ [X]^2\not\subseteq E\big\}$$

is a $\utilde{\Pi}^1_1$-complete (in $\mathcal{P}([\omega]^2)$), tall, non P-ideal.

Proof. Tallness is trivial. If for every $n\in\omega$, we define $E_n=\{\{k,m\}:k\leq n,m\neq k\}\in\mathcal{G}_c$ and $E_n\subseteq^* E\subseteq[\omega]^2$ for every n, then E contains a complete subgraph (see also in [Me09]), hence $\mathcal{G}_c$ is not a P-ideal.

Let $\mathrm{WF}=\{T\in\mathrm{Tree}:[T]=\emptyset\}$ be the $\utilde{\Pi}^1_1$-complete set of well-founded trees. Furthermore, let Tree$'$ be the family of those trees T such that (i) every $t\in T$ is strictly increasing and (ii) if $\{t\in T:n\in\mathrm{ran}(t)\}\neq\emptyset$ then it has a $\subseteq$-minimal element ($n\in\omega$). Then it is not hard to see that Tree$'$ is also closed in $\mathcal{P}(\omega^{<\omega})$ hence Polish. Finally, let $\mathrm{WF}'=\{T\in\mathrm{Tree}':[T]=\emptyset\}$, clearly, it is also Π^1_1.

We will construct Wadge-reductions $\mathrm{WF}\leq_W\mathrm{WF}'\leq_W\mathcal{G}_c$.

$\mathrm{WF}\leq_W\mathrm{WF}'$: Fix an order preserving isomorphism j between $\omega^{<\omega}$ and a $T_0\in\mathrm{Tree}'$. More precisely, for a $t=(k_0,k_1,\dots,k_{m-1})\in\omega^{<\omega}$ let $j(t)=(p^1_{k_0},p^1_{k_0}p^2_{k_1},\dots,p^1_{k_0}p^2_{k_1}\cdots p^m_{k_{m-1}})$ where p_i denotes the ith prime number. Then j is one-to-one, order preserving, and $T_0=j[\omega^{<\omega}]$ is a tree containing strictly increasing sequences. To show that T_0 satisfies (ii), assume that $n \in \mathrm{ran}(j(t))$ for some $n \in \omega$ and $t \in \omega^{<\omega}$. Then, by the definition of j, $n = p^1_{k_0} p^2_{k_1} \dots p^m_{k_{m-1}}$ where $s = (k_0, k_1, \dots, k_{m-1}) \leq t$, and if $n \in \mathrm{ran}(j(t'))$ for some $t' \in \omega^{<\omega}$ then $s \leq t'$, hence $j(s)$ is $\subseteq$-minimal in $\{h \in T_0 : n \in \mathrm{ran}(h)\}$.

The map Tree $\to$ Tree$'$, $T \mapsto j[T]$ is a continuous reduction of WF to WF$'$. Continuity is trivial, and also that $[T] = \emptyset$ iff $[j[T]] \neq \emptyset$, in other words, $T \in$ WF iff $j[T] \in$ WF$'$.

WF$' \leq_W \mathcal{G}_c$: For every $T \in$ Tree$'$ let $E_T = \bigcup\{[\mathrm{ran}(t)]^2 : t \in T\}$. We show that the function $T \mapsto E_T$ is continuous. If $u, v \in \left[[\omega]^2\right]^{<\omega}$ are disjoint then it is easy to see that the preimage of the basic clopen set $[u, v] = \{E \subseteq [\omega]^2 : u \subseteq E, v \cap E = \emptyset\} \subseteq \mathcal{P}([\omega]^2)$ is

$$\big\{T \in \mathrm{Tree}' : \big(\forall\ \{x, y\} \in u\ \exists\ t \in T\ x, y \in \mathrm{ran}(t)\big) \text{ and } \big(\forall\ t \in T\ v \cap [\mathrm{ran}(t)]^2 = \emptyset\big)\big\}.$$

Although, as the collection of the sets satisfying the second part of the condition is a countable intersection of clopen sets, this set seems to be closed (and it is enough to prove that $\mathcal{G}_c$ is $\mathbf{\Pi}^1_1$-complete), actually, it is open in Tree$'$: Let $m = \max(\cup v) + 1$. Then the set $\{T \in \mathrm{Tree}' : \forall\ t \in T\ v \cap [\mathrm{ran}(t)]^2 = \emptyset\}$ is the intersection of Tree$'$ and the clopen set (in $\mathcal{P}(\omega^{<\omega})$)

$$\big[\emptyset, \big\{t \in m^{\leq m} : t \text{ is strictly increasing and } v \cap [\mathrm{ran}(t)]^2 \neq \emptyset\big\}\big].$$

The function $T \mapsto E_T$ is a reduction of WF$'$ to $\mathcal{G}_c$: Clearly, if $T \in$ Tree$'$ and $x \in [T]$ then $X = \mathrm{ran}(x) \in [\omega]^\omega$ shows that $E_T \notin \mathcal{G}_c$ (i.e. $[X]^2 \subseteq E$). Conversely, if $[X]^2 \subseteq E_T$ and $X = \{k_0 < k_1 < \dots\}$, then for every n there is a $t_n \in T$ such that $k_n, k_{n+1} \in \mathrm{ran}(t_n)$, we can assume that t_n is minimal in $\{s \in T : k_{n+1} \in \mathrm{ran}(s)\}$. It yields that $t_0 \subseteq t_1 \subseteq t_2 \subseteq \dots$ is an infinite chain in T. □

In the following example, we show that a seemingly "very" Π^1_2 definition can also give us a $\mathbf{\Pi}^1_1$-complete ideal.

Theorem 3.4. *The ideal*

$$\mathcal{I}_0 = \big\{A \subseteq \omega \times \omega : \forall\ X, Y \in [\omega]^\omega\ \exists\ X' \in [X]^\omega\ \exists\ Y' \in [Y]^\omega\ A \cap (X' \times Y') = \emptyset\big\}$$

is a $\mathbf{\Pi}^1_1$-complete (in $\mathcal{P}(\omega \times \omega)$), tall, non P-ideal.

Proof. Tallness is trivial because injective partial functions from ω to ω belong to $\mathcal{I}_0$. The failure of the P property is also easy: Consider the sets $n \times \omega \in \mathcal{I}$. If for some A we have $n \times \omega \subseteq^* A$ for every n then every vertical section of A is co-finite, and such a set is clearly $\mathcal{I}_0$-positive.

First we show that this ideal is $\mathbf{\Pi}^1_1$, for which the next claim is clearly enough. For $X, Y \in [\omega]^\omega$ define $T^\uparrow(X, Y) = \{(n, k) \in X \times Y : n < k\}$ and $T^\downarrow(X, Y) = \{(n, k) \in X \times Y : n > k\}$.

Claim. *$A \in \mathcal{I}_0$ iff for every infinite X and Y the set A does not contain $T^\uparrow(X, Y)$ or $T^\downarrow(X, Y)$.*

Proof of the Claim. The "only if" part is trivial. Conversely, assume that $A \notin \mathcal{I}_0$, i.e. there exist $X, Y \in [\omega]^\omega$ such that $A \cap (X' \times Y') \neq \emptyset$ for every $X' \in [X]^\omega$ and $Y' \in [Y]^\omega$. Fix increasing enumerations $X = \{x_0 < x_1 < x_2 < \dots\}$ and $Y = \{y_0 < y_1 < y_2 < \dots\}$. By shrinking the sets X and Y, we can assume that $x_0 < y_0 < x_1 < y_1 < \dots$, in particular $X \cap Y = \emptyset$. Consider the following coloring $c : [\omega]^2 \to 2 \times 2$: for $m < n$ let $c(m,n) = (\chi_A(x_m, y_n), \chi_A(x_n, y_m))$ where $\chi_A(x,y) = 1$ iff $(x,y) \in A$.

Applying Ramsey's theorem, there exists an infinite homogeneous subset $S \subseteq \omega$. Let $S = Z \cup W$ be a partition into infinite subsets such that the elements of Z and W follow alternatingly in S. Then the elements of the sets $X' = \{x_m : m \in Z\}$ and $Y' = \{y_n : n \in W\}$ follow alternatingly in ω as well.

S cannot be homogeneous in color $(0,0)$, otherwise $A \cap (X' \times Y') = \emptyset$ would hold. Similarly, if S is homogeneous in color $(1,1)$ then $X' \times Y' \subset A$ and we are done. Now suppose that S is homogeneous in color $(1,0)$ (for $(0,1)$ the same argument works). If $x_m \in X', y_n \in Y'$ and $x_m < y_n$ then $m < n$ because $Z \cap W = \emptyset$. Hence by the homogeneity of S we can conclude $(x_m, y_n) \in A$, so $T^{\uparrow}(X', Y') \subseteq A$. □

Now we show that $\mathcal{I}_0$ is $\underset{\sim}{\Pi}^1_1$-complete. We will use (see [K, 27.B]) that the set

$$S = \{C \in \mathcal{K}(2^\omega) : \forall\, x \in C\ \forall^\infty\, n \in \omega\ x(n) = 0\}$$

is $\underset{\sim}{\Pi}^1_1$-complete where $\mathcal{K}(2^\omega)$ stands for the family of compact subsets of 2^ω equipped with the Hausdorff metric, i.e. with the Vietoris topology, we know that $\mathcal{K}(2^\omega)$ is a compact Polish space.

To finish the proof, we will define a Borel map $\mathcal{K}(2^\omega) \to \mathcal{P}(\omega \times \omega)$, $C \mapsto A_C$ such that $C \in S$ iff $A_C \in \mathcal{I}_0$. Fix an enumeration $\{s_m : m \in \omega\}$ of $2^{<\omega}$, for every $s \in 2^{<\omega}$ define $[s] = \{x \in 2^\omega : s \subseteq x\}$ (a basic clopen subset of 2^ω), and let

$$A_C = \{(m,n) : |s_m| > n,\ s_m(n) = 1, \text{ and } [s_m] \cap C \neq \emptyset\}.$$

For $C \in S$ we show that $A_C \in \mathcal{I}_0$. Let $X, Y \in [\omega]^\omega$ be arbitrary. If the set $\{m \in X : [s_m] \cap C = \emptyset\}$ is infinite then we are done, since

$$A_C \cap (\{m \in X : [s_m] \cap C = \emptyset\} \times Y) = \emptyset.$$

Otherwise, using the compactness of C we can choose an $\{m_0 < m_1 < \dots\} = X' \in [X]^\omega$ and a convergent sequence $(x_i)_{i \in \omega}$ such that $x_i \in [s_{m_i}] \cap C$ for every i. If $x_i \to x$ then $x \in C \in S$ so $x(n) = 0$ for every $n \geq n_0$ for some n_0. If $n \in Y \setminus n_0$ then for every large enough i we have $n < |s_{m_i}|$ and $s_{m_i}(n) = x(n) = 0$, hence the section $\{m : (m,n) \in (A_C \cap (X' \times Y))\}$ is finite. On the other hand, for a fixed m if $|s_m| \leq n$ then $(m,n) \notin A_C$, therefore the section $\{n : (m,n) \in (A_C \cap (X' \times Y))\}$ is also finite. By an easy induction, one can define an $X'' \in [X']^\omega$ and a $Y'' \in [Y]^\omega$ such that $A_C \cap (X'' \times Y'') = \emptyset$.

Now we show that if $C \notin S$ then $A_C \notin \mathcal{I}_0$. Let $x \in C$ be so that $Y = \{n : x(n) = 1\}$ is infinite and let $X = \{m : x \in [s_m]\}$. Now clearly, if $(m,n) \in X \times Y$ then $(m,n) \in A_C$ if and only if $n < |s_m|$. In particular, for every $n \in Y$ the set

$\{m \in X : (m,n) \notin A_C\}$ is finite, and it clearly implies that the rectangle $X \times Y$ witnesses that $A_C \notin \mathcal{I}_0$. □

Remark 3.5. One can give an alternate proof of Theorem 3.3 constructing a Borel reduction of the set C to $\mathcal{G}_c$.

Theorem 3.6. *There exist $\utilde{\Sigma}^1_n$ and $\utilde{\Pi}^1_n$-complete tall ideals for every $n \geq 1$.*

Proof. First we will construct $\utilde{\Sigma}^1_n$-complete ideals. Let $\mathcal{J}$ be a tall Borel ideal, $\mathcal{A}$ be a perfect $\mathcal{J}$-AD family, and let $\mathcal{A}_n$ be a $\utilde{\Sigma}^1_n$-complete subset of the Polish space $\mathcal{A}$. Define $\mathcal{I}_n = \mathrm{id}(\mathcal{J} \cup \mathcal{A}_n)$, i.e. $\mathcal{I}_n$ is the ideal generated by $\mathcal{J} \cup \mathcal{A}_n$. Then $\mathcal{I}_n$ is a tall proper (because $\mathcal{A}_n$ is infinite) ideal. $\mathcal{I}_n$ is $\utilde{\Sigma}^1_n$ because

$$\mathcal{I}_n = \big\{X \subseteq \omega : \exists\, k \in \omega\ \exists\, (A_i)_{i<k} \in \mathcal{A}_n^k\ X \setminus \big(A_0 \cup A_1 \cup \cdots \cup A_{k-1}\big) \in \mathcal{J}\big\}$$

In order to see that $\mathcal{I}_n$ is $\utilde{\Sigma}^1_n$-complete, we know that if B is a $\utilde{\Sigma}^1_n$ set in a Polish space $\mathcal{X}$, then it can be reduced to $\mathcal{A}_n$ with a continuous map $f : \mathcal{X} \to \mathcal{A} \subseteq \mathcal{P}(\omega)$, furthermore applying the trivial observation that $\mathcal{A}_n = \mathcal{I}_n \cap \mathcal{A}$, we obtain that this map is in fact a reduction of B to $\mathcal{I}_n$ as well.

Now we proceed with $\utilde{\Pi}^1_n$ ideals. Again, there exists a $\utilde{\Pi}^1_n$-complete set $\mathcal{B}_n \subseteq \mathcal{A}$. The previous argument gives that the ideal $\mathcal{I}'_n = \mathrm{id}(\mathcal{J} \cup \mathcal{B}_n)$ is $\utilde{\Pi}^1_n$-hard, so it is enough to prove that $\mathcal{I}'_n$ is $\utilde{\Pi}^1_n$. In order to see this just notice that since $\mathcal{A}$ is an $\mathcal{J}$-AD-family, if $\mathcal{I}_0 = \mathrm{id}(\mathcal{J} \cup \mathcal{A})$ then we have

$$X \in \mathcal{I}_0 \setminus \mathcal{I}'_n \ \text{ iff } \ X \in \mathcal{I}_0 \text{ and } \exists\, A \in \mathcal{A} \setminus \mathcal{B}_n\ A \cap X \in \mathcal{J}^+.$$

This implies, as $\mathcal{I}_0$ is clearly $\utilde{\Sigma}^1_1$, that $\mathcal{I}_0 \setminus \mathcal{I}'_n$ is a $\utilde{\Sigma}^1_n$ set, and hence $\mathcal{I}'_n$ is $\utilde{\Pi}^1_n$ (here we used that $\mathcal{I}'_n \subseteq \mathcal{I}_0$). □

The idea of the above proof can be used to construct $\utilde{\Sigma}^0_\alpha$-complete ideals for $\alpha \geq 3$ as well.

4. Proof of Theorem 1.6

Proof. Applying Corollary 1.8, we can fix perfect $\mathcal{I}$-AD families $\mathcal{A}_X$ on every $X \in \mathcal{I}^+$. The statement "$\mathcal{A}_X$ is an $\mathcal{I}$-AD family" is (at most) $\utilde{\Pi}^1_2$ hence absolute because if $\mathcal{A}_X = [T]$ is coded by the perfect tree $T \in \mathrm{Tree}_2 = \{T \subseteq 2^{<\omega} : T \text{ is a tree}\}$ then "$\mathcal{A}_X$ is an $\mathcal{I}$-AD family"≡

$$\forall\, x, y \in [T]\ \big(x \in \mathcal{I}^+ \text{ and } (x = y \text{ or } x \cap y \in \mathcal{I})\big)$$

where of course we are working on 2^ω and $(x \cap y)(n) = x(n) \cdot y(n)$ for every n.

For every $X, Y \in \mathcal{I}^+$ let $B(X,Y) = \{A \in \mathcal{A}_X : A \cap Y \in \mathcal{I}^+\}$. Then it is a continuous preimage of $\mathcal{I}^+$ (under $\mathcal{A}_X \to \mathcal{P}(\omega)$, $A \mapsto A \cap Y$), hence if $\mathcal{I}$ is analytic then $B(X,Y)$ is coanalytic, and similarly, if $\mathcal{I}$ is coanalytic then $B(X,Y)$ is analytic.

Let $\kappa = |\mathfrak{c}^V|^W$ and fix an enumeration $\{X_\alpha : \alpha < \kappa\}$ of the set $\mathcal{I}^+ \cap V$ in W. Working in W, we will construct the desired $\mathcal{I}$-AD refinement $\{A_\alpha : \alpha < \kappa\}$,

$A_\alpha \subseteq X_\alpha$ by recursion on κ. During this process, we will also define a sequence $(B_\alpha)_{\alpha<\kappa}$ in $\mathcal{I}^+$.

Assume that $\{A_\xi : \xi < \alpha\}$ and $(B_\xi)_{\xi<\alpha}$ are done. Let γ_α be minimal such that $B(X_{\gamma_\alpha}, X_\alpha)$ contains a perfect set. This property, namely, that an analytic or coanalytic set $H \subseteq \mathcal{P}(\omega)$ contains a perfect set, is absolute because if it is analytic then "H contains a perfect subset" iff "H is uncountable" is of the form "$\forall f \in \mathcal{P}(\omega)^\omega\ \exists x\ (x \in H$ and $x \notin \operatorname{ran}(f))$" hence it is $\underset{\sim}{\Pi}^1_2$; and if H is coanalytic then "H contains a perfect set" is of the form "$\exists\ T \in \mathrm{Tree}_2$ (T is perfect and $\forall x \in [T]\ x \in H$)" hence it is $\underset{\sim}{\Sigma}^1_2$. In particular, $\gamma_\alpha \leq \alpha$. We also know that if C is a perfect set coded in V, then in W it contains κ many new elements: We know it holds for 2^ω e.g. because of the group structure on it, and we can compute new elements of C along a homeomorphism between C and 2^ω fixed in V. Let

$$B_\alpha \in B(X_{\gamma_\alpha}, X_\alpha) \setminus \big(V \cup \{B_\xi : \xi < \alpha\}\big) \text{ be arbitrary,}$$

and finally, let $A_\alpha = X_\alpha \cap B_\alpha \in \mathcal{I}^+$. We claim that $\{A_\alpha : \alpha < \kappa\}$ is an $\mathcal{I}$-AD family (it is clearly a refinement of $\mathcal{I}^+ \cap V$). Let $\alpha, \beta < \kappa$, $\alpha \neq \beta$.

If $\gamma_\alpha = \gamma_\beta = \gamma$ then $B_\alpha, B_\beta \in \mathcal{A}_{X_\gamma}$ are distinct, and hence $A_\alpha \cap A_\beta \subseteq B_\alpha \cap B_\beta \in \mathcal{I}$ (actually, we can assume that it is finite).

If $\gamma_\alpha < \gamma_\beta$, then because of the minimality of γ_β, we know that $B(X_{\gamma_\alpha}, X_\beta)$ does not contain perfect subsets. It is enough to see that $B(X_{\gamma_\alpha}, X_\beta)$ is the same set in V and W, i.e. if $\psi(x, r)$ is a $\Sigma^1_1(r)$ or $\Pi^1_1(r)$ definition of this set then $\forall x \in W\ (\psi(x,r) \to x \in V)$. Why? Because then $B_\alpha \notin B(X_{\gamma_\alpha}, X_\beta)$ but $B_\alpha \in \mathcal{A}_{X_{\gamma_\alpha}}$, hence it yields that $A_\alpha \cap A_\beta \subseteq B_\alpha \cap X_\beta \in \mathcal{I}$.

The set $K := B(X_{\gamma_\alpha}, X_\beta)$ is analytic or coanalytic and does not contain perfect subsets (neither in V nor in W). Applying the Mansfield-Solovay theorem, we know that $K \subseteq L[r]$ $(r \in V)$. We also know that $(L[r])^V \cap \mathcal{P}(\omega) = (L[r])^W \cap \mathcal{P}(\omega)$ holds because $\omega_1^W \subseteq V$, hence $K^V = K^W$. □

Remark 4.1. It is natural to ask the following: Assume that $V \subseteq W$ are transitive models, W contains new reals, and let C be a perfect set coded in W. Does C contain at least $|\mathfrak{c}^V|^W$ many new elements in W? In other words: Does $|C^W \setminus V|^W \geq |\mathfrak{c}^V|^W$ hold? Surprisingly, the answer is no! Moreover, it is possible that there is a perfect set of groundmodel reals in the extension, see [VW98].

Remark 4.2. What can we say about possible generalizations of Theorem 1.6, for example, can we weaken the condition on the complexity of the ideal? In general, this statement is false. Let $\varphi(x)$ be a Σ^1_2 definition of a Σ^1_2 (i.e. Δ^1_2) prime P-ideal $\mathcal{I}$ in L. (How to construct such an ideal? Using a Δ^1_2-good well-order $\leq$ on $\mathcal{P}(\omega)$, by the most natural recursion, at every stage extending our family with a $\leq$-minimal element which can be added without generating $\mathcal{P}(\omega)$ and also with a $\leq$-minimal pseudounion of the previous elements, avoiding universal quantification by applying goodness, we obtain such an ideal.) We cannot expect that $\varphi(x)$ defines an ideal in general but we can talk about the *generated* ideal: $x \in \mathcal{J}$ iff "$\exists\ y \in \mathcal{I}\ x \subseteq y$" which is Σ^1_2 too. If r is a Sacks real over L, then $\mathcal{J}$ is still a prime P-ideal in $L[r]$ (see [BrJ, Lemma 7.3.48]) hence $\mathcal{J}^+ \cap L$ does not have any $\mathcal{J}$-ADR's in $L[r]$.

5. On the existence of perfect $(\mathcal{I},\mathrm{Fin})$-AD families

First of all, we show that the reverse implication in the first part of Corollary 1.8 does not hold.

Example 5.1. The assumption that there is a perfect $(\mathcal{I},\mathrm{Fin})$-AD family does not imply that $\mathcal{I}$ is meager: Fix a prime ideal $\mathcal{J}$ on ω. For every partition $P=(P_n)_{n\in\omega}$ of ω into finite sets, fix an $X_P\in[\omega]^\omega$ such that $A_P=\bigcup\{P_n : n\in X_P\}\in\mathcal{J}$ (notice that $\mathcal{J}$ cannot be meager); and let the ideal $\mathcal{I}$ on $2^{<\omega}$ be generated by the sets of the form $A'_P=\bigcup\{2^k : k\in A_P\}$.

Clearly, the family $\{\{f\restriction n : n\in\omega\} : f\in 2^\omega\}$ of branches of $2^{<\omega}$ is a perfect AD family. We show that $\{f\restriction n : n\in\omega\}\in\mathcal{I}^+$. Notice that $\{\mathrm{dom}(s) : s\in A'_P\}=A_P\in\mathcal{J}$ for every P. Thus, a set of the form $B_f=\{f\restriction n : n\in\omega\}$ cannot be an element of the ideal because $\{\mathrm{dom}(s) : s\in B_f\}=\omega$.

$\mathcal{I}$ is not meager: Assume the contrary, then by Theorem 1.7 there exists a partition $Q=(Q_n)_{n\in\omega}$ of $2^{<\omega}$ into finite sets such that $\{n\in\omega : Q_n\subseteq A\}$ is finite for every $A\in\mathcal{I}$. Then there is a partition $P=(P_n)_{n\in\omega}$ of ω into finite sets such that for every n there is an m with $Q_m\subseteq\bigcup\{2^k : k\in P_n\}$. We know that $A'_P\in\mathcal{I}$, a contradiction because A'_P contains infinitely many Q_m's.

What can we say if there are perfect $(\mathcal{I},\mathrm{Fin})$-AD families on every $X\in\mathcal{I}^+$? In this case we have only consistent counterexamples.

Theorem 5.2. *Assume that $\mathfrak{b}=\mathfrak{c}$. Then there is a non-meager ideal $\mathcal{I}$ on ω such that there are perfect $(\mathcal{I},\mathrm{Fin})$-AD families on every $X\in\mathcal{I}^+$.*

Proof. Let $[\omega]^\omega=\{X_\alpha : \alpha<\mathfrak{c}\}$ and $\{\text{partitions of }\omega\text{ into finite sets}\}=\{P_\alpha=(P^\alpha_n)_{n\in\omega} : \alpha<\mathfrak{c}\}$ be enumerations. We will construct the desired ideal $\mathcal{I}$ as an increasing union $\bigcup\{\mathcal{I}_\alpha : \alpha<\mathfrak{c}\}$ of ideals by recursion on $\alpha<\mathfrak{c}$. At the αth stage we will make sure that

(i) $\mathcal{I}_\alpha$ is generated by $|\alpha|$ many elements;
(ii) P_α cannot witness that $\mathcal{I}_\alpha$ is meager;
(iii) either X_α belongs to $\mathcal{I}_\alpha$ or there is a perfect $(\mathcal{I}_\alpha,\mathrm{Fin})$-AD family on X_α;
(iv) we do not destroy the $(\mathcal{I}_\beta,\mathrm{Fin})$-AD families we may have constructed in previous stages.

Let $\mathcal{I}_0=\mathrm{Fin}$ and fix a perfect AD family $\mathcal{A}_0$ on X_0. At stage $\alpha>0$ we already have the ideals $\mathcal{I}_\beta$ for every $\beta<\alpha$, let $\mathcal{I}_{<\alpha}=\bigcup\{\mathcal{I}_\beta : \beta<\alpha\}$. We also have perfect $(\mathcal{I}_{<\alpha},\mathrm{Fin})$-AD families $\mathcal{A}_\beta$ on $X_\beta\in\mathcal{I}^+_{<\alpha}$ for certain $\beta\in D_\alpha\subseteq\alpha$.

If we can add X_α to $\mathcal{I}_{<\alpha}$, that is, $\mathcal{A}_\beta\cap\mathrm{id}(\mathcal{I}_{<\alpha}\cup\{X_\alpha\})=\emptyset$ for every $\beta\in D_\alpha$, then let $\mathcal{I}'_\alpha=\mathrm{id}(\mathcal{I}_{<\alpha}\cup\{X_\alpha\})$ and $D'_\alpha=D_\alpha$.

Suppose that we cannot add X_α to $\mathcal{I}_{<\alpha}$, that is, $\mathcal{A}_\beta\cap\mathrm{id}(\mathcal{I}_{<\alpha}\cup\{X_\alpha\})\neq\emptyset$ for some $\beta\in D_\alpha$. Since $\mathcal{I}_{<\alpha}$ is generated by $<\mathfrak{b}=\mathfrak{c}$ many sets, it is an everywhere meager ideal (see [So77] or [Bl10, Thm. 9.10]). We can apply Corollary 1.8 to obtain a perfect $(\mathcal{I}_{<\alpha},\mathrm{Fin})$-AD family $\mathcal{A}_\alpha$ on X_α, let $\mathcal{I}'_\alpha=\mathcal{I}_{<\alpha}$, and let $D'_\alpha=D_\alpha\cup\{\alpha\}$.

Fix a partition $Q=(Q_n)_{n\in\omega}$ of ω into finite sets such that $\{n\in\omega : Q_n\subseteq A\}$ is finite for every $A\in\mathcal{I}'_\alpha$ (we know that $\mathcal{I}'_\alpha$ is meager).

Claim. *There exist partitions* $Q_{\beta,B} = (Q_n^{\beta,B})_{n\in\omega}$ *for every* $\beta \in D'_\alpha$ *and* $B \in \mathcal{I}'_\alpha$ *such that* $A \cap Q_n^{\beta,B} \setminus B \neq \emptyset$ *for every* $\beta \in D'_\alpha$, $A \in \mathcal{A}_\beta$, $B \in \mathcal{I}'_\alpha$, *and* $n \in \omega$.

Proof of the Claim. Let $\beta \in D'_\alpha$ and $B \in \mathcal{I}'_\alpha$. We know that $\mathcal{A}_\beta$ is compact as a subset of $\mathcal{P}(\omega)$. Basic open sets in $\mathcal{P}(\omega)$ are of the form $[s,t] = \{A \subseteq \omega : s \cap A = \emptyset \text{ and } t \subseteq A\}$ for disjoint, finite $s,t \subseteq \omega$. Then $\mathcal{A}_\beta \subseteq \bigcup\{[\emptyset,\{n\}] : n \in \omega \setminus B\}$ because $A \setminus B$ is infinite for every $A \in \mathcal{A}_\beta$. Therefore $\mathcal{A}_\beta \subseteq \bigcup\{[\emptyset,\{n\}] : n \in N_0 \setminus B\}$ for an $N_0 \in \omega$, in particular, $A \cap N_0 \setminus B \neq \emptyset$ for every $A \in \mathcal{A}_\beta$. Let $Q_0^{\beta,B} = [0,N_0)$. We can proceed by the same argument: $\mathcal{A}_\beta \subseteq \bigcup\{[\emptyset,\{n\}] : n \in [N_0,\omega)\setminus B\}$ hence there is an $N_1 > N_0$ such that $\mathcal{A}_\beta \subseteq \bigcup\{[\emptyset,\{n\}] : n \in [N_0,N_1) \setminus B\}$, in other words, $A \cap [N_0,N_1) \setminus B \neq \emptyset$ for every $A \in \mathcal{A}_\beta$. Let $Q_1^{\beta,B} = [N_0,N_1)$ etc. □

Now we have the family $\mathcal{Q} = \{P_\alpha\} \cup \{Q\} \cup \{Q_{\beta,B} : \beta \in D'_\alpha, B \in \mathcal{C}_\alpha\}$ of partitions where $\mathcal{C}_\alpha \subseteq \mathcal{I}'_\alpha$ is a cofinal family, $|\mathcal{C}_\alpha| \leq \max\{|\alpha|,\omega\}$. $|\mathcal{Q}| < \mathfrak{c} = \mathfrak{b}$ hence there is a partition $R = (R_m)_{m\in\omega}$ which dominates all of these partitions, that is, $\forall\ P = (P_n)_{n\in\omega} \in \mathcal{Q}\ \forall^\infty\ m\ \exists\ n\ P_n \subseteq R_m$ (see [Bl10, Thm. 2.10]). Let $Y = \bigcup\{R_{2n} : n \in \omega\}$ and $\mathcal{I}_\alpha = \mathrm{id}(\mathcal{I}'_\alpha \cup \{Y\})$.

Then (i) is clearly satisfied, in order to see (ii) notice that by the fact that the partition R_m was dominating and $P_\alpha \in \mathcal{Q}$, for almost every m there exists an n with $P_n^\alpha \subset R_{2m}$. Condition (iii) is also clear if $X_\alpha \in \mathcal{I}'_\alpha$.

If $X_\alpha \notin \mathcal{I}'_\alpha$ then by definition $\alpha \in D'_\alpha$ so to see (iii) and (iv) we have to show that for every $\beta \in D'_\alpha$ the family $\mathcal{A}_\beta$ is not just an $(\mathcal{I}'_\alpha,\mathrm{Fin})$-AD family, but also an $(\mathcal{I}_\alpha,\mathrm{Fin})$-AD family. In other words, it is enough to check that for every $A \in \mathcal{A}_\beta$ and $B \in \mathcal{I}'_\alpha$ we have $A \setminus (B \cup Y) \neq \emptyset$. Fix such A and B, we can assume that $B \in \mathcal{C}_\alpha$. Then for almost every m, there is an n_m such that $Q_{n_m}^{\beta,B} \subseteq R_{2m+1}$, and by the claim we know that $A \cap Q_{n_m}^{\beta,B} \setminus B \neq \emptyset$. Therefore, $A \setminus (B \cup Y)$ is infinite, hence $\mathcal{A}_\beta \cap \mathcal{I}_\alpha = \emptyset$ for every $\beta \in D'_\alpha$. □

What can we say about ideals on the second level of the projective hierarchy, do there always exist perfect or at least uncountable $(\mathcal{I},\mathrm{Fin})$-AD families? If all $\utilde{\Sigma}^1_2$ and $\utilde{\Pi}^1_2$ sets have the Baire property, then of course, yes because then $\utilde{\Sigma}^1_2$ and $\utilde{\Pi}^1_2$ ideals are meager and we can apply Corollary 1.8. On the other hand, if $\mathcal{I}$ is a Σ^1_2 (i.e. Δ^1_2) prime ideal (e.g. in L) then every $\mathcal{I}$-AD family is a singleton.

Similarly, we can construct a Σ^1_2-ideal $\mathcal{J}$ in L such that there are infinite $\mathcal{J}$-AD families but all of them are countable: Copy the above ideal $\mathcal{I}$ to the elements of a partition $\{P_n : n \in \omega\} \subseteq [\omega]^\omega$ of ω, and let $\mathcal{J}$ be the generated ideal.

This last example is very artificial in the sense that, this ideal is constructed from maximal ideals in a very "obvious" way, many of its restrictions are prime ideals. However, we can construct even more peculiar ideals:

Proposition 5.3. *Suppose that there exists a* $\utilde{\Delta}^1_n$ *prime ideal on* ω *for some* n. *Then there exists a* $\utilde{\Delta}^1_n$ *ideal* $\mathcal{I}$ *such that* $\mathcal{I}$ *is nowhere maximal but every* $\mathcal{I}$*-AD family is countable. In particular, there exists such a* $\utilde{\Delta}^1_2$ *ideal in* L.

Proof. Let $\mathcal{U}$ be an ultrafilter and define $\mu : \mathcal{P}(\omega) \to [0,1]$ as $\mu(A) = \lim_{\mathcal{U}} \frac{|A \cap n|}{n}$ where $\lim_{\mathcal{U}}$ stands for the $\mathcal{U}$-*limit* operation on sequences in topological spaces, that is, $\lim_{\mathcal{U}}(a_n) = a$ iff $\{n \in \omega : a_n \in V\} \in \mathcal{U}$ for every neighbourhood V of a. It is easy to see that if $\overline{\{a_n : n \in \omega\}}$ is compact, then $\lim_{\mathcal{U}}(a_n)_{n \in \omega}$ exists, in particular, μ is defined on every $A \in \mathcal{P}(\omega)$. It is also straightforward to show that μ is a finitely additive non-atomic probability measure on $\mathcal{P}(\omega)$, that is, $\mu(\emptyset) = 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$, $\mu(\omega) = 1$, and if $\mu(X) = \varepsilon > 0$ then for every $\delta \in (0, \varepsilon)$ there is a $Y_\delta \subseteq X$ with $\mu(Y_\delta) = \delta$.

Let $\mathcal{I} = \{A \subseteq \omega : \mu(A) = 0\}$. Then $\mathcal{I}$ is an ideal. $\mathcal{I}$ is nowhere maximal because of μ is non-atomic (in particular, there are infinite $\mathcal{I}$-AD families). We show that every $\mathcal{I}$-AD family is countable. If there was an uncountable $\mathcal{I}$-AD family $\mathcal{A}$, then $\mathcal{A}_n = \{A \in \mathcal{A} : \mu(A) > 1/n\}$ would be uncountable for some $n \in \omega$ and therefore among every n many element of $\mathcal{A}_n$ there would be two with $\mathcal{I}$-positive intersection.

Notice that if $\mathcal{U}$ is $\underset{\sim}{\Delta}^1_n$ ($n \geq 2$) then $\mathcal{I}$ is also $\underset{\sim}{\Delta}^1_n$ because $A \in \mathcal{I}$ iff $\forall\ k \in \omega$ $\{n \in \omega : |A \cap n|/n < 2^{-k}\} \in \mathcal{U}$, and the function $A \mapsto \{n \in \omega : |A \cap n|/n < 2^{-k}\}$ is continuous (for every k). □

6. On $(\mathcal{I}, \mathrm{Fin})$-ADR's

In this section, we study Question 1.11.

Theorem 6.1. *Assume* MA_κ *and let* $\mathcal{I}$ *be an everywhere meager ideal, then every* $\mathcal{H} \in [\mathcal{I}^+]^{\leq \kappa}$ *has an* $(\mathcal{I}, \mathrm{Fin})$*-ADR.*

Proof. Let $\mathcal{H} = \{H_\alpha : \alpha < \kappa\}$ be an enumeration. Define $p \in \mathbb{P} = \mathbb{P}(\mathcal{H})$ iff p is a function, $\mathrm{dom}(p) \in [\kappa]^{<\omega}$, and $p(\alpha) \in [H_\alpha]^{<\omega}$ for every $\alpha \in \mathrm{dom}(p)$; $p \leq q$ iff $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$, $\forall\ \alpha \in \mathrm{dom}(q)$ $p(\alpha) \supseteq q(\alpha)$, and $\forall\ \{\alpha, \beta\} \in [\mathrm{dom}(q)]^2$ $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$.

Then $\mathbb{P}$ is a poset. First of all, we show that $\mathbb{P}$ has the ccc. Let $\{p_\xi : \xi < \omega_1\} \subseteq \mathbb{P}$. Then $\{\mathrm{dom}(p_\xi) : \xi < \omega_1\} \subseteq [\kappa]^{<\omega}$. We can assume that this family forms a Δ-system, $\mathrm{dom}(p_\xi) = D_\xi \cup R$. There are at most ω many functions $R \to \mathrm{Fin}$, hence we can also assume that there is a $q \in \mathbb{P}$ such that $p_\xi \restriction R = q$ for every $\xi < \omega_1$. Clearly, $p_\xi \cup p_\zeta \in \mathbb{P}$ and $p_\xi \cup p_\zeta \leq p_\xi$ for every $\xi, \zeta < \omega_1$.

It is easy to see that for every $\alpha < \kappa$ the set $D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}$. If G is a $\{D_\alpha : \alpha < \kappa\}$-generic filter, then let $F_G : \kappa \to \mathcal{P}(\omega)$, $F_G(\alpha) = \bigcup\{p(\alpha) : p \in G\}$. Clearly, $F_G(\alpha) \subseteq H_\alpha$ for every α.

We show that $F_G(\alpha) \cap F_G(\beta)$ is finite for every distinct $\alpha, \beta < \kappa$. Let $p \in D_\alpha \cap G$, $q \in D_\beta \cap G$, and $r \in G$ be a common lower bound of them. It is easy to see that $F_G(\alpha) \cap F_G(\beta) = r(\alpha) \cap r(\beta)$.

If somehow we can make sure that $F_G(\alpha) \in \mathcal{I}^+$, then we are done because $\{F_G(\alpha) : \alpha < \kappa\}$ will be an $(\mathcal{I}, \mathrm{Fin})$-ADR of $\mathcal{H}$. We show that if G is $(V, \mathbb{P})$-generic then $F_G(\alpha)$ is a Cohen-real in $\mathcal{P}(H_\alpha)$ over V. It is enough because then $F_G(\alpha) \notin \mathcal{I} \restriction H_\alpha$ (we know that $\mathcal{I} \restriction H_\alpha$ is meager) and to show that $V[F_G(\alpha)] \models F_G(\alpha) \notin \mathcal{I} \restriction H_\alpha$, it is enough to use countable many dense sets. Why? For every

α we can fix a countable family $\mathcal{C}_\alpha = \{C^\alpha_n : n \in \omega\}$ of closed nowhere dense subsets of $\mathcal{P}(H_\alpha)$ which covers $\mathcal{I} \restriction H_\alpha$, and hence have countable many dense subsets of the Cohen forcing such that if a filter is generic for this family then the generic real is not covered by any element of $\mathcal{C}_\alpha$. More precisely, we have to translate these dense subsets of the Cohen forcing to dense subsets in $\mathbb{P}$, it can be done by applying the (inverse of the) projection $\mathbb{P} \to \mathbb{C}(H_\alpha)$ defined below.

Fix an $\alpha < \kappa$, let $\mathbb{C}(H_\alpha) = \{s : s$ is a finite partial function form H_α to $2\}$ where $s \le t$ iff $s \supseteq t$ (then $\mathbb{C}(H_\alpha)$ adds a Cohen subset of H_α over V), and define the map $e = e_\alpha : \mathbb{P} \to \mathbb{C}(H_\alpha)$ as follows:

(i) $\mathrm{dom}(e(p)) = \bigcup\{p(\beta) \cap H_\alpha : \beta \in \mathrm{dom}(p)\}$;
(ii) $e(p)(n) = 1$ iff $n \in p(\alpha)$.

We show that e is a *projection* (see e.g. [A11, page 335]) , that is,

(1) e is order-preserving, onto, and $e(\emptyset) = \emptyset$;

(2) $\forall\, p \in \mathbb{P}\ \forall\, s \in \mathbb{C}(H_\alpha)\ \big(s \le e(p) \to \exists\, p' \le p\ e(p') = s\big)$.

Clearly, $e(\emptyset) = \emptyset$. Assume that $p \le q$. Then clearly $\mathrm{dom}(e(p)) \supseteq \mathrm{dom}(e(q))$. If $n \in \mathrm{dom}(e(q))$ and $n \in q(\alpha) \subseteq p(\alpha)$ then $e(q)(n) = e(p)(n) = 1$; if $n \in \mathrm{dom}(e(p))$ and $n \in q(\beta) \setminus q(\alpha)$ for some $\beta \ne \alpha$ then, as $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$, $n \in p(\beta) \setminus p(\alpha)$ and hence $e(q)(n) = e(p)(n) = 0$. This yields that e is indeed order preserving.

To show that e is onto, we have to assume that $H_\alpha \subseteq \bigcup\{H_\beta : \beta \ne \alpha\}$ (and w.l.o.g. we can do so by extending $\mathcal{H}$ to be a cover of ω and adding ω as an element to $\mathcal{H}$). For an $s \in \mathbb{C}(H_\alpha)$ define $p \in \mathbb{P}$ as follows: Fix a finite $D \subseteq \kappa$ containing α such that $\mathrm{dom}(s) \subseteq \bigcup\{H_\beta : \beta \in D\}$, let $\mathrm{dom}(p) = D$, and define $p(\alpha) = s^{-1}(1)$ and $p(\beta) = \{n \in H_\beta \cap H_\alpha : s(n) = 0\}$. Then $e(p) = s$.

To show that e satisfies (2), fix a $p \in \mathbb{P}$, an $s \in \mathbb{C}(H_\alpha)$, and assume that $s \le e(p)$. Define $p' \in \mathbb{P}$ as follows: For every $n \in J = (s \setminus e(p))^{-1}(0)$ pick a $\gamma_n \in \kappa \setminus \{\alpha\}$ such that $n \in H_{\gamma_n}$. Let $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\gamma_n : n \in J\}$ and define $p'(\alpha) = p(\alpha) \cup s^{-1}(1)$, if $\beta \in \mathrm{dom}(p') \setminus \{\alpha\}$ then $p'(\beta) = p(\beta) \cup \{n \in J : \beta = \gamma_n\}$. It is straightforward to see that $p' \in \mathbb{P}$, $p' \le p$, and $e(p') = s$.

We know that if G is $(V, \mathbb{P})$-generic then $e[G]$ generates a $(V, \mathbb{C})$-generic filter G'. Notice that the Cohen real defined from G' is $F_G(\alpha)$, so we are done. □

Unfortunately, at this moment, we do not know whether we really needed Martin's Axiom in the previous theorem or it holds in ZFC. We show that if we attempt to construct a counterexample, that is, say a tall Borel ideal $\mathcal{I}$ and a family $\mathcal{H} \in [\mathcal{I}^+]^{<\mathfrak{c}}$ without a $(\mathcal{I}, \mathrm{Fin})$-ADR, we have to be careful. Let us define the following cardinal invariants of tall ideals on ω: The *star-additivity* of $\mathcal{I}$ is

$$\mathrm{add}^*(\mathcal{I}) = \min\big\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{I} \text{ and } \nexists\, A \in \mathcal{I}\ \forall\, X \in \mathcal{X}\ X \subseteq^* A\big\},$$

the *Fodor number* of $\mathcal{I}$ is

$$F(\mathcal{I}) = \min\big\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{I}^+ \text{ has no } \mathcal{I}\text{-ADR}\big\},$$

and the *star-Fodor number* of $\mathcal{I}$ is

$$F^*(\mathcal{I}) = \min\big\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{I}^+ \text{ has no } (\mathcal{I}, \mathrm{Fin})\text{-ADR}\big\}.$$

Clearly, $\mathcal{J}$ is a P-ideal iff $\mathrm{add}^*(\mathcal{J}) > \omega$. Proposition 1.10 says that $F(\mathcal{J}) = \mathfrak{c}$ whenever $\mathcal{J}$ is everywhere meager; and clearly, $F^*(\mathcal{J}) \leq F(\mathcal{J})$.

Fact 6.2. *If* $\mathrm{add}^*(\mathcal{J}) < F(\mathcal{J})$ *then* $\mathrm{add}^*(\mathcal{J}) < F^*(\mathcal{J})$. *If* $\mathrm{add}^*(\mathcal{J}) = F(\mathcal{J})$ *then* $F(\mathcal{J}) = F^*(\mathcal{J})$.

Proof. Assume that $\mathcal{H} = \{H_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}^+$ where $\kappa = \mathrm{add}^*(\mathcal{J}) < F(\mathcal{J})$. First fix an $\mathcal{J}$-ADR $\{A_\alpha : \alpha < \kappa\}$ of $\mathcal{H}$ ($A_\alpha \subseteq H_\alpha$). Then for every $\alpha < \kappa$ fix a $B_\alpha \in \mathcal{J}$ such that $A_\alpha \cap A_\beta \subseteq^* B_\alpha$ for every $\beta < \alpha$, and let $A'_\alpha = A_\alpha \setminus B_\alpha$. Then $\{A'_\alpha : \alpha < \kappa\}$ is an $(\mathcal{J}, \mathrm{Fin})$-ADR of $\mathcal{H}$. The second statement can be proved by the same argument. □

In particular, if $\mathcal{J}$ is an everywhere meager P-ideal and $F^*(\mathcal{J}) < \mathfrak{c}$, then $F^*(\mathcal{J}) < F(\mathcal{J})$ hence $\mathrm{add}^*(\mathcal{J}) < F(\mathcal{J})$ and so $\omega_1 \leq \mathrm{add}^*(\mathcal{J}) < F^*(\mathcal{J}) < \mathfrak{c}$, therefore $\mathfrak{c} \geq \omega_3$.

7. Mixing reals

In this section, we study two closely related properties of forcing notions, one of which is slightly stronger then "$[\omega]^\omega \cap V$ has an ADR in $V^{\mathbb{P}}$".

Definition 7.1. Let $\mathbb{P}$ be a forcing notion. We say that an $f \in \omega^\omega \cap V^{\mathbb{P}}$ is a *mixing real* over V if $|f[X] \cap Y| = \omega$ for every $X, Y \in [\omega]^\omega \cap V$. If f is one-to-one, then we call it an *injective mixing real* or *mixing injection*.

Clearly, in the definition above, it is enough to require that $f[X] \cap Y \neq \emptyset$ for every $X, Y \in [\omega]^\omega \cap V$.

Proposition 7.2. *Let* $\mathbb{P}$ *be a forcing notion. Then the following are equivalent:*

(i) *There is a mixing real* $f \in \omega^\omega \cap V^{\mathbb{P}}$ *over* V.
(ii) *There is an* $f \in \omega^\omega \cap V^{\mathbb{P}}$ *such that* $f[X] = \omega$ *for all* $X \in [\omega]^\omega \cap V$.
(iii) *There is a partition* $(Y_n)_{n\in\omega}$ *of* ω *into infinite sets in* $V^{\mathbb{P}}$ *such that* $\forall X \in [\omega]^\omega \cap V\ \forall n\ |X \cap Y_n| = \omega$.
(iii)' *There is a partition* $(Y_n)_{n\in\omega}$ *of* ω *into infinite sets in* $V^{\mathbb{P}}$ *such that* $\forall X \in [\omega]^\omega \cap V\ \forall n\ X \cap Y_n \neq \emptyset$.

Proof. (ii)→(i) and (iii)⟷(iii)' are trivial. (ii)⟷(iii)' because let $Y_n = f^{-1}(n)$ (and vice versa). Finally, (i) implies (ii): Fix a partition $(C_n)_{n\in\omega}$ of ω into infinite sets in V and let $g : \omega \to \omega$, $g \restriction C_n \equiv n$. If f is a mixing real over V, then $h = g \circ f$ has the required property. □

(iii) says that mixing reals can be seen as "infinite splitting paritions". Recall that a set $S \subseteq \omega$ is a *splitting real* over V if $|X \cap S| = |X \setminus S| = \omega$ for every $X \in [\omega]^\omega \cap V$, in other words, $P = \{S, \omega \setminus S\}$ is a partition of ω such that $\forall X \in [\omega]^\omega \cap V\ \forall Y \in P\ |X \cap Y| = \omega$.

Why is this property relevant to almost-disjoint refinements? Fix an AD family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ in V, and let $\{X_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$ in V. If $f \in \omega^\omega \cap V^{\mathbb{P}}$ is a mixing injection over V, then the family $\{f[A_\alpha] \cap X_\alpha : \alpha < \mathfrak{c}\} \in V^{\mathbb{P}}$ is an ADR of $[\omega]^\omega \cap V$.

Proposition 7.3. *Let* $\mathbb{P}$ *be a forcing notion.*

(i) *If $\mathbb{P}$ adds random reals then it adds mixing reals.*
(ii) *If $\mathbb{P}$ adds dominating reals, then it adds mixing reals.*
(iii) *If $\mathbb{P}$ adds Cohen reals then it adds mixing injections.*
(iv) *If $\mathbb{P}$ adds mixing injections then it adds unbounded reals.*
(v) *If $\mathbb{P}$ has the Laver-property, then it does not add injective mixing reals.*

Proof. (i): Let λ be the usual probability measure on ω^ω, that is, λ is uniquely determined by the values $\lambda([s]) = 2^{-s(0)-s(1)-\cdots-s(n-1)-n}$ where $s: n \to \omega$ and $[s] = \{f \in \omega^\omega : s \subseteq f\}$. If $\mathcal{N}_\lambda = \{A \subseteq \omega^\omega : \lambda(A) = 0\}$, then it is well-know that $\mathrm{Borel}(\omega^\omega)/\mathcal{N}_\lambda$ is forcing equivalent to the random forcing. It is enough to see that the set $A_{X,Y} = \{f \in \omega^\omega : |f[X] \cap Y| < \omega\}$ is a null set in ω^ω for every $X, Y \in [\omega]^\omega$: $A_{X,Y} = \bigcup_{n\in\omega}\{f \in \omega^\omega : f[X] \cap Y \subseteq n\}$ and if $X = \{x_k : k \in \omega\}$ and $n \in \omega$ then $\{f : f[X] \cap Y \subseteq n\} = \{f : \forall\, k\ f(x_k) \in n \cup (\omega \setminus Y)\}$. Clearly, $\sum\{2^{-m-1} : m \in n \cup (\omega \setminus Y)\} = \varepsilon < 1$ and hence $\lambda(\{f : f[X] \cap Y \subseteq n\}) \leq \lim_{k\to\infty} \varepsilon^k = 0$.

(ii): Trivial modification of the proof of the fact (see e.g. [Hb, Fact 20.1]) that adding a dominating real implies adding a splitting real works here as well: Adding a dominating real is equivalant to adding a dominating partition $(P_n)_{n\in\omega}$ of ω into finite sets (see [Bl10, Thm. 2.10]), that is, for every partition $(Q_m)_{m\in\omega} \in V$ of ω into finite sets, $\forall^\infty\, n\ \exists\, m\ Q_m \subseteq P_n$. Now any infinite partition of ω containing of unions of infinitely many P_n's satisfy (iii) from Proposition 7.2.

(iii): We can talk about *injective Cohen-reals*. Simply consider the forcing notion $(\mathrm{Inj}, \supseteq)$ where $\mathrm{Inj} = \{s \in \omega^{<\omega} : s \text{ is one-to-one}\}$, or the forcing notion $(\mathrm{Borel}(\mathrm{INJ}) \setminus \mathcal{M}(\mathrm{INJ}), \subseteq)$ where $\mathrm{INJ} = \{f \in \omega^\omega : f \text{ is one-to-one}\}$ is a nowhere dense closed subset on ω^ω and $\mathcal{M}(\mathrm{INJ})$ is the meager ideal on this Polish space. It is not difficult to see that these forcing notions are forcing equivalent to the Cohen forcing (moreover, INJ is homemomorphic to ω^ω).

If c is an injective Cohen-real over V, then c is mixing: For every $X, Y \in [\omega]^\omega$, the set $A'_{X,Y} = A_{X,Y} \cap \mathrm{INJ} = \bigcup_{n\in\omega} \{f \in \mathrm{INJ} : f[X] \cap Y \subseteq n\}$ is meager because $\{f \in \mathrm{INJ} : f[X] \cap Y \subseteq n\}$ is closed and nowhere dense in INJ.

(iv): Let $f \in \mathrm{INJ} \cap V^{\mathbb{P}}$ be a mixing injection and assume on the contrary that there is a strictly increasing $g \in \omega^\omega \cap V$ such that $f, f^{-1} < g$ (where of course $f^{-1} < g$ means that $f^{-1}(k) < g(k)$ for every $k \in \mathrm{ran}(f)$).

We define $X = \{x_k : k \in \omega\}, Y = \{y_k : k \in \omega\} \in [\omega]^\omega$ in V as follows: $x_0 = 0$, $y_0 = g(0)$, $x_n = \max\{g(y_k) : k < n\}$, and $y_n = g(x_n)$. Suppose that $f(x_k) = y_l$ for some $k, l \in \omega$. If $k \leq l$ then

$$f(x_k) < g(x_k) = g\big(\max_{m<k} g(y_m)\big) \leq g\big(\max_{m<l} g(y_m)\big) = x_l < g(x_l) = y_l,$$

a contradiction. Now, if $k > l$ then

$$x_k = f^{-1}(y_l) < g(y_l) \leq \max\{g(y_m) : m < k\} = x_k$$

which is again impossible. Thus, $f[X] \cap Y = \emptyset$, so f cannot be a mixing injection.

(v): Fix a sequence $(a_n)_{n\in\omega} \in \omega^\omega \cap V$ satisfying $a_{n+1} - a_n > (n+2)2^{n+1}$ and $a_0 > 1$. Assume that $p \Vdash \dot{f} \in \mathrm{INJ}$. Let $\dot{g}$ be a $\mathbb{P}$-name for a function on ω such that $p \Vdash \dot{g}(n) = \dot{f} \cap (a_n \times a_n) = \{(k,l) \in a_n \times a_n : f(k) = l\}$ for every n (in particular, $p \Vdash$"$\dot{g}(n)$ is an injective partial function from a_n to a_n"). Then $p \Vdash \dot{g} \in \prod_{n\in\omega} \mathcal{P}(a_n \times a_n)$ hence, applying the Laver property of our forcing notion to the name $\dot{g}$ for a function from ω to $[\omega \times \omega]^{<\omega}$, there is a $q \le p$ and a "slalom" $S : \omega \to \big[[\omega \times \omega]^{<\omega}\big]^{<\omega}$ in V which catches $\dot{g}$, that is, $S(n) \subseteq \mathcal{P}(a_n \times a_n)$, $|S(n)| \le 2^n$, and $q \Vdash \dot{g}(n) \in S(n)$ for every n. Without loss of generality we can assume that all elements of $S(n)$ are injective partial functions $a_n \to a_n$.

Working in V, we will define the sets $X = \{x_n : n \in \omega\}, Y = \{y_n : n \in \omega\} \in [\omega]^\omega$ by recursion on n such that $q \Vdash \dot{f}[X] \cap Y = \emptyset$.

Let $x_0 \in a_0$ be arbitrary. We know that there is a $y_0 \in a_0$ such that $(x_0, y_0) \notin \bigcup S(0)$ (a function cannot cover $\{(x_0, k) : k < a_0\}$).

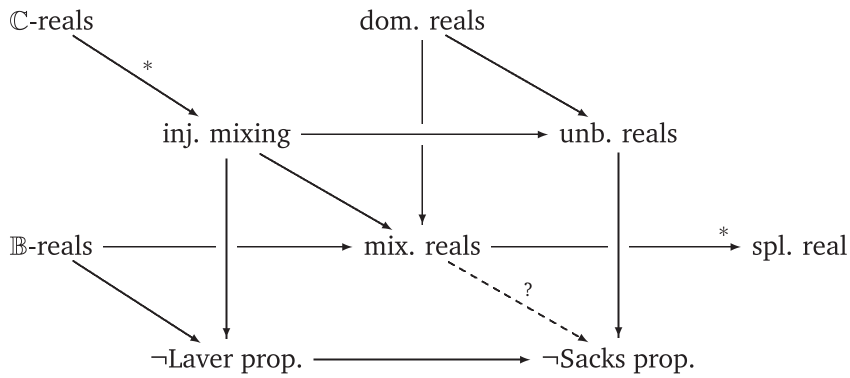
Assume that we already have $X_n = \{x_k : k \le n\}$ and $Y_n = \{y_k : k \le n\}$ such that $(X_n \times Y_n) \cap \bigcup_{k\le n} \bigcup S(k) = \emptyset$. There is an $x_{n+1} \in a_{n+1} \setminus a_n$ such that

$$\big\{s(x_{n+1}) : s \in S(n+1), x_{n+1} \in \mathrm{dom}(s)\big\} \cap Y_n = \emptyset.$$

Why? If for every $m \in a_{n+1} \setminus a_n$ there is an $s_m \in S(n+1)$ such that $s_m(m) \in Y_n$ then there is a set $H \in [a_{n+1} \setminus a_n]^{n+2}$ such that $s_m = s$ does not depend on $m \in H$ (because $|a_{n+1} \setminus a_n| > (n+2)2^{n+1}$ and $|S(n+1)| \le 2^{n+1}$). But it would mean that $H \subseteq \mathrm{dom}(s)$ and $|s[H]| \le |Y_n| = n+1$ which is a contradiction because s is injective.

We also want to fix a $y_{n+1} \in a_{n+1} \setminus a_n$ such that $y_{n+1} \ne s(x_k)$ for any $k \le n+1$, $s \in S(n+1)$ if $x_k \in \mathrm{dom}(s)$. The set of forbidden values is of size at most $2^{n+1}(n+2)$ hence there is such a y_{n+1}. □

In the diagram below, we summarize logical implications between classical properties of forcing notions and the ones we defined above. We will show that arrows without an $*$ above them are strict (i.e. not equivalences), and that there are no other implications between these properties. The arrow $\dashrightarrow$ with question mark means that we do not know whether this implication holds (but the reverse implication is false). Of course, $\mathbb{C}$ stands for the Cohen forcing, $\mathbb{B}$ is the random forcing, and to keep the diagram small, we did not put "$\mathbb{P}$ adds ..." and "$\mathbb{P}$ has the ..." before the properties we deal with.



The non-trivial non-implications in the diagram are the following:

- ¬Laver prop. $\not\to$ splitting reals: The infinitely equal forcing $\mathbb{EE}$ is ω^ω-bounding, preserves P-points (hence cannot add splitting reals), and $\Vdash_{\mathbb{E}}$"$2^\omega \cap V$ is a null set" (see [BrJ, Lemma 7.4.13-15]). $\mathbb{EE}$ cannot have the Laver property because otherwise it would have the Sacks property as well but then it could not force $2^\omega \cap V$ to be of measure zero (it follows from e.g. [BrJ, Thm. 2.3.12]).
- unbounded reals $\not\to$ splitting reals: The Miller forcing (see [BrJ, 7.3.E]).
- spl. reals $\not\to$ ¬Sacks prop.: The Silver forcing adds splitting reals (see [Hb, Lemma 2.3]) and it is straightforward to show that it satisfies the Sacks property.

We list the remaining questions in the next section.

8. Related questions

We already presented $\mathbf{\Sigma}^1_n$- and $\mathbf{\Pi}^1_n$-complete ideals but our construction was pretty artificial.

Question 8.1. Can we define "natural" $\mathbf{\Sigma}^1_n$- and $\mathbf{\Pi}^1_n$-complete ideals?

Question 8.2. Assume that V, W and $\mathcal{I}$ are as in Theorem 1.6. Does there exist an $(\mathcal{I}, \mathrm{Fin})$-ADR of $\mathcal{I}^+ \cap V$ in W? Or at least an $\mathcal{I}$-ADR $\{A_X : X \in \mathcal{I}^+ \cap V\} \in W$ such that for every distinct $X, Y \in \mathcal{I}^+ \cap V$ (using the notiations from the proof of Theorem 1.6) there is a $B_{X,Y} \in \mathcal{I} \cap V$ such that $A_X \cap A_Y \subseteq B_{X,Y}$?

Question 8.3. Does there exist a non-meager ideal $\mathcal{I}$ (in ZFC) such that there are perfect $(\mathcal{I}, \mathrm{Fin})$-AD families on every $X \in \mathcal{I}^+$?

In Example 5.3, assuming that there is a $\mathbf{\Delta}^1_2$ ultrafilter, we constructed a $\mathbf{\Delta}^1_2$ ideal $\mathcal{I}$ such that every $\mathcal{I}$-AD family is countable but $\mathcal{I}$ is nowhere maximal.

Question 8.4. Is it consistent that there are no $\mathbf{\Delta}^1_2$ ultrafilters but there is a $\mathbf{\Sigma}^1_2$ ideal $\mathcal{I}$ such that every $\mathcal{I}$-AD family is countable but $\mathcal{I}$ is nowhere maximal?

A remark to Question 8.4: We know (see [BrJ, Thm. 9.3.9 (2)]) that if there are no dominating reals over $L[r]$ for any $r \in \omega^\omega$, then there is a Σ^1_2 unbounded hence non-meager filter. If every $\underset{\sim}{\Delta}^1_2$ set is Lebesgue measurable or has the Baire property, then there are no $\underset{\sim}{\Delta}^1_2$ ultrafilters. For instance, these conditions above hold in the Cohen and random models over $V = L$ (see [BrJ, Thm. 9.2.1]). In these models a non-meager Σ^1_2 ideal $\mathcal{I}$ must be nowhere maximal (otherwise a restriction of $\mathcal{I}$ would be a Δ^1_2 prime ideal). It would be interesting to know the possible sizes of $\mathcal{I}$-AD families in these models.

Question 8.5. Is it consistent that for some (tall) Borel (P-)ideal $\mathcal{I}$ a family $\mathcal{H} \in [\mathcal{I}^+]^{<\mathfrak{c}}$ does not have an $(\mathcal{I}, \mathrm{Fin})$-ADR (i.e. $F^*(\mathcal{I}) < \mathfrak{c}$)?

Question 8.6. Does adding mixing injections imply adding Cohen reals?

Question 8.7. Does the Sacks property of a forcing notion imply that it does not add mixing reals?

Proposition 7.2 motivates the following notion: Let $n \geq 2$. We say that a forcing notion adds an *n-splitting partition*, if there is a partition $(Y_k)_{k<n}$ of ω into infinite sets in $V^{\mathbb{P}}$ such that $|X \cap Y_k| = \omega$ for every $X \in [\omega]^\omega \cap V$ and $k < n$. In particular, adding 2-splitting partitions is the same as adding splitting reals, and adding ω-splitting (infinite splitting) partitions is equivalent to adding mixing reals.

It is easy to see that if $\mathbb{P}$ adds a splitting real then the n stage iteration of $\mathbb{P}$ adds a 2^n-splitting partition. In fact, splitting reals and n-splitting partitions cannot be separated in terms of cardinal invariants. Let us denote $\mathfrak{s}_n$ ($2 \leq n < \omega$) the least size of a family $\mathcal{S}_n$ of partitions of ω into n many infinite sets such that

$$(*) \qquad \forall\, X \in [\omega]^\omega\ \exists\, P = (P_k)_{k<n} \in \mathcal{S}_n\ \forall\, k < n\ |X \cap P_k| = \omega.$$

Of course, this definition makes sense for $n = \omega$ as well but $\mathfrak{s}_\omega$ stands for an already defined and studied cardinal invariant. To avoid confusions, let us denote this cardinal by $\mathfrak{s}_{\mathrm{mix}}$.

Then $\mathfrak{s}_n = \mathfrak{s} = \mathfrak{s}_2$ for every $2 \leq n < \omega$. For the non-trivial direction, assume that we have a family $\mathcal{S}$ of splitting partitions of size $\mathfrak{s}$ and consider all possible "$(n-1)$-long iterated nestings" of these partitions. For example, if $n = 3$ then to every pair $(P = (P_0, P_1), Q = (Q_0, Q_1))$ of partitions from $\mathcal{S}$ we associate a partition of ω into three infinite sets as follows: Let $e_0 : \omega \to Q_0$ be the increasing bijection and take the partition $(e_0[P_0], e_0[P_1], Q_1)$. We obtain $\mathfrak{s}^{n-1} = \mathfrak{s}$ many partitions of ω into n many infinite sets, the family $\mathcal{S}_n$ of these partitions satisfies $(*)$, and hence $\mathfrak{s}_n \leq \mathfrak{s}$.

Question 8.8. Does adding n-splitting partitions ($2 \leq n < \omega$) imply adding $(n+1)$-splitting partitions?

Question 8.9. Is $\mathfrak{s}_{\mathrm{mix}} = \mathfrak{s}$? Does adding splitting reals (or n-splitting partitions for every n) imply adding mixing reals? What can we say about the Silver forcing? (It is straightforward to see that it adds n-splitting partitions for every n.)

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3.8 Cofinalities of Marczewski ideals

Jörg Brendle, Yurii Khomskii and Wolfgang Wohofsky, Colloquium Mathematicum, to appear.

Originally, this was a project between Wohofsky and Khomskii. Having established some basic ideas, we involved Brendle in the collaboration who proved the important results Theorem 14, Theorem 16 and Proposition 17.

- ***Definition 4 and Proposition 5****: Khomskii and Wohofsky*
- ***Proposition 7****: Khomskii*
- ***Proposition 9****: Khomskii*
- ***Propositoin 10****: Khomskii and Wohofsky*
- ***Theorem 14****: Brendle*
- ***Theorem 16****: Brendle*
- ***Proposition 17 and Corollary 18****: Brendle*

The paper was mostly written by Brendle, with some proofs of Section 2 taken from earlier notes by Khomskii.

Cofinalities of Marczewski-like ideals

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Abstract

We show in ZFC that the cofinalities of both the Miller ideal m^0 (the σ-ideal naturally related to Miller forcing $\mathbb{M}$) and the Laver

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ideal ℓ^0 (related to Laver forcing $\mathbb{L}$) are larger than the size of the continuum $\mathfrak{c}$.

1 Introduction

The purpose of this note is to prove (in ZFC) that the ideals naturally related to Laver forcing $\mathbb{L}$ and to Miller forcing $\mathbb{M}$, the *Laver ideal* ℓ^0 and the *Miller ideal* m^0, have cofinality strictly larger than $\mathfrak{c}$, the size of the continuum (Corollary 18 below). We will phrase our result in a more general framework and show that $\mathsf{cof}(t^0) > \mathfrak{c}$ holds for all tree ideals t^0 derived from tree forcings $\mathbb{T}$ satisfying a certain property (Theorem 13 in Section 3). This was known previously for the Marczewski ideal s^0 [JMS] and the nowhere Ramsey ideal r^0 [Ma], but it is unclear whether the method of proof for these two ideals works for ℓ^0 and m^0 (see the discussion in Section 2), and our approach is more general. We emphasize that while there is a close connection to the corresponding forcing notions, our results are in ZFC and no knowledge of forcing theory is required for understanding them.

For $n \leq \omega$ let $\omega^n = \{f : n \to \omega\}$ be the collection of all functions from n to ω or, equivalently, the collection of all sequences of length n of natural numbers. ω^ω is the *Baire space*, and $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$ is the collection of finite sequences of natural numbers. A subset T of $\omega^{<\omega}$ is called a *tree* (or a *subtree* of $\omega^{<\omega}$) if it is closed under initial segments, that is, if $s \in T$ and $n \in \omega$ then $s{\restriction}n \in T$. For a tree T, $[T] = \{x \in \omega^\omega : x{\restriction}n \in T \text{ for all } n \in \omega\}$ denotes the *set of branches* through T.

Definition 1 (Combinatorial tree forcing)**.** A collection $\mathbb{T}$ of subtrees of $\omega^{<\omega}$ is a *combinatorial tree forcing* if

1. $\omega^{<\omega} \in \mathbb{T}$,
2. (closure under subtrees) if $T \in \mathbb{T}$ and $s \in T$, then the tree $T_s = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$ also belongs to $\mathbb{T}$,
3. (large disjoint antichains) there is a continuous function $f : \omega^\omega \to 2^\omega$ such that for all $x \in 2^\omega$, $f^{-1}(\{x\})$ is the set of branches of a tree in $\mathbb{T}$,
4. (homogeneity) if $T \in \mathbb{T}$, then there is an order-preserving injection $i : \omega^{<\omega} \to T$ such that the map $g : \omega^\omega \to [T]$ given by $g(x) = \bigcup\{i(x{\restriction}n) : n \in \omega\}$ is a homeomorphism and for any subtree $S \subseteq \omega^{<\omega}$, $S \in \mathbb{T}$ iff the closure of $i(S)$ under initial segments belongs to $\mathbb{T}$.

$\mathbb{T}$ is partially ordered by inclusion, that is, for $S, T \in \mathbb{T}$, $S \leq T$ if $S \subseteq T$.

Homogeneity says that the partial order looks the same below each element. In view of homogeneity, "large disjoint antichains" implies that

5. each $T \in \mathbb{T}$ splits into continuum many trees with pairwise disjoint sets of branches, that is, there are $T_\alpha \in \mathbb{T}$, $\alpha < \mathfrak{c}$, with $T_\alpha \subseteq T$ such that $[T_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$.

Recall here that $S, T \in \mathbb{T}$ are *incompatible* if there is no $U \in \mathbb{T}$ with $U \leq S, T$. A set $\mathcal{T} \subseteq \mathbb{T}$ is an *antichain* if any two distinct elements of $\mathcal{T}$ are incompatible. An antichain $\mathcal{T}$ is a *maximal antichain* if for each $S \in \mathbb{T}$ there is $T \in \mathcal{T}$ compatible with S. A partial order $\mathbb{T}$ satisfies the *countable chain condition* (*ccc* for short) if every antichain in $\mathbb{T}$ is at most countable. Clause 5 then says that there are $\mathfrak{c}$-sized antichains in a combinatorial tree forcing $\mathbb{T}$ so that $\mathbb{T}$ is not ccc and some classical forcing notions like Cohen and random forcing do not fit into this framework.

For partial orders whose elements are subtrees of $2^{<\omega}$ like Sacks forcing $\mathbb{S}$, an analogous definition applies, with $\omega^{<\omega}$ and ω^ω replaced by $2^{<\omega}$ and the *Cantor space* 2^ω, respectively.

Definition 2 (Tree ideal)**.** The *tree ideal* t^0 associated with the combinatorial tree forcing $\mathbb{T}$ consists of all $X \subseteq \omega^\omega$ such that for all $T \in \mathbb{T}$ there is $S \leq T$ with $X \cap [S] = \emptyset$.

For a tree T (in $\omega^{<\omega}$ or $2^{<\omega}$) and $t \in T$, $\mathrm{succ}_T(t) = \{n \in \omega : t\hat{\ }n \in T\}$ is the *set of successors* of t in T. A node $s \in T$ is called a *splitting node* of T if $\mathrm{succ}_T(s)$ has at least two elements. The stem of T, $\mathrm{stem}(T)$, is the smallest splitting node. A subtree $T \subseteq 2^{<\omega}$ is a *Sacks tree* (or *perfect tree*) if for each $t \in T$ there is a splitting node $s \supseteq t$ in T. *Sacks forcing* $\mathbb{S}$, the collection of all Sacks trees, is a combinatorial tree forcing: "large disjoint antichains" is witnessed by $f : 2^\omega \to 2^\omega$ given by $f(x)(n) = x(2n)$ for $x \in 2^\omega$ and $n \in \omega$. The *Marczewski ideal* s^0 is the corresponding tree ideal. A subtree $T \subseteq \omega^{<\omega}$ is a *Laver tree* [La] if for all $t \in T$ containing $\mathrm{stem}(T)$, $\mathrm{succ}_T(t)$ is infinite. $T \subseteq \omega^{<\omega}$ is a *Miller tree* [Mi] (or *superperfect tree*) if for all $t \in T$ there is $s \supseteq t$ in T such that $\mathrm{succ}_T(s)$ is infinite. *Laver forcing* $\mathbb{L}$ (*Miller forcing* $\mathbb{M}$, respectively) is the collection of all Laver trees (Miller trees, resp.). Both are combinatorial tree forcings in the above sense for $f : \omega^\omega \to 2^\omega$ given by $f(x)(n) = x(n) \mod 2$ for all $x \in \omega^\omega$ and $n \in \omega$ witnesses "large disjoint antichains". The *Laver* and *Miller ideals* ℓ^0 and m^0 are the corresponding tree ideals. For basic facts about such tree ideals, like non-inclusion between different ideals, see e.g. [Br].

Definition 3 (Cofinality of an ideal)**.** Given an ideal $\mathcal{I}$ on ω^ω or 2^ω, its *cofinality* $\mathsf{cof}(\mathcal{I})$ is the smallest cardinality of a family $\mathcal{J} \subseteq \mathcal{I}$ such that every member of $\mathcal{I}$ is contained in a member of $\mathcal{J}$.

A family like $\mathcal{J}$ in this definition is said to be a *basis* of $\mathcal{I}$ (or: *cofinal* in $\mathcal{I}$).

While the topic of our work are cofinalities of tree ideals, we note that other cardinal invariants of tree ideals t^0, such as the *additivity* $\mathsf{add}(t^0)$ (the least size of a subfamily $\mathcal{J} \subseteq t^0$ whose union is not in t^0) and the *covering number* $\mathsf{cov}(t^0)$ (the least size of a subfamily $\mathcal{J} \subseteq t^0$ whose union is ω^ω) have been studied as well. If there is a fusion argument for $\mathbb{T}$, t^0 is a σ-ideal, and one has $\omega_1 \leq \mathsf{add}(t^0) \leq \mathsf{cov}(t^0) \leq \mathfrak{c}$, while the exact value of these two cardinals depends on the model of set theory. Furthermore, by "large disjoint antichains", the *uniformity* $\mathsf{non}(t^0)$ of a tree ideal t^0 (the smallest cardinality of a subset of ω^ω not belonging to t^0) is always equal to $\mathfrak{c}$. Since $\mathsf{cof}(\mathcal{I}) \geq \mathsf{non}(\mathcal{I})$ for any non-trivial ideal $\mathcal{I}$, $\mathsf{cof}(t^0) \geq \mathfrak{c}$ follows, and the main problem about cofinalities of tree ideals is whether they can be equal to $\mathfrak{c}$ or must be strictly above $\mathfrak{c}$.

The question whether $\mathsf{cof}(\ell^0)$ and $\mathsf{cof}(m^0)$ are larger than $\mathfrak{c}$ was discussed in private communication with M. Dečo and M. Repický, and Repický [Re] in the meantime used our result to obtain a characterization of $\mathsf{cof}(\ell^0)$ as $\mathfrak{d}((\ell^0)^{\mathfrak{c}})$.

2 The disjoint maximal antichain property

Definition 4. Let $\mathbb{T}$ be a combinatorial tree forcing. $\mathbb{T}$ has the *disjoint maximal antichain property* if there is a maximal antichain $(T_\alpha : \alpha < \mathfrak{c})$ in $\mathbb{T}$ such that $[T_\alpha] \cap [T_\beta] = \emptyset$ for all $\alpha \neq \beta$.

The following has been known for some time (see also [Re, Theorem 1.2]).

Proposition 5. *Assume $\mathbb{T}$ has the disjoint maximal antichain property. Then $cf(\mathsf{cof}(t^0)) > \mathfrak{c}$.*

Proof. Let $(T_\alpha : \alpha < \mathfrak{c})$ be a disjoint maximal antichain in $\mathbb{T}$. Also let $\kappa = cf(\mathsf{cof}(t^0))$ and assume $\kappa \leq \mathfrak{c}$. We shall derive a contradiction. Assume $\mathcal{X}_\alpha \subseteq t^0$, $\alpha < \kappa$, are of size $< \mathsf{cof}(t^0)$. We shall show that $\mathcal{X} = \bigcup\{\mathcal{X}_\alpha : \alpha < \kappa\}$ is not cofinal in t^0. By homogeneity of the tree forcing $\mathbb{T}$, we know that $\mathcal{X}_\alpha$ is not cofinal below T_α, that is, there is $X_\alpha \in t^0$, $X_\alpha \subseteq [T_\alpha]$, such that

$X_\alpha \not\subseteq Y$ for all $Y \in \mathcal{X}_\alpha$. Let $X = \bigcup\{X_\alpha : \alpha < \kappa\}$. By disjointness of the maximal antichain, we see that $X \in t^0$. Obviously $X \not\subseteq Y$ for all $Y \in \mathcal{X}$, and we are done. □

Note that for only showing $\mathsf{cof}(t^0) > \mathfrak{c}$, the homogeneity of the forcing is not needed (that is, properties 1, 2, and 5 of Definition 1 are enough).

Definition 6. Let $\mathbb{T}$ be a combinatorial tree forcing. $\mathbb{T}$ has the *incompatibility shrinking property* if for any $T \in \mathbb{T}$ and any family $(S_\alpha : \alpha < \mu)$, $\mu < \mathfrak{c}$, in $\mathbb{T}$ such that S_α is incompatible with T for all α, one can find $T' \leq T$ such that $[T']$ is disjoint from all the $[S_\alpha]$.

For the next proof recall that a set $\mathcal{D}$ in a partial order $\mathbb{T}$ is *dense* if for all $S \in \mathbb{T}$ there is $T \leq S$ belonging to $\mathcal{D}$. If $\mathcal{T}$ is a maximal antichain in $\mathbb{T}$, then the set $\mathcal{D} = \{S \in \mathbb{T} : S \leq T \text{ for some } T \in \mathcal{T}\}$ is easily seen to be dense.

Proposition 7. *Let $\mathbb{T}$ be a combinatorial tree forcing. The incompatibility shrinking property for $\mathbb{T}$ implies the disjoint maximal antichain property for $\mathbb{T}$. In fact, it implies that any maximal antichain can be refined to a disjoint maximal antichain.*

Proof. Let $(T_\alpha : \alpha < \mathfrak{c})$ be a dense set of trees in $\mathbb{T}$ all of which lie below a given maximal antichain of size $\mathfrak{c}$. We construct $A \subseteq \mathfrak{c}$ of size $\mathfrak{c}$ and $\{S_\alpha : \alpha \in A\} \subseteq \mathbb{T}$ such that

- $S_\alpha \leq T_\alpha$ for $\alpha \in A$,
- if $\alpha \notin A$, then T_α is compatible with some S_β for $\beta < \alpha$ with $\beta \in A$,
- $[S_\alpha] \cap [S_\beta] = \emptyset$ for $\alpha \neq \beta$ from A.

Clearly, these conditions imply that $(S_\alpha : \alpha \in A)$ is a disjoint maximal antichain. Also A must necessarily have size $\mathfrak{c}$.

Suppose we are at stage $\alpha < \mathfrak{c}$ of the construction. If T_α is compatible with some S_β where $\beta < \alpha$, $\beta \in A$, let $\alpha \notin A$, and we are done. If this is not the case, let $\alpha \in A$. By the incompatibility shrinking property we find $T' = S_\alpha$ as required. □

Say that a Laver tree $T \subseteq \omega^{<\omega}$ is a *Mathias tree* if $\text{stem}(T)$ is a strictly increasing sequence and there is an infinite $A \subseteq \omega$ such that for all $t \in T$ containing $\text{stem}(T)$, $\text{succ}_T(t) = A \setminus (t(|t|-1)+1)$. *Mathias forcing* $\mathbb{R}$ is the collection of Mathias trees, and the ideal r^0 of *nowhere Ramsey* sets is the corresponding tree ideal. A Sacks tree $T \subseteq 2^{<\omega}$ is a *Silver tree* if there are an infinite $B \subseteq \omega$ and $g : \omega \setminus B \to 2$ such that $t \in T$ iff $t{\restriction}(\omega \setminus B) = g{\restriction}|t|$ (in particular, $t \in T$ is a splitting node iff $|t| \in B$). *Silver forcing* $\mathbb{V}$ is the collection of Silver trees, and v^0 is the corresponding *Silver ideal*. Both $\mathbb{R}$ and $\mathbb{V}$ are combinatorial tree forcings (in the case of $\mathbb{R}$, in Definition 1, use the collection of strictly increasing finite sequences and the space of strictly increasing functions instead of $\omega^{<\omega}$ and the Baire space ω^ω, respectively).

Example 8. Sacks forcing $\mathbb{S}$, Mathias forcing $\mathbb{R}$, and Silver forcing $\mathbb{V}$ have the incompatibility shrinking property and thus also the disjoint maximal antichain property.

To see this, simply use that for any two incompatible $S, T \in \mathbb{S}$, $[S] \cap [T]$ is at most countable, while in the case of $\mathbb{V}$, this intersection is finite, and for $\mathbb{R}$, even empty.

From this we obtain that $cf(\mathsf{cof}(s^0)) > \mathfrak{c}$ [JMS, Theorem 1.3], that $cf(\mathsf{cof}(r^0)) > \mathfrak{c}$ [Ma], and that $cf(\mathsf{cof}(v^0)) > \mathfrak{c}$.

We also note that if t^0 is a σ-ideal (which is the case if there is a fusion argument for $\mathbb{T}$), then the continuum hypothesis CH implies the incompatibility shrinking property and thus also the disjoint maximal antichain property. For Laver and Miller forcings, a weaker hypothesis is sufficient.

Proposition 9. *Assume $\mathfrak{b} = \mathfrak{c}$. Then Laver forcing $\mathbb{L}$ has the incompatibility shrinking property and thus also the disjoint maximal antichain property.*

Recall here that the *unbounding number* $\mathfrak{b}$ is the least size of an $\mathcal{F} \subseteq \omega^\omega$ that is unbounded in the partial order $(\omega^\omega, \leq^*)$, where $f \leq^* g$ if $f(n) \leq g(n)$ holds for all but finitely many $n \in \omega$. The *dominating number* $\mathfrak{d}$ is the least size of an $\mathcal{F} \subseteq \omega^\omega$ that is cofinal in $(\omega^\omega, \leq^*)$. It is well-known and easy to see that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Proof. Fix $T \in \mathbb{L}$, $\mu < \mathfrak{c}$, and any family $(S_\alpha : \alpha < \mu)$ in $\mathbb{L}$ such that S_α is incompatible with T for all α. Since $T \cap S_\alpha$ does not contain a Laver tree, by [GRSS, Lemma 2.3], there is a function $g_\alpha : \omega^{<\omega} \to \omega$ such that if $x \in [T] \cap [S_\alpha]$, then there are infinitely many n with $x(n) < g_\alpha(x{\restriction}n)$. By $\mathfrak{b} = \mathfrak{c}$, there is $f : \omega^{<\omega} \to \omega$ eventually dominating all g_α. Let $T' = \{s \in T : s(n) > f(s{\restriction}n)$ for all $n \in \mathrm{dom}(s)$ beyond the stem of $T\}$. Clearly T' is still a Laver tree with the same stem as T. Furthermore, $[T'] \cap [S_\alpha] = \emptyset$ for if x belonged to the intersection, we would have $x(n) < g_\alpha(x{\restriction}n)$ for infinitely many n and $x(n) > f(x{\restriction}n)$ for all n beyond the stem of T', a contradiction. □

A similar argument which we leave to the reader shows:

Proposition 10. *Assume* $\mathfrak{d} = \mathfrak{c}$. *Then Miller forcing* $\mathbb{M}$ *has the incompatibility shrinking property and thus also the disjoint maximal antichain property.*

Question 11. *Do* $\mathbb{L}$ *or* $\mathbb{M}$ *have the disjoint maximal antichain property in ZFC?*

3 The selective disjoint antichain property

We now consider a property weaker than the disjoint maximal antichain property which is sufficient to show that the cofinalities of the Laver ideal ℓ^0 and the Miller ideal m^0 are larger than $\mathfrak{c}$ in ZFC.

Definition 12. Let $\mathbb{T}$ be a combinatorial tree forcing. $\mathbb{T}$ has the *selective disjoint antichain property* if there is an antichain $(T_\alpha : \alpha < \mathfrak{c})$ in $\mathbb{T}$ such that

- $[T_\alpha] \cap [T_\beta] = \emptyset$ for all $\alpha \neq \beta$,
- for all $T \in \mathbb{T}$ there is $S \leq T$ such that
  - either $S \leq T_\alpha$ for some $\alpha < \mathfrak{c}$,
  - or $|[S] \cap [T_\alpha]| \leq 1$ for all $\alpha < \mathfrak{c}$.

We note that for our applications, it would be enough to have $|[S] \cap [T_\alpha]| \leq \aleph_0$ in the last clause.

Theorem 13. *Assume* $\mathbb{T}$ *has the selective disjoint antichain property. Then* $cf(\mathsf{cof}(t^0)) > \mathfrak{c}$.

Proof. Let $(T_\alpha : \alpha < \mathfrak{c})$ be a selective disjoint antichain in $\mathbb{T}$. Also assume that $(S_\beta : \beta < \mathfrak{c})$ is a list of all trees S in $\mathbb{T}$ such that $|[S] \cap [T_\alpha]| \leq 1$ for all $\alpha < \mathfrak{c}$. Put $\kappa = cf(\mathsf{cof}(t^0))$ and assume $\kappa \leq \mathfrak{c}$. Also assume $\mathcal{X}_\alpha \subseteq t^0$, $\alpha < \kappa$, are of size $< \mathsf{cof}(t^0)$. As in the proof of Proposition 5, we shall show that $\mathcal{X} = \bigcup\{\mathcal{X}_\alpha : \alpha < \kappa\}$ is not cofinal in t^0.

By "large disjoint antichains", we find $T'_\alpha \leq T_\alpha$ such that $[T'_\alpha] \cap \bigcup_{\beta<\alpha}[S_\beta] = \emptyset$. By homogeneity, there is $X_\alpha \in t^0$ with $X_\alpha \subseteq [T'_\alpha]$ such that $X_\alpha \not\subseteq Y$ for all $Y \in \mathcal{X}_\alpha$. Let $X = \bigcup\{X_\alpha : \alpha < \kappa\}$. Obviously $X \not\subseteq Y$ for all $Y \in \mathcal{X}$. We need to show that X belongs to t^0. To this end, let $T \in \mathbb{T}$.

First assume there is $S \leq T$ such that $S \leq T_\alpha$ for some $\alpha < \mathfrak{c}$. Then $[S] \cap X \subseteq X_\alpha$. Since $X_\alpha \in t^0$, there is $S' \leq S$ such that $[S'] \cap X_\alpha = \emptyset$, and $[S'] \cap X = \emptyset$ follows.

Next assume there is $S \leq T$ such that $|[S] \cap [T_\alpha]| \leq 1$ for all $\alpha < \mathfrak{c}$. Then $S = S_\beta$ for some $\beta < \mathfrak{c}$. By construction, we know that $X_\alpha \cap [S_\beta] = \emptyset$ for $\alpha > \beta$. Hence $[S_\beta] \cap X \subseteq \bigcup_{\alpha \leq \beta} [S_\beta] \cap [T_\alpha]$ and therefore $|[S_\beta] \cap X| < \mathfrak{c}$. Using again "large disjoint antichains", we see that there is $S' \leq S_\beta$ such that $[S'] \cap X = \emptyset$, as required. This completes the proof of the theorem. $\square$

Again note that for only showing $\mathsf{cof}(t^0) > \mathfrak{c}$, the homogeneity of the forcing is not needed (that is, properties 1, 2, and 5 of Definition 1 are enough).

The next property of a combinatorial tree forcing $\mathbb{T}$ implies that $\mathbb{T}$ adds a minimal real and, in fact, standard proofs of minimality go via this property.

Definition 14. Let $\mathbb{T}$ be a combinatorial tree forcing. $\mathbb{T}$ has the *constant or one-to-one property* if for all $T \in \mathbb{T}$ and all continuous $f : [T] \to 2^\omega$, there is $S \leq T$ such that $f \restriction [S]$ is either constant or one-to-one.

It is known that both Miller forcing and Laver forcing have the constant or one-to-one property. For the former, this is implicit in work of Miller [Mi, Section 2], for the latter, in work of Gray [Gra] (see also [Gro, Theorems 2 and 7] for similar arguments). These results are formulated in terms of minimality. For completeness' sake, we include a proof of the more difficult case of Laver forcing in our formulation. Note also that the result for $\mathbb{M}$ is a trivial consequence of the result for $\mathbb{L}$.

Theorem 15 (Miller)**.** *Miller forcing* $\mathbb{M}$ *has the constant or one-to-one property.*

Theorem 16 (Gray)**.** *Laver forcing* $\mathbb{L}$ *has the constant or one-to-one property.*

Proof. Fix f and T. The pure decision property of Laver forcing (see, e.g., [BJ, Lemma 7.3.32]) implies:

Claim 16.1. Let $n \in \omega$ and $\tau \in T$ with $\mathrm{stem}(T) \subseteq \tau$. There are $T' \leq_0 T_\tau$ and $s \in 2^n$ such that $[T'] \subseteq f^{-1}([s])$.

Here, $T' \leq_0 T$ if $T' \leq T$ and $\mathrm{stem}(T) = \mathrm{stem}(T')$.

Claim 16.2. Let $\tau \in T$ with $\mathrm{stem}(T) \subseteq \tau$. There are $T' \leq_0 T_\tau$ and $x = x_\tau \in 2^\omega$ such that if $(k^n_\tau : n \in \omega)$ is the increasing enumeration of $\mathrm{succ}_{T'}(\tau)$ then $[T'_{\tau^\frown k^n_\tau}] \subseteq f^{-1}([x \restriction (|\tau| + n)])$.

Proof. Using Claim 16.1, construct a $\leq_0$-decreasing sequence $(S^n : n \in \omega)$ with $S^0 \leq_0 T_\tau$ and a $\subset$-increasing sequence $(s^n \in 2^{n+|\tau|} : n \in \omega)$ such that $[S^n] \subseteq f^{-1}([s^n])$ for all n. Let $k^n_\tau = \min(\mathrm{succ}_{S^n}(\tau) \setminus (k^{n-1}_\tau + 1))$ where we put $k^{-1}_\tau = -1$. Let T' be such that $\mathrm{succ}_{T'}(\tau) = \{k^n_\tau : n \in \omega\}$ and $T'_{\tau\hat{}k^n_\tau} = S^n_{\tau\hat{}k^n_\tau}$. Also let $x = \bigcup_n s^n \in 2^\omega$. Then $T' \leq_0 T_\tau$ and $[T'_{\tau\hat{}k^n_\tau}] = [S^n_{\tau\hat{}k^n_\tau}] \subseteq [S^n] \subseteq f^{-1}([s^n]) = f^{-1}([x{\restriction}(|\tau| + n)])$. □

By Claim 16.2 and a fusion argument we see

Claim 16.3. There are $T' \leq_0 T$, $(x_\tau : \tau \in T', \mathrm{stem}(T) \subseteq \tau)$, and $((k^n_\tau : n \in \omega) : \tau \in T', \mathrm{stem}(T) \subseteq \tau)$ such that $(k^n_\tau : n \in \omega)$ is the increasing enumeration of $\mathrm{succ}_{T'}(\tau)$ for all τ and $[T'_{\tau\hat{}k^n_\tau}] \subseteq f^{-1}([x_\tau{\restriction}(|\tau| + n)])$ for all n and all τ. In particular $[T'_\tau] \subseteq f^{-1}([x_\tau{\restriction}|\tau|])$ for all τ.

The properties of the x_τ imply in particular that $x_{\tau\hat{}k^n_\tau}$ converges to x_τ as n goes to infinity. Now define a rank function for $\tau \in T'$ as follows.

- $\rho(\tau) = 0 \iff \exists^\infty k \in \mathrm{succ}_{T'}(\tau)$ such that $x_{\tau\hat{}k} \neq x_\tau$,
- for $\alpha > 0$, $\rho(\tau) = \alpha \iff \neg\rho(\tau) < \alpha \ \wedge\ \exists^\infty k \in \mathrm{succ}_{T'}(\tau)\ (\rho(\tau\hat{}k) < \alpha)$.

By the convergence property of the x_τ, we see that $\rho(\tau) = 0$ implies in particular that the set $\{x_{\tau\hat{}k^n_\tau} : n \in \omega\}$ is infinite.

Case 1. $\rho(\tau) = \infty$ for some $\tau \in T'$ (i.e., the rank is undefined).
Then we can easily construct a Laver tree $S \leq T'$ such that $\mathrm{stem}(S) = \tau$ and $x_\sigma = x_\tau$ for all $\sigma \in S$ with $\sigma \supseteq \tau$. We claim that $f{\restriction}[S]$ is constant with value x_τ. Indeed let $y \in [S]$. Fix $k \geq |\tau|$. By construction $y \in [S_{y{\restriction}k}] \subseteq f^{-1}([x_\tau{\restriction}k])$. Since this holds for all k, $f(y) = x_\tau$, and we are done.

Case 2. $\rho(\tau)$ is defined for all $\tau \in T'$.
Recall that $F \subseteq T'$ is a *front* if for all $y \in [T']$ there is a unique n with $y{\restriction}n \in F$. We build a subtree S of T' by specifying fronts F_n, $n \in \omega$, such that for every $\sigma \in F_{n+1}$ there is a (necessarily unique) $\tau \in F_n$ with $\tau \subset \sigma$. That is, S will be the tree generated by the fronts: $\sigma \in S$ iff there are $n \in \omega$ and $\tau \in F_n$ with $\sigma \subseteq \tau$. Additionally, we shall guarantee that there are $s_\tau \subseteq x_\tau$ for $\tau \in \bigcup_n F_n$ such that

- if $\sigma \neq \sigma'$ both are in F_n then $[s_\sigma] \cap [s_{\sigma'}] = \emptyset$,
- $[S_\tau] \subseteq f^{-1}([s_\tau])$ for $\tau \in \bigcup_n F_n$,
- if $\sigma \subset \tau$ with $\sigma \in F_n$ and $\tau \in F_{n+1}$ then $s_\sigma \subset s_\tau$,
- if $\sigma \subset \tau$ with $\sigma \in F_n$ and $\tau \in F_{n+1}$ then for every k with $|\sigma| \leq k < |\tau|$, $x_{\tau{\restriction}k} = x_\sigma$ and $\rho(\sigma) > \rho(\tau{\restriction}|\sigma| + 1) > ... > \rho(\tau{\restriction}|\tau| - 1) = 0$.

We first verify that this is enough to guarantee that $f\restriction[S]$ is one-to-one. If $y, y' \in [S]$ are distinct, then there are $n, i, i' \in \omega$ such that $y\restriction i$ and $y'\restriction i'$ are distinct elements of F_n. Then $y \in f^{-1}([s_{y\restriction i}])$, $y' \in f^{-1}([s_{y'\restriction i'}])$ by the second clause, and $[s_{y\restriction i}]$ and $[s_{y'\restriction i'}]$ are disjoint by the first clause. Hence $f(y) \neq f(y')$ as required. Thus it suffices to construct the F_n and s_τ.

$n = 0$. We let $F_0 = \{\mathrm{stem}(T')\} = \{\mathrm{stem}(S)\}$. Also let $s_{\mathrm{stem}(S)} = x_{\mathrm{stem}(S)}\restriction|\mathrm{stem}(S)|$.

Suppose F_n and s_σ for $\sigma \in F_n$ have been constructed. We shall construct F_{n+1}, s_σ for $\sigma \in F_{n+1}$, as well as the part of the tree S in between F_n and F_{n+1}. Fix $\sigma \in F_n$. By A^n_σ we denote the part of S between σ and F_{n+1}, that is, $A^n_\sigma = \{\tau \in S : \sigma \subseteq \tau \text{ and } \tau \subset \upsilon \text{ for some } \upsilon \text{ in } F_{n+1}\}$. A^n_σ will be constructed recursively so as to satisfy the forth clause above.

Put σ into A^n_σ. Suppose some $\tau \supseteq \sigma$ has been put into A^n_σ, $x_\tau = x_\sigma$ and, in case $\tau \supset \sigma$, $\rho(\sigma) > \rho(\tau)$. In case $\rho(\tau) = 0$, no successor of τ will be in A^n_σ and the successors of τ will belong to F_{n+1}, as explained below. If $\rho(\tau) > 0$, then $x_{\tau\hat{}k} = x_\tau$ for almost all $k \in \mathrm{succ}_{T'}(\tau)$ and $\rho(\tau\hat{}k) < \rho(\tau)$ for infinitely many $k \in \mathrm{succ}_{T'}(\tau)$. Hence we can prune the successor level of τ to $\mathrm{succ}_S(\tau)$ such that $x_{\tau\hat{}k} = x_\tau$ and $\rho(\tau\hat{}k) < \rho(\tau)$ for all $k \in \mathrm{succ}_S(\tau)$. The forth clause is clearly satisfied. This completes the construction of A^n_σ.

Now fix $\tau \in A^n_\sigma$ with $\rho(\tau) = 0$. By pruning $\mathrm{succ}_{T'}(\tau)$, if necessary, we may assume without loss of generality that the $x_{\tau\hat{}k^m_\tau}$, $m \in \omega$, are all pairwise distinct and converge to $x_\tau = x_\sigma$ and that, in fact, there is a strictly increasing sequence $(i^m_\tau : m \in \omega)$ such that $i^m_\tau = \min\{i : x_{\tau\hat{}k^m_\tau}(i) \neq x_\tau(i)\}$. Unfixing τ, we may additionally assume that if $\tau \neq \tau'$ are both in A^n_σ of rank 0 and $m, m' \in \omega$, then $i^m_\tau \neq i^{m'}_{\tau'}$. Finally we may assume that all such i^m_τ are larger than $|s_\sigma|$. This means in particular that $s_\sigma \subseteq x_{\tau\hat{}k^m_\tau}$ for all τ and m because $s_\sigma \subseteq x_\sigma = x_\tau$. Now choose $s_{\tau\hat{}k^m_\tau} \subseteq x_{\tau\hat{}k^m_\tau}$ such that $|s_{\tau\hat{}k^m_\tau}| > i^m_\tau$. Then $s_\sigma \subset s_{\tau\hat{}k^m_\tau}$ and the $s_{\tau\hat{}k^m_\tau}$ for distinct pairs (τ, m) with $\tau \in A^n_\sigma$ of rank 0 and $m \in \omega$ are pairwise incompatible.

Unfix $\sigma \in F_n$. Let $F_{n+1} = \{\tau\hat{}k^m_\tau : \tau \in A^n_\sigma \text{ for some } \sigma \in F_n, \rho(\tau) = 0, \text{ and } m \in \omega\}$. The third clause is immediate. To see the first clause, take distinct $\tau, \tau' \in F_{n+1}$. There are $\sigma, \sigma' \in F_n$ such that $\sigma \subset \tau$ and $\sigma' \subset \tau'$. If $\sigma \neq \sigma'$, then $[s_\tau] \cap [s_{\tau'}] = \emptyset$ because $[s_\sigma] \cap [s_{\sigma'}] = \emptyset$ and $s_\sigma \subset s_\tau$ and $s_{\sigma'} \subset s_{\tau'}$. If $\sigma = \sigma'$, then $[s_\tau] \cap [s_{\tau'}] = \emptyset$ by the construction in the previous paragraph. Finally, to see the second clause, by pruning T'_τ for $\tau \in F_{n+1}$ if necessary, we may assume $[T'_\tau] \subseteq f^{-1}([s_\tau])$ (see Claim 16.3). This completes the recursive construction and the proof of the theorem. □

Proposition 17. *Assume $\mathbb{T}$ is a combinatorial tree forcing with the constant* *or one-to-one property. Then* $\mathbb{T}$ *has the selective disjoint antichain property.*

Proof. Let $f : \omega^\omega \to 2^\omega$ be a continuous function witnessing "large disjoint antichains" of $\mathbb{T}$. Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of 2^ω. Let $T_\alpha \in \mathbb{T}$ be such that $[T_\alpha] = f^{-1}(\{x_\alpha\})$. We check that $(T_\alpha : \alpha < \mathfrak{c})$ witnesses the selective disjoint antichain property. Clearly $[T_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$. Given $T \in \mathbb{T}$, find $S \leq T$ such that $f \restriction [S]$ is constant or one-to-one. In the first case, $S \leq T_\alpha$ for some α, and in the second case, $|[S] \cap [T_\alpha]| \leq 1$ for all α, and we are done. □

We are finally ready to complete the proof of the main result of this note.

Corollary 18. $cf(\mathsf{cof}(\ell^0)) > \mathfrak{c}$ *and* $cf(\mathsf{cof}(m^0)) > \mathfrak{c}$*.*

Proof. This follows from Theorem 16, Theorem 15, Proposition 17, and Theorem 13. □

4 Problems

For some natural tree forcings, we still do not know whether the cofinality of the corresponding tree ideal is larger than $\mathfrak{c}$ in ZFC. A Miller tree $T \subseteq \omega^{<\omega}$ is a *full splitting Miller tree* if whenever $s \in T$ is a splitting node then $s\hat{}n \in T$ for all $n \in \omega$. *Full splitting Miller forcing* $\mathbb{FM}$, originally introduced by [NR] (see also [KL]), consists of all full splitting Miller trees, and fm^0 is the *full splitting Miller ideal.* $\mathbb{FM}$ is also a combinatorial tree forcing.

Question 19. *Is* $\mathsf{cof}(fm^0) > \mathfrak{c}$*?*

By the discussion in Section 2 (before Proposition 9), we know this is true under CH.

More generally, one may ask:

Question 20. *Are there combinatorial tree forcings* $\mathbb{T}$ *which consistently fail to have the disjoint maximal antichain property? Which consistently fail to satisfy* $\mathsf{cof}(t^0) > \mathfrak{c}$*? For which* t^0 *consistently has a Borel basis?*

Note that the existence of a Borel basis implies $\mathsf{cof}(t^0) = \mathfrak{c}$. By the above comment fm^0 has no Borel basis under CH, but this is open in ZFC. Question 20 is also of interest for tree forcings which do not necessarily satisfy all the clauses of Definition 1, e.g., for non-homogeneous forcing notions.

By [JMS, Theorems 1.4 and 1.5], we know that $\mathsf{cof}(s^0)$ can consistently assume arbitrary values $\leq 2^{\mathfrak{c}}$ whose cofinality is larger than $\mathfrak{c}$ and it is easy to see that the same arguments work for other tree ideals like m^0 and ℓ^0. (In these models CH holds.)

Question 21. *Can we consistently separate the cofinalities of different tree ideals? E.g., are* $\mathsf{cof}(s^0) < \mathsf{cof}(m^0)$ *or* $\mathsf{cof}(m^0) < \mathsf{cof}(s^0)$ *consistent?*

Added to the revised version: Shelah and Spinas [SS] recently proved the consistency of, e.g., $\mathsf{cof}(m^0) < \mathsf{cof}(s^0)$ and $\mathsf{cof}(\ell^0) < \mathsf{cof}(s^0)$. The consistency of $\mathsf{cof}(s^0) < \mathsf{cof}(m^0)$, however, remains open.

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3.9 Filter-Laver measurability

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Filter-Laver Measurability

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Abstract

We study σ-ideals and regularity properties related to the "filter-Laver" and "dual-filter-Laver" forcing partial orders. An important innovation which enables this study is a dichotomy theorem proved recently by Miller [Mil].

1 Introduction

In this paper, F will always be a filter on ω (or a suitable countable set), containing at least the cofinite sets. We will use F^- to refer to the ideal of all $a \subseteq \omega$ such that $\omega \setminus a \in F$, and F^+ to the collection of $a \subseteq \omega$ such that $a \notin F^-$. Cof and Fin denote the filter of cofinite subsets of ω and the ideal of finite subsets of ω, respectively.

Definition 1.1. An *F-Laver tree* is a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending $\text{stem}(T)$, $\text{Succ}_T(\sigma) \in F$. An *$F^+$-Laver-tree* is a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending $\text{stem}(T)$, $\text{Succ}_T(\sigma) \in F^+$. We use $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$ to denote the partial orders of F-Laver and F^+-Laver trees, respectively, ordered by inclusion.

If $F = \mathsf{Cof}$ then $\mathbb{L}_{F^+}$ is the standard *Laver forcing* $\mathbb{L}$, and $\mathbb{L}_F$ is (a version of) the standard *Hechler forcing* $\mathbb{D}$. Both $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$ have been used as forcing notions in the literature, see, e.g., [Gro87]. As usual, the generic real added by these forcings can be defined as the limit of the stems of conditions in the generic filter. It is easy to see that in both cases, this generic real is dominating. It is also known that if F is not an ultrafilter, then $\mathbb{L}_F$ adds a Cohen real, and if F is an ultrafilter, then $\mathbb{L}_F$ adds a Cohen real if and only if F is not a *nowhere dense ultrafilter* (see Definition 4.9). Moreover, $\mathbb{L}_F$ is σ-centered and hence satisfies the ccc, and it is known that $\mathbb{L}_{F^+}$ satisfies Axiom A (see [Gro87, Theorem] and Lemma 2.5 (3)).

In this paper, we consider σ-ideals and regularity properties naturally related to $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$, and study the regularity properties for sets in the low projective hierarchy, following ideas from [BL99, Ike10, Kho12]. An important technical innovation is a dichotomy theorem proved recently by Miller in [Mil] (see Theorem 3.6), which allows us to simplify the σ-ideal for $\mathbb{L}_{F^+}$ when restricted to Borel sets, while having a $\mathbf{\Sigma}^1_2$ definition regarding the membership of Borel sets in it.

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One question may occur to the reader of this paper: why are we not considering the *filter-Mathias* forcing alongside the filter-Laver forcing, when clearly the two forcing notions (and their derived σ-ideals and regularity properties) are closely related? The answer is that, although the basic results from Section 2 do indeed hold for filter-Mathias, there is no corresponding dichotomy theorem like Theorem 3.6. In fact, by a result of Sabok [Sab12], even the σ-ideal corresponding to the *standard* Mathias forcing is not a $\mathbf{\Sigma}^1_2$-ideal on Borel sets, implying that even in this simple case, there is no hope of a similar dichotomy theorem. It seems that in the Mathias case, a more subtle analysis is required.

In Section 2 we give the basic definitions and prove some easy properties. In Section 3 we present Miller's dichotomy and the corresponding σ-ideal. In Section 4 we study direct relationships that hold between the regularity properties regardless of the complexity of F, whereas in Section 5 we prove stronger results under the assumption that F is an analytic filter.

2 $(\mathbb{L}_F)$- and $(\mathbb{L}_{F^+})$-measurable sets.

In [Ike10], Ikegami provided a natural framework for studying σ-ideals and regularity properties related to tree-like forcing notions, generalising the concepts of *meager* and *Baire property*. This concept proved to be very useful in a number of circumstances, see, e.g., [Kho12, Lag14, KL15].

Definition 2.1. Let $\mathbb{P}$ be $\mathbb{L}_F$ or $\mathbb{L}_{F^+}$ and let $A \subseteq \omega^\omega$.

1. $A \in \mathcal{N}_\mathbb{P}$ iff $\forall T \in \mathbb{P}\, \exists S \leq T\ ([S] \cap A = \varnothing)\}$.
2. $A \in \mathcal{I}_\mathbb{P}$ iff A is contained in a countable union of sets in $\mathcal{N}_\mathbb{P}$.
3. A is $\mathbb{P}$-*measurable* iff $\forall T \in \mathbb{P}\, \exists S \leq T\ ([S] \subseteq^* A$ or $[S] \cap A =^* \varnothing)$, where $\subseteq^*$ and $=^*$ stands for "modulo a set in $\mathcal{I}_\mathbb{P}$".

Lemma 2.2. *The collection $\{[T] \mid T \in \mathbb{L}_F\}$ forms a topology base. The resulting topology refines the standard topology and the space satisfies the Baire category theorem (i.e., $[T] \notin \mathcal{I}_{\mathbb{L}_F}$ for all $T \in \mathbb{L}_F$).*

Proof. Clearly, for all $S, T \in \mathbb{L}_F$ the intersection $S \cap T$ is either empty or an $\mathbb{L}_F$-condition. A basic open set in the standard topology trivially corresponds to a tree in $\mathbb{L}_F$. For the Baire category theorem, let A_n be nowhere dense and, given an arbitrary $T \in \mathbb{L}_F$, build a sequence $T = T_0 \geq T_1 \geq T_2 \geq \dots$ with strictly increasing stems such that $[T_n] \cap A_n = \varnothing$ for all n. Then the limit of the stems is an element in $[T] \setminus \bigcup_n A_n$. □

We use $\tau_{\mathbb{L}_F}$ to denote the topology on ω^ω generated by $\{[T] \mid T \in \mathbb{L}_F\}$. Clearly $\mathcal{N}_{\mathbb{L}_F}$ is the collection of $\tau_{\mathbb{L}_F}$-nowhere dense sets and $\mathcal{I}_{\mathbb{L}_F}$ the collection of $\tau_{\mathbb{L}_F}$-meager sets. Moreover, we recall the following fact, which is true in arbitrary topologal spaces (the proof is similar to [Kec95, Theorem 8.29]):

Fact 2.3. *Let $\mathcal{X}$ be any topological space, and $A \subseteq \mathcal{X}$. Then the following are equivalent:*

1. *A satisfies the Baire property.*
2. *For every basic open O there is a basic open $U \subseteq O$ such that $U \subseteq^* A$ or $U \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ refer to "modulo meager".*

In particular, $A \subseteq \omega^\omega$ is $\mathbb{L}_F$-measurable iff A satisfies the $\tau_{\mathbb{L}_F}$-Baire property.

What about the dual forcing $\mathbb{L}_{F^+}$? Notice that a topological approach cannot work in general:

Lemma 2.4. *The collection $\{[T] \mid T \in \mathbb{L}_{F^+}\}$ forms a topology base iff F is an ultrafilter.*

Proof. If F is not an ultrafilter, fix Z such that $Z \in F^+$ and $(\omega \setminus Z) \in F^+$ and consider trees $S, T \in \mathbb{L}_{F^+}$ defined so that $\forall \sigma \in S\ (\mathrm{Succ}_S(\sigma) = Z \cup \{0\})$ and $\forall \tau \in T\ (\mathrm{Succ}_T(\tau) = (\omega \setminus Z) \cup \{0\})$. □

Instead, to study $\mathbb{L}_{F^+}$, we rely on combinatorial methods familiar from Laver forcing. For every n, define $\leq_n$ by:

$$S \leq_n T \ :\Leftrightarrow\ S \leq T \text{ and } S \cap \omega^{\leq k+n} = T \cap \omega^{\leq k+n},$$

where $k = |\mathrm{stem}(T)|$. If $T_0 \geq_0 T_1 \geq_1 \dots$ is a decreasing sequence then $T := \bigcap_n T_n \in \mathbb{L}_{F^+}$ and $T \leq T_n$ for every n.

Lemma 2.5. *Let F be a filter on ω. Then:*

1. *$\mathbb{L}_{F^+}$ has* pure decision*, i.e., for every ϕ and every $T \in \mathbb{L}_{F^+}$, there is $S \leq_0 T$ such that $S \Vdash \phi$ or $S \Vdash \neg\phi$.*
2. *For all $A \subseteq \omega^\omega$, the following are equivalent:*
   (a) *$A \in \mathcal{N}_{\mathbb{L}_{F^+}}$,*
   (b) *$\forall T \in \mathbb{L}_{F^+}\ \exists S \leq_0 T\ ([S] \cap A = \varnothing)$.*
3. *$\mathcal{N}_{\mathbb{L}_{F^+}} = \mathcal{I}_{\mathbb{L}_{F^+}}$.*
4. *For all $A \subseteq \omega^\omega$, the following are equivalent:*
   (a) *A is $(\mathbb{L}_{F^+})$-measurable,*
   (b) *$\forall T \in \mathbb{L}_{F^+}\ \exists S \leq T\ ([S] \subseteq A$ or $[S] \cap A = \varnothing)$,*
   (c) *$\forall T \in \mathbb{L}_{F^+}\ \exists S \leq_0 T\ ([S] \subseteq A$ or $[S] \cap A = \varnothing)$.*
5. *The collection of $(\mathbb{L}_{F^+})$-measurable sets forms a σ-algebra.*

Proof. Since many of the arguments here are similar, we prove the first assertion and only sketch the others.

1. Fix ϕ and T and let $u := \mathrm{stem}(T)$. For $\sigma \in T$ extending u, say:

   - σ is *positive-good* if $\exists S \leq_0 T{\uparrow}\sigma$ such that $S \Vdash \phi$,
   - σ is *negative-good* if $\exists S \leq_0 T{\uparrow}\sigma$ such that $S \Vdash \neg\phi$,
   - σ is *bad* if neither of the above holds.

   Here "$T{\uparrow}\sigma$" denotes $\{\tau \in T \mid \tau \subseteq \sigma \vee \sigma \subseteq \tau\}$.

   We claim that u is good, completing the proof. Assume that u is bad. Partition $\mathrm{Succ}_T(u)$ into Z_0, Z_1 and Z_2 by setting $n \in Z_0$ iff $u^\frown\langle n\rangle$ is positive-good, $n \in Z_1$ iff $u^\frown\langle n\rangle$ is negative-good, and $n \in Z_2$ iff $u^\frown\langle n\rangle$ is bad. One of the three components must be in F^+. But if it is Z_0, then let $S_n \leq_0 T{\uparrow}(u^\frown \langle n \rangle)$ be such that $S_n \Vdash \phi$, and then take $S := \bigcup_{n \in Z_0} S_n$. Then we get $S \leq_0 T{\uparrow}u$ and $S \Vdash \phi$, thus u is positive-good contrary to assumption. Analogously, if Z_1 is in F^+ then u is negative-good contrary to assumption. Hence, Z_2 must be in F^+. Now, for each $n \in Z_2$, use the same argument to obtain an F^+-positive set $Z_{2,2}$ of successors of $u^\frown \langle n \rangle$ such that for all $m \in Z_{2,2}$, $u^\frown \langle n, m \rangle$ is bad, and so on.

This way we construct a tree $T^* \leq T$ such that all $\sigma \in T^*$ are bad. But there is a $T^{**} \leq T^*$ deciding ϕ, which means that $\text{stem}(T^{**})$ is either positive-good or negative-good, leading to a contradiction.

2. Let $A \in \mathcal{N}_{\mathbb{L}_{F^+}}$, fix T, and let $u = \text{stem}(T)$. For $\sigma \in T$ extending u, say that σ is *good* if $\exists S \leq_0 T{\uparrow}\sigma$ such that $[S] \cap A = \varnothing$, and σ is *bad* otherwise. By the same argument as above we prove that u is good.

3. Suppose $A_n \in \mathcal{N}_{\mathbb{L}_{F^+}}$ for all n. Fix $T \in \mathbb{L}_{F^+}$. Clearly it is enough to produce a fusion sequence $T \geq_0 T_0 \geq_1 T_1 \geq_2 \dots$ such that for all n, $[T_n] \cap A_n = \varnothing$. So suppose we have constructed T_n. Let $\{u_i \mid i < \omega\}$ enumerate all the nodes in T_n of length $|\text{stem}(T_n)| + n$. For each u_i, use (2) to find $S_i \leq_0 T_n{\uparrow}u_i$ with $[S_i] \cap A_{n+1} = \varnothing$. Let $T_{n+1} := \bigcup_i S_i$. Then clearly $T_{n+1} \leq_n T_n$ and $[T_{n+1}] \cap A_{n+1} = \varnothing$ as required.

4. For $(a) \Rightarrow (b)$, use the fact that $\mathcal{I}_{\mathbb{L}_{F^+}} = \mathcal{N}_{\mathbb{L}_{F^+}}$. For $(b) \Rightarrow (c)$, use the same argument as in (1).

5. It suffices to show closure under countable unions. Suppose A_n is $\mathbb{L}_{F^+}$-measurable and fix $T \in \mathbb{L}_{F^+}$. If for one n, there is $S \leq T$ with $[S] \subseteq A_n$ then we are done. Otherwise (using the equivalence from (4)) for every n, there is $S \leq T$ such that $[S] \cap A_n = \varnothing$. Then an argument like in (3) shows that there is $S \leq T$ such that $[S] \cap \bigcup_n A_n = \varnothing$. □

Remark 2.6. Note that an argument like in (4) above in fact shows that $\mathbb{L}_{F^+}$ satisfies a stronger form of properness, namely, for all countable elementary models $M \prec \mathcal{H}_\theta$ and all $T \in \mathbb{L}_{F^+}$, there exists $S \leq T$ such that every $x \in [S]$ is $\mathbb{L}_{F^+}$-generic over M.

Again it is interesting to ask whether any of the "simplifications" (1)–(4) from the above Lemma might go through for $\mathbb{L}_F$, too.

Lemma 2.7. *If we replace $\mathbb{L}_{F^+}$ with $\mathbb{L}_F$ in Lemma 2.5, then the statements (1)–(4) are all equivalent to each other, and equivalent to the statement "F is an ultrafilter".*

Proof. If F is not an ultrafilter, let Z be such that $Z \in F^+$ and $(\omega \setminus Z) \in F^+$, let $A_n := \{x \in \omega^\omega \mid \forall m \geq n\, (x(m) \in Z)\}$ and $A = \bigcup_n A_n = \{x \in \omega^\omega \mid \forall^\infty m\, (x(m) \in Z)\}$. Also, x_G denotes the $\mathbb{L}_F$-generic real. We leave it to the reader to verify that

- the statement "$x_G(0) \in Z$" cannot be decided by any $\mathbb{L}_F$-condition with empty stem (falsifying (1)),
- $Z^\omega \in \mathcal{N}_{\mathbb{L}_F}$ but for every $T \in \mathbb{L}_F$ with empty stem we have $[T] \cap Z^\omega \neq \varnothing$ (falsifying (2)),
- $A_n \in \mathcal{N}_{\mathbb{L}_F}$ for all n, but $A \notin \mathcal{N}_{\mathbb{L}_F}$ (falsifying (3)), and
- A is $\mathbb{L}_F$-measurable (see Theorem 2.8), but for every $T \in \mathbb{L}_F$ we have $[T] \not\subseteq A$ and $[T] \cap A \neq \varnothing$ (falsifying (4)). □

Thus, the situation can be neatly summarized as follows: when F is *not* an ultrafilter, $\mathbb{L}_F$ generates a topology but does not satisfy properties 1–4 from Lemma 2.5, while $\mathbb{L}_{F^+}$ satisfies those properties but does not generate a topology. $\mathbb{L}_F$-measurability is the Baire propery in the $\tau_{\mathbb{L}_F}$-topology, whereas $\mathbb{L}_{F^+}$-measurability is the "Marczewski"-property corresponding to the partial order $\mathbb{L}_{F^+}$, and $\mathcal{I}_{\mathbb{L}_{F^+}}$ is the "Marczewski"-ideal corresponing to $\mathbb{L}_{F^+}$.

In the interesting scenario when F is an ultrafilter everything coincides, and the ideal $\mathcal{I}_{\mathbb{L}_F}$ of $\tau_{\mathbb{L}_F}$-meager sets is the same as the ideal of $\tau_{\mathbb{L}_F}$-nowhere dense sets. In this context, the ideal has been studied by Louevau in [Lou76] and is sometimes called the *Louveau ideal*.

Theorem 2.8. *Let F be a filter on ω. Every analytic and co-analytic set $A \subseteq \omega^\omega$ is both $\mathbb{L}_F$-measurable and $\mathbb{L}_{F^+}$-measurable.*

Proof. Since $\tau_{\mathbb{L}_F}$ refines the standard topology on ω^ω, analytic (co-analytic) sets are also analytic (co-analytic) in $\tau_{\mathbb{L}_F}$. By classical results, such sets have the $\tau_{\mathbb{L}_F}$-Baire property.

For $\mathbb{L}_{F^+}$, suppose A is analytic, defined by a $\Sigma^1_1(r)$ formula ϕ. Let $T \in \mathbb{L}_{F^+}$. Let $S \leq T$ be a stronger condition forcing $\phi(\dot{x}_G)$ or $\neg\phi(\dot{x}_G)$, without loss of generality the former. Let M be a countable elementary submodel of a sufficiency large $\mathcal{H}_\theta$ with $S, r, F \in M$. By Remark 2.6, we can find an $S' \leq S$ such that all $x \in [S']$ are $\mathbb{L}_{F^+} \cap M$-generic over M. Then for all such x we have $M[x] \models \phi(x)$. By $\boldsymbol{\Sigma}^1_1$-absoluteness, $\phi(x)$ is really true. Thus we have $[S'] \subseteq A$. The co-analytic case is analogous. □

A different (forcing-free) proof of the second assertion will follow from Theorem 3.6.

From the above it follows that there we have dense embeddings $\mathbb{L}_F \hookrightarrow_d \mathsf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{L}_F}$ and $\mathbb{L}_{F^+} \hookrightarrow_d \mathsf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{L}_{F^+}}$.

Definition 2.9. Let $\boldsymbol{\Gamma}$ be a projective pointclass. The notation $\boldsymbol{\Gamma}(\mathbb{L}_F)$ and $\boldsymbol{\Gamma}(\mathbb{L}_{F^+})$ abbreviates the propositions "all sets of complexity $\boldsymbol{\Gamma}$ are $\mathbb{L}_F$-measurable" and "all sets of complexity $\boldsymbol{\Gamma}$ are $\mathbb{L}_{F^+}$-measurable", respectively.

The statements $\boldsymbol{\Sigma}^1_2(\mathbb{L}_F)$ and $\boldsymbol{\Sigma}^1_2(\mathbb{L}_{F^+})$ are independent of ZFC, and we will study the exact strength of these statements in Section 4 (for arbitrary F) and Section 5 (for definable F).

3 A dichotomy theorem for $\mathbb{L}_{F^+}$

While $\mathcal{I}_{\mathbb{L}_F}$ is a ccc Borel-generated ideal exhibiting many familiar properties, $\mathcal{I}_{\mathbb{L}_{F^+}}$ is a "Marczewski-style" ideal, which is not Borel-generated and rather difficult to study. The rest of the paper depends crucially on the dichotomy result presented in this section, which simplifies the ideal $\mathcal{I}_{\mathbb{L}_{F^+}}$ when it is restricted to Borel sets. The proof, as well as several key insights, are due to Arnold Miller [Mil]. For motivation, recall the *Laver dichotomy*, originally due to Goldstern et al [GRSS95].

Definition 3.1. If $f : \omega^{<\omega} \to \omega$ and $x \in \omega^\omega$, we say that x *strongly dominates* f if $\forall^\infty n\, (x(n) \geq f(x{\restriction}n))$. A family $A \subseteq \omega^\omega$ is called *strongly dominating* if for every $f : \omega^{<\omega} \to \omega$ there exists $x \in A$ which strongly dominates f. $\mathcal{D}$ denotes the ideal of sets A which are *not* strongly dominating.

It is easy to see that if $T \in \mathbb{L}$ then $[T] \notin \mathcal{D}$, and the classical result [GRSS95, Lemma 2.3] shows that if A is analytic, then either $A \in \mathcal{D}$ or there is a Laver tree T such that $[T] \subseteq A$. The ideal $\mathcal{D}$ was discovered independently by Zapletal (cf. [Zap03, Lemma 3.3.]) and was studied, among others, in [DR13, Deč15]. Generalising this, we obtain the following definitions:

Definition 3.2. Let F be a filter on ω. If $\varphi : \omega^{<\omega} \to F$ and $x \in \omega^\omega$, we say that x *F-dominates* φ iff $\forall^\infty n\, (x(n) \in \varphi(x{\restriction}n))$. A family $A \subseteq \omega^\omega$ is *F-dominating* if for every $\varphi : \omega^{<\omega} \to F$ there exists $x \in A$ which dominates φ. $\mathcal{D}_{F^+}$ denotes the ideal of sets A which are not F-dominating. In other words:

$$A \in \mathcal{D}_{F^+} \;:\Longleftrightarrow\; \exists \varphi : \omega^{<\omega} \to F \;\; \forall x \in A \;\; \exists^\infty n\, (x(n) \notin \varphi(x{\restriction}n)).$$

In the above context, the terminology "F-dominates" might seem inappropriate, but we choose it in order to retain the analogy with Definition 3.1. Note that $\mathcal{D} = \mathcal{D}_{\mathsf{Cof}^+}$.

Lemma 3.3. *$\mathcal{D}_{F^+}$ is a σ-ideal.*

Proof. Suppose $A_i \in \mathcal{D}_{F^+}$ for $i < \omega$. Let φ_i witness this for each i, and define φ by setting $\varphi(\sigma) := \bigcap_{i<|\sigma|} \varphi_i(\sigma)$. We claim that φ witnesses that $A = \bigcup_{i<\omega} A_i \in \mathcal{D}_{F^+}$. Pick $x \in A$. There is i such that $x \in A_i$, hence for infinitely many n we have $x(n) \notin \varphi_i(x{\restriction}n)$. But if $n > i$ then $\varphi(x{\restriction}n) \subseteq \varphi_i(x{\restriction}n)$. Therefore, for infinitely many n we also have $x(n) \notin \varphi(x{\restriction}n)$. □

Lemma 3.4. *Let $A \subseteq \omega^\omega$. The following are equivalent:*

1. $A \in \mathcal{D}_{F^+}$.
2. $\forall \sigma \in \omega^{<\omega}\, \exists T \in \mathbb{L}_F$ *with* $\mathrm{stem}(T) = \sigma$, *such that* $[T] \cap A = \varnothing$.
3. $\forall S \in \mathbb{L}_F\, \exists T \leq_0 S\, ([S] \cap A = \varnothing)$

Proof. The equivalence between 2 and 3 is clear, so we prove the equivalence between 1 and 2.

First, note that if $\varphi : \omega^{<\omega} \to F$ and $\sigma \in \omega^{<\omega}$, then there is a unique $T_{\sigma,\varphi} \in \mathbb{L}_F$ such that $\mathrm{stem}(T_{\sigma,\varphi}) = \sigma$ and $\forall \tau \supseteq \sigma$, $\mathrm{Succ}_{T_{\sigma,\varphi}}(\tau) = \varphi(\tau)$. Conversely, for every $T \in \mathbb{L}_F$ with $\mathrm{stem}(T) = \sigma$, there exists a (not unique) φ such that $T = T_{\sigma,\varphi}$.

Now suppose $A \in \mathcal{D}_{F^+}$, as witnessed by φ, and let $\sigma \in \omega^{<\omega}$. Then $A \cap [T_{\sigma,\varphi}] = \varnothing$, since if $x \in A \cap [T_{\sigma,\varphi}]$ then $\forall n > |\sigma|\, (x(n) \in \varphi(x{\restriction}n))$, contrary to the assumption.

Conversely, suppose for every σ there is $T_\sigma \in \mathbb{L}_F$ such that $\mathrm{stem}(T_\sigma) = \sigma$ and $A \cap [T_\sigma] = \varnothing$. For each σ, let $\varphi_\sigma : \omega^{<\omega} \to F$ be such that $T_\sigma = T_{\sigma,\varphi_\sigma}$. Then define $\varphi : \omega^{<\omega} \to F$ by

$$\varphi(\sigma) = \bigcap_{\tau \subseteq \sigma} \varphi_\tau(\sigma).$$

We claim that φ witnesses that $A \in \mathcal{D}_{F^+}$. Let $x \in A$ be arbitrary. Let $\sigma \subseteq x$. Then $x \notin [T_\sigma] = [T_{\sigma,\varphi_\sigma}]$, hence, there is $n > |\sigma|$ such that $x(n) \notin \varphi_\sigma(x{\restriction}n)$. But by definition, since $\sigma \subseteq x{\restriction}n$, we have $\varphi(x{\restriction}n) \subseteq \varphi_\sigma(x{\restriction}n)$. Therefore also $x(n) \notin \varphi(x{\restriction}n)$. □

The following are easy consequences of the above; the proofs are left to the reader.

Lemma 3.5.

1. $\mathcal{D}_{F^+} \subseteq \mathcal{N}_{\mathbb{L}_F}$.

2. $\mathcal{D}_{F^+} \subseteq \mathcal{I}_{\mathbb{L}_{F^+}}$ *(in particular, if* $T \in \mathbb{L}_{F^+}$ *then* $[T] \notin \mathcal{D}_{F^+}$*).*

3. *If* F *is an ultrafilter then* $\mathcal{D}_{F^+} = \mathcal{N}_{\mathbb{L}_F} = \mathcal{I}_{\mathbb{L}_F} = \mathcal{I}_{\mathbb{L}_{F^+}}$.

4. *If* F *is not an ultrafilter then there is a closed witness to* $\mathcal{D}_{F^+} \neq \mathcal{N}_{\mathbb{L}_F}$.

Theorem 3.6 (Miller)**.** *For every analytic* A*, either* $A \in \mathcal{D}_{F^+}$ *or there is* $T \in \mathbb{L}_{F^+}$ *such that* $[T] \subseteq A$.

Proof. See the proof of Theorem 3 and the comment after Theorem 8 in [Mil].[1] We need a slight modification of this proof: rather than talking about trees *with empty stem*, we consider trees with a fixed stem σ. If $A \notin \mathcal{D}_{F^+}$, then by Lemma 3.4 (2), there exists $\sigma \in \omega^{<\omega}$ such that for all $S \in \mathbb{L}_F$ with stem$(S) = \sigma$, $[S] \cap A \neq \varnothing$. By applying the same argument as in [Mil, Theorem 3], we obtain a $T \in \mathbb{L}_{F^+}$ (with stem$(T) = \sigma$) such that $[T] \subseteq A$. □

Remark 3.7. As a direct consequence of this theorem, we obtain an alternative (forcing-free) proof of the second part of Theorem 2.8. Namely: let A be analytic and let $T \in \mathbb{L}_{F^+}$ be arbitrary, so $A \cap [T]$ is analytic. If there exists $S \in \mathbb{L}_{F^+}$ with $[S] \subseteq A \cap [T]$ we are done, and if $A \cap [T] \in \mathcal{D}_{F^+}$, use Lemma 3.4 to find a tree $U \in \mathbb{L}_F$ with stem$(U) =$ stem(T) and $[U] \cap A \cap [T] = \varnothing$. Notice that $T \cap U \in \mathbb{L}_{F^+}$, so we are done.

Also, we now have a dense embedding $\mathbb{L}_{F^+} \hookrightarrow_d \mathsf{Borel}(\omega^\omega)/\mathcal{D}_{F^+}$, with $\mathcal{D}_{F^+}$ being a Borel-generated σ-ideal which is far easier to study than $\mathcal{I}_{\mathbb{L}_{F^+}}$. This will be of particular importance in Section 5 where we look at analytic filters.

4 Direct implications

We first look at some straightforward implications between various statements of the form $\mathbf{\Gamma}(\mathbb{L}_F)$, $\mathbf{\Gamma}(\mathbb{L}_{F^+})$ and $\mathbf{\Gamma}(\mathbb{P})$ for other well-known forcings $\mathbb{P}$. Here $\mathbf{\Gamma}$ denotes an arbitrary *boldface pointclass*, i.e., a collection of subsets of ω^ω closed under continuous pre-images and intersections with closed sets. No further assumptions on the complexity of F are required.

Recall the following reducibility relations for filters on a countable set:

Definition 4.1. Let F, G be filters on dom(F) and dom(G), respectively. We say that:

1. G is *Katetov-reducible* to F, notation $G \leq_K F$, if there is a map $\pi : \text{dom}(F) \to \text{dom}(G)$ such that $a \in G \Rightarrow \pi^{-1}[a] \in F$.

2. G is *Rudin-Keisler-reducible* to F, notation $G \leq_{RK} F$, if there is a map $\pi : \text{dom}(F) \to \text{dom}(G)$ such that $a \in G \Leftrightarrow \pi^{-1}[a] \in F$.

Remark 4.2. Note that $G \leq_K F$ and $G \leq_{RK} F$ are equivalent to the reducibility relation between ideals (i.e., between G^- and F^-). Also, it is clear that if π witnesses $G \leq_K F$, then $a \in F^+ \Rightarrow \pi[a] \in G^+$, and if π witnesses $G \leq_{RK} F$ then, in addition, $a \in F \Rightarrow \pi[a] \in G$.

[1] Here we should also note that Miller's Theorem 3 is, in fact, a direct consequence of Goldstern et al's dichotomy [GRSS95, Lemma 2.3]. However, the point is that its generalisation to filters does not follow from the proof in [GRSS95], which uses infinite games and determinacy. Miller's proof, on the other hand, uses only classical methods and generalises directly to filters.

Notation 4.3. We use the following slight abuse of notation: if F is a filter and $a \in F^+$, then $F \restriction a$ denotes the set $\{b \subseteq a \mid (a \setminus b) \in F^-\}$. In other words, $F \restriction a$ is the filter with $\mathrm{dom}(F \restriction a) = a$ which is dual to the ideal $(F^-)\restriction a$.

Definition 4.4. A filter F is called *K-uniform* if for every $a \in F^+$, $F \restriction a \leq_K F$.

Lemma 4.5. *Suppose $G \restriction a \leq_K F$ for all $a \in G^+$. Then $\mathbf{\Gamma}(\mathbb{L}_{F^+}) \Rightarrow \mathbf{\Gamma}(\mathbb{L}_{G^+})$. In particular, this holds if G is K-uniform and $G \leq_K F$.*

Proof. Let $A \in \mathbf{\Gamma}$ and $T \in \mathbb{L}_{G^+}$ arbitrary. For all $\sigma \in T$ extending $\mathrm{stem}(T)$, let $X_\sigma := \mathrm{Succ}_T(\sigma)$ and fix π_σ witnessing $G \restriction X_\sigma \leq_K F$. Define $f' : \omega^{<\omega} \to \omega^{<\omega}$ by $f'(\varnothing) := \mathrm{stem}(T)$ and $f'(\tau^\frown \langle n \rangle) := f'(\tau)^\frown \langle \pi_{f'(\tau)}(n) \rangle$, and let $f : \omega^\omega \to \omega^\omega$ be the limit of f'. Let $A' := f^{-1}[A]$. Then $A \in \mathbf{\Gamma}$, so by assumption there is an $S \in \mathbb{L}_{F^+}$ such that $[S] \subseteq A'$ or $[S] \cap A' = \varnothing$, without loss of generality the former.

By assumption, we know that for every $\sigma \in S$ extending $\mathrm{stem}(S)$, $\pi_{f'(\sigma)}[\mathrm{Succ}_S(\sigma)] \in G^+$. To make sure that the image under f is the set of branches through an $\mathbb{L}_{G^+}$-tree, prune S to $S^* \subseteq S$, so that $\mathrm{stem}(S^*) = \mathrm{stem}(S)$, and for all $\sigma \in S^*$ extending $\mathrm{stem}(S^*)$, $\pi_{f'(\sigma)}[\mathrm{Succ}_{S^*}(\sigma)] = \pi_{f'(\sigma)}[\mathrm{Succ}_S(\sigma)]$, and $\pi_{f'(\sigma)} \restriction \mathrm{Succ}_{S^*}(\sigma)$ is injective. Then $f[S^*]$ is the set of branches through an $\mathbb{L}_{F^+}$-tree, and moreover $f[S^*] \subseteq [T] \cap A$. □

Lemma 4.6. *Suppose $G \restriction a \leq_K F$ for all $a \in G^+$. Then $\mathbf{\Gamma}(\mathbb{L}_F) \Rightarrow \mathbf{\Gamma}(\mathbb{L}_{G^+})$. In particular, if F is K-uniform then $\mathbf{\Gamma}(\mathbb{L}_F) \Rightarrow \mathbf{\Gamma}(\mathbb{L}_{F^+})$*

Proof. Let $A \in \mathbf{\Gamma}$ and $T \in \mathbb{L}_{G^+}$ be arbitrary. Let f and $A' := f^{-1}[A]$ be as above. By the same argument, it suffices to find $S \in \mathbb{L}_{F^+}$ such that $[S] \subseteq A'$ or $[S] \cap A' = \varnothing$.

By assumption, there is an $\mathbb{L}_F$-tree U with $[U] \setminus A' \in \mathcal{I}_{\mathbb{L}_F}$ or $[U] \cap A' \in \mathcal{I}_{\mathbb{L}_F}$, without loss of generality the former. Since $\mathcal{I}_{\mathbb{L}_F}$ is Borel-generated, let B be a Borel $\mathcal{I}_{\mathbb{L}_F}$-positive set such that $B \subseteq A' \cap [U]$. By Lemma 3.5 B is also $\mathcal{D}_{F^+}$-positive. But then, by Theorem 3.6 there exists an $S \in \mathbb{L}_{F^+}$ such that $[S] \subseteq B$, which completes the proof. □

Lemma 4.7. *Suppose $G \leq_{RK} F$. Then $\mathbf{\Gamma}(\mathbb{L}_F) \Rightarrow \mathbf{\Gamma}(\mathbb{L}_G)$.*

Proof. Let π witness $G \leq_{RK} F$ and let $f : \omega^\omega \to \omega^\omega$ be defined by $f(x)(n) := \pi(x(n))$. Clearly f is continuous in the standard sense. Moreover, we claim the following:

Claim. *f is continuous and open as a function from $(\omega^\omega, \tau_{\mathbb{L}_F})$ to $(\omega^\omega, \tau_{\mathbb{L}_G})$.*

Proof. If $[T]$ is a basic open set in $\tau_{\mathbb{L}_G}$, then $T \in \mathbb{L}_G$ and so $f^{-1}[T]$ is a union of $\mathbb{L}_F$-trees (one for each f-preimage of the stem of T), so it is open in $\tau_{\mathbb{L}_F}$. Conversely, if $[S]$ is basic open in $\tau_{\mathbb{L}_F}$, then $S \in \mathbb{L}_F$. Although $f[S]$ is not necessarily the set of branches through an $\mathbb{L}_G$-tree, we can argue as follows: given $y \in f[S]$, let $x \in [S]$ be such that $f(x) = y$. Then prune S to S^* in a similar way as in the proof of Lemma 4.5, in such a way that the function π restricted to $\mathrm{Succ}_{S^*}(\sigma)$ is injective for each σ while the image $\pi[\mathrm{Succ}_{S^*}(\sigma)]$ remains unchanged. Moreover, we can do this so that $x \in [S^*]$. Then $f[S^*]$ is indeed the set of branches through an $\mathbb{L}_G$-tree, and moreover $y \in f[S^*] \subseteq f[S]$. Since this can be done for every $y \in f[S]$, it follows that $f[S]$ is open in $\tau_{\mathbb{L}_G}$. □ (Claim)

From this, it is not hard to conclude that if $A \in \mathcal{I}_{\mathbb{L}_G}$ then $f^{-1}[A] \in \mathcal{I}_{\mathbb{L}_F}$. To complete the proof, let $A \in \mathbf{\Gamma}$ and let O be $\tau_{\mathbb{L}_G}$-open. It suffices to find a non-empty $\tau_{\mathbb{L}_G}$-open $U \subseteq O$ such that $U \subseteq^* A$ or $U \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ refers to "modulo $\mathcal{I}_{\mathbb{L}_G}$.

Let $A' := f^{-1}[A]$ and $O' := f^{-1}[O]$. Since A' has the Baire property in $\tau_{\mathbb{L}_F}$, there is an open $U' \subseteq O'$ such that $U' \subseteq^* A'$ or $U' \cap A' =^* \varnothing$ (wlog the former) where $\subseteq^*$ and $=^*$ refers to "modulo $\mathcal{I}_{\mathbb{L}_F}$". Then there is a Borel set B such that $B \notin \mathcal{I}_{\mathbb{L}_F}$ and $B \subseteq A' \cap U'$. Hence $f[B]$ is an analytic subset of $A \cap O$, and by the Claim, $f[B] \notin \mathcal{I}_{\mathbb{L}_G}$. By the $\tau_{\mathbb{L}_G}$-Baire property of analytic sets, there is an $\tau_{\mathbb{L}_G}$-open U such that $U \subseteq^* f[B]$. Hence $U \cap O \subseteq^* A'$, which completes the proof. □

The relationships established in the above three lemmas are summarised in Figure 1.

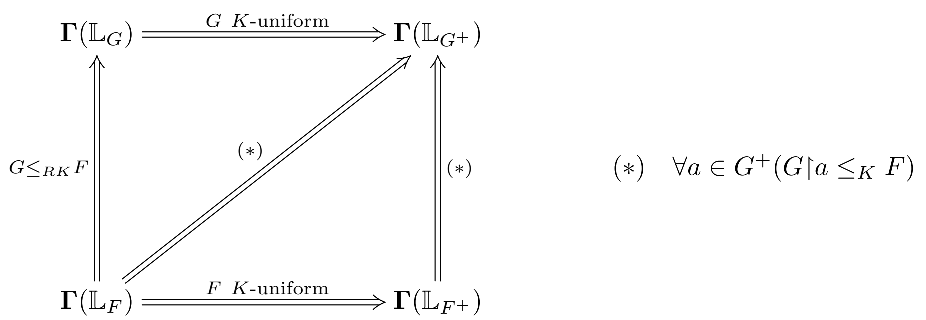

Figure 1: Implications between the properties for filters F and G.

In particular, since $\mathsf{Cof}{\restriction}a \leq_K F$ holds for every F and every infinite a, we obtain the following corollary:

Corollary 4.8. *$\mathbf{\Gamma}(\mathbb{L}_F) \Rightarrow \mathbf{\Gamma}(\mathbb{L})$ and $\mathbf{\Gamma}(\mathbb{L}_{F^+}) \Rightarrow \mathbf{\Gamma}(\mathbb{L})$ for all F.*

Next, we look at the relationship between $\mathbb{L}_F$-measurability and the classical Baire property. In accordance to common usage, we denote the statement "all sets in $\mathbf{\Gamma}$ have the Baire property" by $\mathbf{\Gamma}(\mathbb{C})$ ($\mathbb{C}$ denoting the Cohen forcing partial order). It is known that if F is not an ultrafilter then $\mathbb{L}_F$ adds a Cohen real. Specifically, if Z is such that $Z \notin F$ and $(\omega \setminus Z) \notin F$, and $f : \omega^\omega \to 2^\omega$ is defined by

$$f(x)(n) := \begin{cases} 1 & \text{if } x(n) \in Z \\ 0 & \text{if } x(n) \notin Z \end{cases}$$

then f is continuous with the property that if A is meager then $f^{-1}[A] \in \mathcal{I}_{\mathbb{L}_F}$.

Concerning ultrafilters, the following is known.

Definition 4.9. Let $\mathsf{NWD} \subseteq 2^{<\omega}$ denote the ideal of *nowhere dense subsets of* $2^{<\omega}$, that is, those $H \subseteq 2^{<\omega}$ such that $\forall \sigma\, \exists \tau \supseteq \sigma\, \forall \rho \supseteq \tau\, (\rho \notin H)$. An ultrafilter U is called *nowhere dense* iff $\mathsf{NWD} \not\leq_K U^-$.

It is known that $\mathbb{L}_U$ adds a Cohen real iff U is not a nowhere dense ultrafilter. Specifically, if U is not nowhere dense and $\pi : \omega \to 2^{<\omega}$ is a witness to $\mathsf{NWD} \leq_K U^-$, then we can define a continuous function $f : \omega^\omega \to 2^\omega$ by $f(x) := \pi(x(0))^\frown \pi(x(1))^\frown \dots$. We leave it to the reader to verify that if A is meager then $f^{-1}[A] \in \mathcal{I}_{\mathbb{L}_U}$. This easily leads to the following:

Lemma 4.10. *If F is not an ultrafilter, or a non-nowhere dense ultrafilter, then $\mathbf{\Gamma}(\mathbb{L}_F) \Rightarrow \mathbf{\Gamma}(\mathbb{C})$.*

Proof. In either case, we have a continuous $f : \omega^\omega \to 2^\omega$ such that f-preimages of meager sets are $\mathcal{I}_{\mathbb{L}_F}$-small, as above. Let $A \in \mathbf{\Gamma}$ and $\sigma \in 2^{<\omega}$ arbitrary. Let φ be a homeomorphism from 2^ω to $[\sigma]$ and $A' := (\varphi \circ f)^{-1}[A]$. Then $A' \in \mathbf{\Gamma}$, so let B be a Borel $\mathcal{I}_{\mathbb{L}_F}$-positive set with $B \subseteq A'$ or $B \cap A' = \varnothing$, without loss the former. Then $(\varphi \circ f)[B]$ is an analytic non-meager subset of $A \cap [\sigma]$, so there exists $[\tau] \subseteq [\sigma]$ such that $[\tau] \subseteq^* A$, which is sufficient. □

Finally, an argument from [Mil] yields the following implication. Recall that a set $A \subseteq [\omega]^\omega$ is *Ramsey* iff there exists $H \in [\omega]^\omega$ such that $[H]^\omega \subseteq A$ or $[H]^\omega \cap A = \varnothing$.

Lemma 4.11. *If U is an ultrafilter then $\mathbf{\Gamma}(\mathbb{L}_U) \Rightarrow \mathbf{\Gamma}(\text{Ramsey})$.*

Proof. In fact, we prove a stronger statement: if $A \subseteq \omega^{\uparrow\omega}$ (strictly increasing sequences) is $\mathbb{L}_U$-measurable then $\{\text{ran}(x) \mid x \in A\}$ is Ramsey. First note that, by Lemma 2.5 (4), there exists a $T \in \mathbb{L}_U$ with empty stem, such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$. Also, without loss of generality, we can assume that $[T] \subseteq \omega^{\uparrow\omega}$.

Now proceed inductively:

- Let $n_0 \in \text{Succ}_T(\varnothing)$ be arbitrary.
- Let $n_1 \in \text{Succ}_T(\varnothing) \cap \text{Succ}_T(\langle n_0 \rangle)$.
- Let $n_2 \in \text{Succ}_T(\varnothing) \cap \text{Succ}_T(\langle n_0 \rangle) \cap \text{Succ}_T(\langle n_1 \rangle) \cap \text{Succ}_T(\langle n_0, n_1 \rangle)$.
- etc.

Since U is a filter we can always continue this process and make sure that for any k, any subsequence of the sequence $\langle n_0, \dots, n_k \rangle$ is an element of T. It then follows that any infinite subsequence of the sequence $\langle n_i \mid i < \omega \rangle$ is an element of $[T]$. This is exactly what we need. □

If U is not an ultrafilter, then the above result does not hold in general. For example, considering the cofinite filter, both implications $\mathbf{\Gamma}(\mathbb{L}) \Rightarrow \mathbf{\Gamma}(\text{Ramsey})$ and $\mathbf{\Gamma}(\mathbb{D}) \Rightarrow \mathbf{\Gamma}(\text{Ramsey})$ are consistently false for $\mathbf{\Gamma} = \mathbf{\Delta}^1_2$ (see [FFK14, Section 6]).

5 Analytic filters

In this section, we focus on analytic filters (or ideals). This is important if we want the forcings to be definable, and if we want to apply results from [Ike10, Kho12]. Note that just for absoluteness of the forcing, it would have been sufficient to consider $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$ filters, by Shoenfield absoluteness. However, we also require the ideals and other related notions to have a sufficiently low complexity. For this reason, in this section the following assumption will hold:

Assumption. F is an analytic filter on ω.

It is clear that the statement "$T \in \mathbb{L}_F$" is as complex as F itself. Recall from [BJ95, Section 3.6]) that a forcing notion is *Suslin ccc* if it is ccc and the statements "$T \in \mathbb{L}_F$", "$T \perp S$" and "$S \leq T$" are $\mathbf{\Sigma}^1_1$-relations on the codes of trees. The following is clear:

Fact 5.1. *Let F be analytic. Then $\mathbb{L}_F$ is a Suslin ccc forcing notion.*

Lemma 5.2. *Let F be analytic. Then the ideals $\mathcal{I}_{\mathbb{L}_F}$ and $\mathcal{D}_{F^+}$ are $\mathbf{\Sigma}^1_2$ on Borel sets (i.e., the membership of Borel sets in the ideal is a $\mathbf{\Sigma}^1_2$-property on the Borel codes).*

Proof. A Borel set B is in $\mathcal{D}_{F^+}$ iff $\exists\varphi : \omega^{<\omega} \to F\ \forall x\ (x \in B \to \exists^\infty n\ (x(n) \notin \varphi(x\restriction n)))$. This is easily seen to be a $\mathbf{\Sigma}^1_2$ statement if F is $\mathbf{\Sigma}^1_1$.

For $\mathcal{I}_{\mathbb{L}_F}$, let B be a Borel set. Notice that B is $\tau_{\mathbb{L}_F}$-nowhere dense iff there exists a $\tau_{\mathbb{L}_F}$-open dense set O such that $B \cap O = \varnothing$, iff there is a maximal antichain $A \subseteq \mathbb{L}_F$ such that $B \cap \bigcup\{[T] \mid T \in A\} = \varnothing$. By the ccc, one such maximal antichain can be coded by a real. The resulting computation yields a $\mathbf{\Sigma}^1_2$ statement. □

In [BHL05, Ike10] the concept of *quasi-generic real* was introduced—a real avoiding all Borel sets in a certain σ-ideal coded in the ground model. This concept coincides with generic reals for ccc ideals, but yields a weaker concept for other (combinatorial) ideals, see e.g. [Kho12, Section 2.3].

In the case of $\mathbb{L}_F$, "quasi-generic reals" are the $\mathbb{L}_F$-generic ones, whereas in the case of $\mathbb{L}_{F^+}$, they have a simple characterisation due to the combinatorial ideal $\mathcal{D}_{F^+}$.

Lemma 5.3. *Let M be a model of set theory. A real x is $\mathbb{L}_F$-generic over M iff $x \notin B$ for every Borel set $B \in \mathcal{I}_{\mathbb{L}_F}$ with code in M.*

Proof. See [Kho12, Lemma 2.3.2]. □

Definition 5.4. Let M be a model of set theory. We will call a real $x \in \omega^\omega$ *F-dominating over M* if for every $\varphi : \omega^{<\omega} \to F$ with $\varphi \in M$, x F-dominates φ, i.e., $\forall^\infty n\ (x(n) \in \varphi(x\restriction n))$ (note that the statement $\varphi : \omega^{<\omega} \to F$ is absolute for between M and larger models).

Lemma 5.5. *Let M be a model of set theory with $\omega_1 \subseteq M$. A real x is F-dominating over M iff $x \notin B$ for every Borel set $B \in \mathcal{D}_{F^+}$ with code in M.*

Proof. This is easy to verify from the definition, using $\mathbf{\Sigma}^1_2$-absoluteness between M and V and the fact that $B \in \mathcal{D}_{F^+}$ is a $\mathbf{\Sigma}^1_2$-statement for Borel sets. □

As an immediate corollary of the above and the general framework from [Ike10] and [Kho12], we immediately obtain the following four characterizations for ($\mathbb{L}_F$)- and ($\mathbb{L}_{F^+}$)-measurability.

Corollary 5.6. *Let F be an analytic filter. Then:*

1. $\mathbf{\Delta}^1_2(\mathbb{L}_F) \iff \forall r \in \omega^\omega\ \forall T \in \mathbb{L}_F\ \exists x \in [T]$ (x *is* $\mathbb{L}_F$*-generic over* $L[r]$).
2. $\mathbf{\Sigma}^1_2(\mathbb{L}_F) \iff \forall r \in \omega^\omega\ \{x \mid x$ *not* $\mathbb{L}_F$*-generic over* $L[r]\} \in \mathcal{I}_{\mathbb{L}_F}$.
3. $\mathbf{\Delta}^1_2(\mathbb{L}_{F^+}) \iff \forall r \in \omega^\omega\ \forall T \in \mathbb{L}_{F^+}\ \exists x \in [T]$ (x *is* F*-dominating over* $L[r]$).
4. $\mathbf{\Sigma}^1_2(\mathbb{L}_{F^+}) \iff \forall r \in \omega^\omega\ \{x \mid x$ *not* F*-dominating over* $L[r]\} \in \mathcal{I}_{\mathbb{L}_{F^+}}$.

Proof. See [Ike10, Theorem 4.3 and Theorem 4.4] and [Kho12, Theorem 2.3.7 and Corollary 2.3.8]. Note that both ideals $\mathbb{L}_F$ and $\mathcal{D}_{F^+}$ are $\mathbf{\Sigma}^1_2$, the forcings have absolute definitions and are proper, so the above results can be applied. □

We are interested in more elegant characterizations of the four above statements.

Theorem 5.7. $\mathbf{\Sigma}^1_2(\mathbb{L}_F) \iff \forall r \in \omega^\omega\ (\omega_1^{L[r]} < \omega_1)$.

The proof uses methods similar to [LR95, Theorem 6.2] (see also [BL99, Theorem 5.11]). It follows using a series of definitions and lemmas.

Definition 5.8. For every open dense set $D \subseteq \mathbb{L}_F$, define a *rank function* $\mathrm{rk}_D : \omega^{<\omega} \to \omega_1$ by

- $\mathrm{rk}_D(\sigma) := 0$ iff there is $T \in D$ with $\mathrm{stem}(T) = \sigma$ and
- $\mathrm{rk}_D(\sigma) := \alpha$ iff $\mathrm{rk}_D(\sigma) \not< \alpha$ and $\exists Z \in F^+ \, \forall n \in Z \, (\mathrm{rk}_D(\sigma^\frown \langle n \rangle) < \alpha)$.

A standard argument shows that $\mathrm{rk}_D(\sigma)$ is well-defined for every σ.

Definition 5.9. An (F^-)-mad family is a collection $\mathcal{A} \subseteq F^+$ such that $\forall a \neq b \in \mathcal{A}$ $(a \cap b) \in F^-$, and $\forall a \in F^+$ there exists $b \in \mathcal{A}$ such that $(a \cap b) \in F^+$.

Fact 5.10. *For every analytic filter F, there exists an (F^-)-mad family of size $2^{\aleph_0}$.*

Proof. See [FKV, Corollary 1.8]. □

Lemma 5.11. *Let $\mathcal{A}$ be an (F^-)-mad family. For each $a \in \mathcal{A}$, let $X_a := \{x \in \omega^\omega \mid \mathrm{ran}(x) \cap a = \varnothing\} \in \mathcal{N}_{\mathbb{L}_F}$. Then, for any $X \in \mathcal{I}_{\mathbb{L}_F}$, the collection $\{a \in \mathcal{A} \mid X_a \subseteq X\}$ is at most countable.*

Proof. Let $X \subseteq \bigcup_n X_n$ where X_n are closed nowhere dense in $\tau_{\mathbb{L}_F}$, and let $D_n := \{T \mid [T] \cap X_n = \varnothing\}$. Then the D_n are open dense in $\mathbb{L}_F$. Consider a countable elementary submodel N of some sufficiently large $\mathcal{H}_\theta$ containing $\mathcal{A}$, the D_n, and the defining parameter of F (i.e., the $r \in \omega^\omega$ such that $F \in \Sigma^1_1(r)$). The proof will be completed by showing that if $a \in \mathcal{A} \setminus N$, then there exists $x \in X_a \cap \bigcap_n \bigcup \{[T] \mid T \in D_n\}$, hence $x \in X_a \setminus X$.

Sublemma. For every D_n, every $a \in \mathcal{A} \setminus N$, and every $T \in \mathbb{L}_F$, if $\mathrm{ran}(\mathrm{stem}(T)) \cap a = \varnothing$ then there exists $S \leq T$ with $S \in D_n$ and such that $\mathrm{ran}(\mathrm{stem}(S)) \cap a = \varnothing$ as well.

Proof. Let $Y := \{\tau \in T \mid \mathrm{stem}(T) \subseteq \tau \text{ and } \mathrm{ran}(\tau) \cap a = \varnothing\}$. Let $\tau \in Y$ be of least D_n-rank. We claim that $\mathrm{rk}_{D_n}(\tau) = 0$, which completes the proof. Towards contradiction, assume $\mathrm{rk}_{D_n}(\tau) = \alpha > 0$ and let $Z \in F^+$ witness this. By elementarity and using the fact that all relevant objects are in N and F is absolute for N as well, it follows that $Z \in N$.

By elementarity and absoluteness of F, $N \models$ "$\mathcal{A}$ is an (F^-)-mad family", hence there exists $b \in \mathcal{A} \cap N$ such that $Z \cap b \in F^+$. Since $b \neq a$, it follows that $b \cap a \in F^-$, so there exists $n \in (Z \setminus a)$. Then $\tau^\frown \langle n \rangle$ is an element of Y with D_n-rank less than α, contradicting the minimality of τ. □ (Sublemma)

Now, it is clear that we can inductively apply the sublemma to find a sequence $T_0 \geq T_1 \geq T_2 \geq \dots$, with strictly increasing stems, such that $T_n \in D_n$ for every n, and moreover $\mathrm{ran}(\mathrm{stem}(T_n)) \cap a = \varnothing$ for every n. Then $x := \bigcup_n \mathrm{stem}(T_n)$ has all the required properties, i.e., $x \in X_a \setminus X$. □

Proof of Theorem 5.7. We need to prove the equivalence between

1. $\forall r \, \{x \mid x \text{ not } \mathbb{L}_F\text{-generic over } L[r]\} \in \mathcal{I}_{\mathbb{L}_F}$ and
2. $\forall r \, (\omega_1^{L[r]} < \omega_1)$.

By Lemma 5.3, the former statement is equivalent to $\forall r \bigcup\{B \mid B$ is a Borel $\mathcal{I}_{\mathbb{L}_F}$-small set with code in $L[r]\} \in \mathcal{I}_{\mathbb{L}_F}$. The direction from 2 to 1 is thus immediate.

Conversely, fix r and assume that $\omega_1^{L[r]} = \omega_1$. Let $\mathcal{A}$ be an (F^-)-mad family such that $|\mathcal{A} \cap L[r]| = \omega_1$ (this can be done by extending an (F^-)-almost disjoint family of size ω_1 in $L[r]$). For every $a \in \mathcal{A} \cap L[r]$, X_a is a Borel $\mathcal{I}_{\mathbb{L}_F}$-small set with code in $L[r]$. If 1 was true, then in V there would be an $X \in \mathcal{I}_{\mathbb{L}_F}$ such that $X_a \subseteq X$ for all such a, contradicting Lemma 5.11. □

Remark 5.12. The same argument yields $\mathrm{add}(\mathcal{I}_{\mathbb{L}_F}) = \omega_1$ and $\mathrm{cof}(\mathcal{I}_{\mathbb{L}_F}) = 2^{\aleph_0}$ for analytic filters (where add and cof denote the *additivity* and *cofinality* numbers of the ideal, respectively).

Next, we consider $\mathbf{\Delta}^1_2(\mathbb{L}_F)$ and $\mathbf{\Delta}^1_2(\mathbb{L}_{F^+})$. In [BS99, Theorem 2], the covering number of $\mathcal{I}_{\mathbb{L}_U}$ for an ultrafilter U was determined to be the minimum of $\mathfrak{b}$ and a certain combinatorial characteristic of U called $\pi\mathfrak{p}(U)$. This was generalised by Hrusak and Minami in [HM14, Theorem 2] to arbitrary filters. Similar proofs yield characterisations of $\mathbf{\Delta}^1_2(\mathbb{L}_F)$ and $\mathbf{\Delta}^1_2(\mathbb{L}_{F^+})$.

Definition 5.13. Let M be a model of set theory and F an analytic filter. We say that a real $C \in [\omega]^\omega$ is

1. *F-pseudointersecting over M* if $C \subseteq^* a$ for all $a \in F \cap M$.
2. *F-separating over M* if it is F-pseudointersecting over M, and additionally, for all $b \in (F^+) \cap M$, $|C \cap b| = \omega$.

We use the shorthand "$\exists F$-pseudoint" and "$\exists F$-sep" to abbreviate the statements "$\forall r \in \omega^\omega \, \exists C$ (C is F-pseudointersecting/separating over $L[r]$)".

Question 5.14. *Are there natural regularity properties equivalent to "$\exists F$-pseudoint" and "$\exists F$-sep" for $\mathbf{\Delta}^1_2$ sets of reals?*

Recall that $\mathbf{\Sigma}^1_2(\mathbb{C})$ is equivalent to $\mathbf{\Delta}^1_2(\mathbb{C}) \wedge \mathbf{\Delta}^1_2(\mathbb{L})$ and equivalent to $\mathbf{\Delta}^1_2(\mathbb{D})$, where $\mathbb{C}, \mathbb{L}$ and $\mathbb{D}$ stand for the Baire property, Laver- and Hechler-measurability, respectively. Also, recall that $\mathbf{\Delta}^1_2(\mathbb{C})$ is equivalent to the existence of Cohen reals over $L[r]$, $\mathbf{\Delta}^1_2(\mathbb{L})$ is equivalent to the existence of dominating reals over $L[r]$, and $\mathbf{\Delta}^1_2(\mathbb{D})$ is equivalent to the existence of Hechler-generic reals over $L[r]$. See [BL99, Theorem 4.1 and Theorem 5.8].

The next two theorems would have a cleaner look if the forcings $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$ were homogeneous in a certain sense. Since in general they might not be, one of the direction in the following two theorems requires an additional assumption. Say that a filter F is *principal** iff there is an a such that $F = \{x \subseteq \mathrm{dom}(F) \mid a \subseteq^* x\}$.

Theorem 5.15. $\mathbf{\Delta}^1_2(\mathbb{L}_F) \implies \mathbf{\Sigma}^1_2(\mathbb{C}) \wedge \exists F$-sep. *If F is not principal*, then the converse direction holds as well.*

Proof. By Corollary 4.8 and Lemma 4.10, we know that $\mathbf{\Delta}^1_2(\mathbb{L}_F)$ implies $\mathbf{\Delta}^1_2(\mathbb{L})$ and $\mathbf{\Delta}^1_2(\mathbb{C})$, which in turns implies $\mathbf{\Sigma}^1_2(\mathbb{C})$ as mentioned above. Moreover, a standard density argument shows that $\mathbb{L}_F$ generically adds an F-separating real, specifically, if x is $\mathbb{L}_F$-generic then $\mathrm{ran}(x)$ is F-separating.

For the converse direction, first note the following: by an easy argument (see [MZ16, Lemma 3]), every filter F which is not principal*, and every $X \in F$, there exists a bijection $\pi : \omega \to X$

such that for all $a \subseteq X$, $a \in F \iff \pi^{-1}[a] \in F$. We leave it to the reader to verify that this implies homogeneity of $\mathbb{L}_F$, in the sense that if there exists an $\mathbb{L}_F$-generic real then there also exists an $\mathbb{L}_F$-generic real inside T for every $T \in \mathbb{L}_F$. Therefore, by Corollary 5.6 (1), it suffices to prove that for every $r \in \omega^\omega$ there exists an $\mathbb{L}_F$-generic real over $L[r]$.

So, fix $r \in \omega^\omega$ and let C be F-separating over $L[r]$. Let $\mathbb{D}_C$ denote Hechler forcing as defined on C^ω (i.e., the conditions are trees in $C^{<\omega}$ with branching into all of C except for finitely many points). Clearly $\mathbb{D}_C$ is isomorphic to the ordinary Hechler forcing. Notice that for every $T \in \mathbb{L}_F \cap L[r]$, if $\mathrm{ran}(\mathrm{stem}(T)) \subseteq C$ then $T \cap C^{<\omega} \in \mathbb{D}_C$.

For every $D \in L[r]$ dense in $\mathbb{L}_F$, let $D' := \{T \cap C^{<\omega} \mid T \in D \text{ and } \mathrm{ran}(\mathrm{stem}(T)) \subseteq C\}$. We claim that D' is predense in $\mathbb{D}_C$. Let $S \in \mathbb{D}_C$ be arbitrary, with $\sigma := \mathrm{stem}(S)$. Recall the rank-function from Definition 5.8. Since $D \in L[r]$, we consider the rank function rk_D as defined inside $L[r]$. If $\mathrm{rk}_D(\sigma) = 0$ then there is $T \in D$ with $\mathrm{stem}(T) = \sigma$, hence S and T are compatible. Otherwise, let $\mathrm{rk}_D(\sigma) = \alpha$. By definition of rk_D and the fact that rk_D is in $L[r]$, there exists $Z \in F^+$ with $Z \in L[r]$, such that $\mathrm{rk}_D(\sigma^\frown \langle n \rangle) < \alpha$ for all $n \in Z$. Since $\mathrm{Succ}_S(\sigma) \cap Z$ is also in F^+, by assumption, there is $n \in C \cap \mathrm{Succ}_S(\sigma) \cap Z$. Continuing this process, we arrive at some τ extending σ, such that $\tau \in S$, $\mathrm{ran}(\tau) \subseteq C$ and $\mathrm{rk}_D(\tau) = 0$. Then we are done as before.

By the remark above, $\mathbf{\Sigma}^1_2(\mathbb{C})$ implies $\mathbf{\Delta}^1_2(\mathbb{D})$, which implies the existence of Hechler-generic reals. In particular, there is a $d \in C^\omega$ which is $\mathbb{D}_C$-generic real over $L[r][C]$. But then d is $\mathbb{L}_F$-generic over $L[r]$, since for every $D \in L[r]$ dense in $\mathbb{L}_F$, we find $T \in D$ with $d \in [T \cap C^{<\omega}]$. □

A similar argument can be used to simplify $\mathbf{\Delta}^1_2(\mathbb{L}_{F^+})$. Here, the homogeneity of $\mathbb{L}_{F^+}$ is guaranteed by the K-uniformity of F.

Theorem 5.16. $\mathbf{\Delta}^1_2(\mathbb{L}_{F^+}) \implies \mathbf{\Delta}^1_2(\mathbb{L}) \wedge \exists F\text{-}\mathsf{pseudoint}$. *If F is K-uniform, then the converse implication holds.*

Proof. By Corollary 4.8 we know that $\mathbf{\Delta}^1_2(\mathbb{L}_{F^+}) \Rightarrow \mathbf{\Delta}^1_2(\mathbb{L})$. Let x be F-dominating over $L[r]$ and let $C := \mathrm{ran}(x)$. For each $a \in F \cap L[r]$ let φ be the function given by $\varphi(\sigma) := a \setminus |\sigma|$ for all $\sigma \in \omega^{<\omega}$. Since $\forall^\infty n\ (x(n) \in \varphi(x \restriction n))$, clearly C is infinite and $C \subseteq^* a$.

Conversely, assume that F is K-uniform. We leave it to the reader to verify that, if $T \in \mathbb{L}_{F^+}$, then there exists a continuous function $f : \omega^\omega \to [T]$ such that f-preimages of $\mathcal{D}_{F^+}$-small sets are $\mathcal{D}_{F^+}$-small. In particular, the statements

- $\exists x\ (x \text{ is } F\text{-dominating over } L[r])$, and
- for all $T \in \mathbb{L}_{F^+}\ \exists x \in [T]\ (x \text{ is } F\text{-dominating over } L[r])$

are equivalent. Therefore, by Corollary 5.6 (3), it suffices to prove that for every r there exists an F-dominating real over $L[r]$.

So, fix $r \in \omega^\omega$, and let C be F-pseudointersecting over $L[r]$. For each $\varphi : \omega^{<\omega} \to F$ from $L[r]$, define $g_\varphi : \omega^{<\omega} \to \omega$ by $g_\varphi(\sigma) := \min\{n \mid C \setminus n \subseteq \varphi(\sigma)\}$. Then all g_φ are in $L[r][C]$, by $\mathbf{\Delta}^1_2(\mathbb{L})$ there is a dominating real g over $L[r][C]$, so, in particular, g dominates all g_φ. Let $x \in \omega^\omega$ be such that $x(n) \in C$ and $x(n) \geq g(x \restriction n)$ for every n. Clearly for every $\varphi \in L[r]$ we have $\forall^\infty n\ (x(n) \in \varphi(x \restriction n))$, hence x is F-dominating over $L[r]$. This suffices by what we mentioned above. □

Currently, we do not have a similarly elegant characterization for $\boldsymbol{\Sigma}^1_2(\mathbb{L}_{F^+})$.

Question 5.17. *Is there a characterization of* $\boldsymbol{\Sigma}^1_2(\mathbb{L}_{F^+})$ *similar to the above? Is* $\boldsymbol{\Sigma}^1_2(\mathbb{L}_F)$ *equivalent to* $\boldsymbol{\Delta}^1_2(\mathbb{L}_{F^+})$*?*

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3.10 Definable Maximal Independent Families

Jörg Brendle, Yurii Khomskii, preprint.

This research started as a joint project during the workshop "New challenges in iterated forcing" in Norwich, in November 2015. The main results (Theorem 4.2 and 5.2) were essentially proved during that time. The project lay dormant for a while but some more results were proved recently during the research visit of Brendle to Hamburg in June 2017. The authorship is as follows:

1. ***Theorem 2.1**: Brendle*
2. ***Lemma 3.4**: Collaboration*
3. ***Lemma 3.6**: Collaboration*
4. ***Corollary 3.8**: Collaboration*
5. ***Theorem 4.2**: Collaboration*
6. ***Theorem 5.2**: Collaboration*

The paper was written by Khomskii.

Definable Maximal Independent Families

Jörg Brendle* and Yurii Khomskii†

July 15, 2017

Abstract

We study maximal independent families for sets in the projective hierarchy. Our main result shows that in the Cohen model, there are no maximal independent families. We also consider a new cardinal invariant related to the question of destroying and preserving maximal independent families.

1 Introduction

In descriptive set theory, one often looks at objects defined in a con-constructive way, such as ultrafilters, Bernstein-type sets, maximal almost disjoint families etc., and asks the question "how low in the projective hierarchy do such objects first appear"? In this paper, we look at maximal independent families, a close relative of the maximal almost disjoint families studied in this way by the same authors in [1].

Definition 1.1. A family $\mathcal{I} \subseteq [\omega]^\omega$ is called *independent* if whenever we choose finite disjoint $F, G \subseteq \mathcal{I}$, we get

$$\sigma(F;G) := \left(\bigcap_{A \in F} A\right) \cap \left(\bigcap_{B \in G} (\omega \setminus B)\right) \text{ is infinite.}$$

A family $\mathcal{I} \subseteq [\omega]^\omega$ is called a *maximal independent family* (m.i.f.) if it is independent and maximal with regard to this property.

Note that maximality of $\mathcal{I}$ is equivalent to:

$$\forall X \in [\omega]^\omega \; \exists F \in [\mathcal{I}]^{<\omega} \; \exists G \in [\mathcal{I} \setminus F]^{<\omega} \; (\sigma(F;G) \subseteq^* X \;\vee\; \sigma(F;G) \cap X =^* \varnothing).$$

By identifying the space $[\omega]^\omega$ with 2^ω via characteristic functions, one can consider independent families as subsets of the reals and study their complexity in the projective hieararchy.

Remark 1.2. If $\mathcal{I}$ is a $\mathbf{\Sigma}^1_n$ m.i.f. then it is $\mathbf{\Delta}^1_n$.

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Proof. Suppose $\mathcal{I}$ is a $\mathbf{\Sigma}^1_n$ m.i.f. Then $\forall X \in [\omega]^\omega$:

$$X \notin \mathcal{I} \iff \exists F \in [\mathcal{I}]^{<\omega}\ \exists G \in [\mathcal{I} \setminus F]^{<\omega}$$
$$(X \notin F \wedge X \notin G \wedge (\sigma(F;G) \subseteq^* X \vee \sigma(F;G) \cap X =^* \varnothing)).$$

The last statement is easily seen to be $\mathbf{\Sigma}^1_n$. □

Theorem 1.3 (Miller; [2]). *There is no analytic m.i.f.*

An analysis of Miller's proof shows the following stronger result: $\mathbf{\Sigma}^1_n(\mathbb{C}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f., for all n, where we use "$\mathbf{\Sigma}^1_n(\mathbb{C})$" to denote the statement "all $\mathbf{\Sigma}^1_n$ sets have the Baire property". In particular, it follows that in the Cohen model there is no $\mathbf{\Sigma}^1_2$ m.i.f., that in the Solovay and the Shelah model (for projective Baire Property without inaccessible) there is no m.i.f. at all, and that $\mathsf{AD} \Rightarrow$ there is no m.i.f.

In this paper, we prove a much stronger result, namely, that in the Cohen model there is no projective m.i.f. Since $\mathbf{\Sigma}^1_2(\mathbb{C})$ is false in the Cohen model, this will show that the above implication cannot be reversed in general.

On the other hand, it is easy to construct a m.i.f. by induction using a wellorder of the reals. In particular, it is easy to see that in L, there exists a $\mathbf{\Sigma}^1_2$ m.i.f. In [2], Miller used sophisticated coding techniques to show that, in fact, there is a $\mathbf{\Pi}^1_1$ m.i.f. in L. Building on an idea due to Asger Törnquist [4], we will show that in fact this proof is unneccessary, since one can show directly in ZFC that if there exists a $\mathbf{\Sigma}^1_2$ m.i.f. then there exists a $\mathbf{\Pi}^1_1$ m.i.f.

The paper is structured as follows: in Section 2, we prove the implication mentioned above. In Section 3 we present a break-down of Miller's original proof necessary for further development. In Section 4 we prove the main theorem about projective m.i.f.'s in the Cohen model, and in Section 5 we study a cardinal invariant related to the question of preserving or destroying a m.i.f.

2 $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_1$ m.i.f's

Theorem 2.1. *If there exists a $\mathbf{\Sigma}^1_2$ m.i.f. then there exists a $\mathbf{\Pi}^1_1$ m.i.f.*

Proof. Suppose $\mathcal{I}_0$ is a $\mathbf{\Sigma}^1_2$ maximal independent family. Let $F_0 \subseteq ([\omega]^\omega)^2$ be a $\mathbf{\Pi}^1_1$ set such that $\mathcal{I}_0$ is the projection of F_0. Consider the space $\omega \,\dot\cup\, 2^{<\omega}$ as a disjoint union, and consider the mapping

$$g: \begin{array}{l} ([\omega]^\omega)^2 \longrightarrow \mathscr{P}(\omega \,\dot\cup\, 2^{<\omega}) \\ (x,y) \longmapsto x \cup \{\chi_y{\restriction}n \mid n < \omega\} \end{array}$$

where χ_y is the characteristic function of y. It is not hard to see that g is a continuous function (in the sense of the space $\mathscr{P}(\omega \,\dot\cup\, 2^{<\omega})$).

By $\mathbf{\Pi}^1_1$-uniformization, there exists a $\mathbf{\Pi}^1_1$ set $F \subseteq F_0$ which is the graph of a function, i.e., $\forall x \in \mathcal{I}_0\ \exists! y\ ((x,y) \in F)$. We let $\mathcal{I} := g[F]$ and claim that $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ m.i.f.

To see that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$, note that for $z \in [\omega \,\dot\cup\, 2^{<\omega}]^\omega$, there is an explicit way to recover x and y such that $g(x,y) = z$, if such x and y exist. More precisely: for $B \subseteq 2^{<\omega}$, let $\lim(B) := \{y \in 2^\omega \mid \forall n\ (y{\restriction}n \in B)\}$. Note that if B is infinite then $\lim(B) \neq \varnothing$. Then we can say the following: $z \in \mathcal{I}$ if and only if

1. $\forall y, y' \ (y \in \lim(z \cap 2^{<\omega}) \wedge y' \in \lim(z \cap 2^{<\omega}) \ \rightarrow \ y = y')$, and
2. $\forall y \ (y \in \lim(z \cap 2^{<\omega}) \ \rightarrow \ (z \cap \omega, y) \in F)$.

This gives a $\mathbf{\Pi}^1_1$ definition of $\mathcal{I}$.

To see that $\mathcal{I}$ is independent, suppose we have $z_1, \dots z_n$ and $w_1, \dots w_\ell \in \mathcal{I}$, the z's being different from the w's. Write $a_i := z_i \cap \omega$ and $b_j := w_j \cap \omega$. Then all a_i and b_j are in $\mathrm{dom}(F) = \mathcal{I}_0$, and moreover, since F is a function, the a_i's are different from the b_j's. But then we have that $\sigma(z_1, \dots, z_n; w_1, \dots, w_\ell) \supseteq \sigma(a_1, \dots, a_n; b_1, \dots, b_\ell)$ is infinite, since the latter set is infinite by the independence of $\mathcal{I}_0$.

To show maximality of $\mathcal{I}$, suppose $W \in [\omega \mathbin{\dot\cup} 2^{<\omega}]^\omega$ and $W \notin \mathcal{I}$. Let $A := W \cap \omega$. By maximality of $\mathcal{I}_0$, there are $a_1, \dots, a_n$ and different $b_1, \dots, b_\ell$ such that $\sigma(a_1, \dots, a_n, A; \ b_1, \dots b_\ell)$ is finite or $\sigma(a_1, \dots, a_n; \ b_1, \dots b_\ell \cup \{A\})$ is finite, w.l.o.g. the former. Then there are $z_1, \dots, z_n$ and different $w_1, \dots, w_\ell$ such that $a_i = z_i \cap \omega$ and $b_j = w_j \cap \omega$. To make sure that the "$2^{<\omega}$-part" of the z_i's and the w_j's does not make the intersection infinite, we pick two additional $t_0 \neq t_1 \in \mathcal{I}$, different from the z_i's and the w_j's. Let $t_0 = g(x_0, y_0)$ and $t_1 = g(x_1, y_1)$. If $y_0 = y_1$, then $(t_0 \setminus t_1) \cap 2^{<\omega} = \varnothing$, hence $\sigma(x_1, \dots, x_n, W, t_0; \ b_1, \dots b_\ell, t_1)$ is finite. If, on the other hand, $y_0 \neq y_1$, then the sets $\{\chi_{y_0} \restriction n \ | \ n < \omega\}$ and $\{\chi_{y_1} \restriction n \ | \ n < \omega\}$ are almost disjoint, so $(t_0 \cap t_1) \cap 2^{<\omega}$ is finite. In that case, $\sigma(x_1, \dots, x_n, W, t_0, t_1; \ b_1, \dots b_\ell)$ is finite. So in any case, $\mathcal{I} \cap \{W\}$ is not independent, completing the proof. □

3 Perfect almost disjoint and almost covering sets

Next, we turn our attention to Miller's original proof of the non-existence of analytic m.i.f.'s., using it to prove a stronger result and breaking it down a bit, using the following definition.

Definition 3.1. A tree $T \subseteq 2^{<\omega}$ is called *perfect almost disjoint* (*perfect a.d.*) if it is a perfect tree and $\forall x, y \in [T] \ \{n \mid x(n) = y(n) = 1\}$ is finite. A tree $S \subseteq 2^{<\omega}$ is called *perfect almost covering* (*perfect a.c.*) if it is a perfect tree and $\forall x, y \in [T] \ \{n \mid x(n) = y(n) = 0\}$ is finite.

Definition 3.2.

1. A set $A \subseteq 2^\omega$ satisfies the *perfect-a.d.-a.c. property*, abbreviated by $\mathbb{S}_{ad\text{-}ac}$, if there exists a perfect a.d. tree T with $[T] \subseteq A$, or there exists a perfect a.c. tree S with $[S] \cap A = \varnothing$.
2. A set $A \subseteq 2^\omega$ satisfies the *perfect-a.c.-a.d. property*, abbreviated by $\mathbb{S}_{ac\text{-}ad}$, if there exists a perfect a.c. tree S with $[S] \subseteq A$, or there exists a perfect a.d. tree T with $[T] \cap A = \varnothing$.

Question 3.3. *Do the statements $\mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac})$ or $\mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad})$ have any interesting characterisations and/or has anything like this ever been studied previously, for example for $\mathbf{\Gamma} = \mathbf{\Delta}^1_2$ or $\mathbf{\Gamma} = \mathbf{\Sigma}^1_2$?*

Lemma 3.4. $\mathbf{\Gamma}(\mathbb{C}) \Rightarrow \mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac}) \wedge \mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad})$ *for any projective pointclass* $\mathbf{\Gamma}$.

Proof. Let $A \subseteq 2^\omega$ be in $\mathbf{\Gamma}$. If A has the Baire property, in particular there is a basic open set $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$ (here $\subseteq^*$ and $=^*$ denote modulo meager). If we assume the former, we will find *both* a perfect a.d. tree T and a perfect a.c. tree S such that

$[T] \subseteq A$ and $[S] \subseteq A$. Analogously, if we assume the latter, we will find *both* a perfect a.d. tree T and a perfect a.c. tree S such that $[T] \cap A = \varnothing$ and $[S] \cap A = \varnothing$. Thus, it is clear that in effect we prove both $\mathbf{\Gamma}(\mathbb{S}_{ad\text{-}ac})$ and $\mathbf{\Gamma}(\mathbb{S}_{ac\text{-}ad})$. We only show how to construct the perfect almost disjoint tree T in the former case; the other cases are similar.

So, assume $[s] \subseteq^* A$ and let B_n be nowhere dense so that $[s] \setminus A \subseteq \bigcup_n B_n$.

- Let $s_\varnothing$ be an extension of s with at least one (new) non-zero digit, and such that $[s_\varnothing] \cap B_0 = \varnothing$. Let $k_0 := |s_\varnothing|$.
- Let $s_{\langle 0 \rangle}$ be an extension of $s_\varnothing$, with at least one new non-zero digit, and such that $[s_{\langle 0 \rangle}] \cap B_1 = \varnothing$. Let $k_1 := |s_{\langle 0 \rangle}|$.
- Let $s_{\langle 1 \rangle}$ be an extension of $s_\varnothing$ consisting only of 0's on the interval $[k_0, k_1)$, followed by an arbitrary extension with at least one non-zero digit, and such that $[s_{\langle 1 \rangle}] \cap B_1 = \varnothing$. Let $k_2 := |s_{\langle 1 \rangle}|$.
- Let $s_{\langle 0,0 \rangle}$ be an extension of $s_{\langle 0 \rangle}$, consisting only of 0's on the interval $[k_1, k_2)$, followed by an arbitrary extension with at least one non-zero digit, and such that $[s_{\langle 0,0 \rangle}] \cap B_2 = \varnothing$. Let $k_3 := |s_{\langle 0,0 \rangle}|$.
- Let $s_{\langle 0,1 \rangle}$ be an extension of $s_{\langle 0 \rangle}$, consisting only of 0's on the interval $[k_1, k_3)$, followed by an arbitrary extension with at least one non-zero digit, and such that $[s_{\langle 0,1 \rangle}] \cap B_2 = \varnothing$. Let $k_4 := |s_{\langle 0,0 \rangle}|$.
- Continue in the same way: $s_{\sigma ^\frown \langle i \rangle}$ extends s_σ with only 0's until the largest k_j which has been defined, followed by an arbitrary extension with at least one non-zero digit, such that $[s_{\sigma ^\frown \langle i \rangle}] \cap B_{|\sigma|+1} = \varnothing$.

Finally let T be the tree generated by $\{s_\sigma \mid \sigma \in 2^{<\omega}\}$. This is a perfect tree (because of the "new non-zero digit"), and clearly $[T] \subseteq [s] \cap A$. The construction clearly guarantees that $[T]$ is an almost disjoint tree.

To construct the perfect almost covering tree S in A, proceed analogously replacing "0" by "1" in the proof above. $\square$

Remark 3.5. An equivalent formulation of the above lemma is: "for every countable model M there exists a perfect almost disjoint set and perfect almost covering set of Cohen reals over M".

Lemma 3.6. *$\mathbf{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f. and $\mathbf{\Sigma}^1_n(\mathbb{S}_{ac\text{-}ad}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f..*

Proof. We prove both statements simultaneously. Let $\mathcal{I}$ be $\mathbf{\Sigma}^1_n$, and assume, towards contradiction, that $\mathcal{I}$ is a m.i.f. Let

$$\begin{aligned} H &:= \{X \mid \exists F \in [\mathcal{I}]^{<\omega}\ \exists G \in [\mathcal{I} \setminus F]^{<\omega}\ (\sigma(F;G) \subseteq^* X)\} \\ K &:= \{X \mid \exists F \in [\mathcal{I}]^{<\omega}\ \exists G \in [\mathcal{I} \setminus F]^{<\omega}\ (\sigma(F;G) \cap X =^* \varnothing)\} \end{aligned}$$

Then both H and K are $\mathbf{\Sigma}^1_n$ sets. Moreover, by maximality of $\mathcal{I}$, $[\omega]^\omega = H \cup K$.

Assume that $\mathbf{\Sigma}^1_n(\mathbb{S}_{ad\text{-}ac})$ was true. Then, applying this property to H, we either obtain a perfect almost disjoint tree T with $[T] \subseteq H$, or a perfect almost coverting tree S with $[S] \cap H = \varnothing$, hence $[S] \subseteq K$ (note that here, and in the rest of the proof, we identify subsets of ω with their characteristic function).

Alternatively, assume that $\mathbf{\Sigma}^1_n(\mathbb{S}_{ac\text{-}ad})$ was true. Then, applying this property to K, we either obtain a perfect almost covering tree S with $[S] \subseteq K$, or a perfect almost disjoint tree T with $[T] \cap K = \varnothing$, and therefore $[T] \subseteq H$.

In both cases, the proof proceeds analogously.

First assume there is a perfect almost disjoint T with $[T] \subseteq H$. For each $X \in [T]$ let F_X, G_X witness the fact that $X \in H$, and apply the Δ-systems Lemma to find distinct $X, Y \in [T]$ such that $(F_X \cup F_Y) \cap (G_X \cup G_Y) = \varnothing$. Then $\sigma(F_X \cup F_Y; G_X \cup G_Y) \subseteq^* X \cap Y =^* \varnothing$, contradicting the independence of $\mathcal{I}$.

Similarly, assume there is a perfect almost covering S with $[S] \subseteq K$, and proceed analogously. Then we obtain $\sigma(F_X \cup F_Y; G_X \cup G_Y) \cap (X \cup Y) =^* \varnothing$. But by assumption $(X \cup Y) =^* \omega$ so this implies that $\sigma(F_X \cup F_Y; G_X \cup G_Y) =^* \varnothing$, again contradicting the independence of $\mathcal{I}$. □

Corollary 3.7. *$\mathbf{\Sigma}^1_n(\mathbb{C}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$ m.i.f.*

Remark 3.8. A curious aspect of this corollary is that the proof can proceed either via $\mathbb{S}_{ad\text{-}ac}$ or via $\mathbb{S}_{ac\text{-}ad}$; in fact, considering just any one of these dichotomy properties would be sufficient (see Figure 1), and in the proof of Lemma 3.6 it would be sufficient for just H or just K to be $\mathbf{\Sigma}^1_2$.

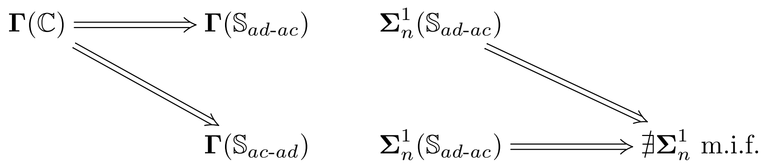

Figure 1: Impications in ZFC.

Question 3.9. *Can we strengthen Lemma 3.6 to $\mathbf{\Delta}^1_n(\mathbb{S}_{ad\text{-}ac}) \Rightarrow \nexists \mathbf{\Sigma}^1_n$-m.i.f.? (Note that $\mathbf{\Delta}^1_n(\mathbb{S}_{ad\text{-}ac})$ and $\mathbf{\Delta}^1_n(\mathbb{S}_{ac\text{-}ad})$ are equivalent).*

4 Projective m.i.f's

The general question is: in which models do m.i.f.'s of complexity $\mathbf{\Gamma}$ exist? A recent abstract result of Schrittesser [3] shows:

Fact 4.1 (Schrittesser 2016). *In the iterated Sacks model (of any length) starting from L, there exists a (lightface) Δ^1_2 m.i.f.*

Theorem 4.2. *In the Cohen model there are no projective m.i.f.'s*

What we actually show is that in the Cohen model all projective sets (and even all sets in $L(\mathbb{R})$) satisfy $\mathbb{S}_{ad\text{-}ac}$ and $\mathbb{S}_{ac\text{-}ad}$. The main point is the following Lemma, closely related to Lemma 3.4:

Lemma 4.3. *If $c \in [s]$ is Cohen over V, then in $V[c]$ there exists a perfect almost disjoint set and a perfect almost covering set of Cohen reals over V, contained in $[s]$.*

Proof. Let $\mathbb{P}$ denote the partial order consisting of finite trees $T \subseteq 2^{<\omega}$ with the following property: $\exists k_0 < k_1 < \cdots < k_\ell$ such that $T \subseteq 2^{\leq k_\ell}$, and for every $i < \ell$, there is at most one $t \in T$ where $t \restriction [k_i, k_{i+1})$ is not constantly 0 (notice that the tree constructed in the proof of Lemma 3.4 has this property). The trees are ordered by end-extension.

Notice that $\mathbb{P}$ generically adds a perfect tree T_G, defined as the limit of the trees in G. Moreover, using an analogous idea to Lemma 3.4, we can see that T is almost disjoint and that every $x \in [T_G]$ is Cohen-generic over the ground model.

Since $\mathbb{P}$ is countable, it is isomorphic to Cohen forcing. Therefore, if $V[c]$ is a Cohen extension of V, it is also a $\mathbb{P}$-generic extension of V, so there exists a perfect almost disjoint set $[T_G]$ of Cohen reals. W.l.o.g. T_G can be assumed to be within $[s]$.

To obtain a perfect almost covering set of Cohen reals in $[s]$, apply the same argument with "0" replaced by "1". $\square$

Proof of Theorem 4.2. Let $W := V^{\mathbb{C}_\kappa}$ (for any κ), and let A be a set in W defined by a formula $\phi(x)$ with real or ordinal parameters, w.l.o.g. all of which are in V (so we can forget about them). In W, let c be Cohen over V, and assume w.l.o.g. that $\phi(c)$. Then $V[c] \models$ "$p \Vdash_{\mathbb{Q}} \phi(\check{c})$", where $\mathbb{Q}$ is the remainder forcing leading from $V[c]$ to W and p is some $\mathbb{Q}$-condition. However, since $\mathbb{C}_\kappa$ is the product forcing, $\mathbb{Q}$ is isomorphic to $\mathbb{C}_\kappa$. Moreover, since $\mathbb{C}_\kappa$ is homogeneous we can assume that p is the trivial condition, hence we really have:

$$V[c] \models \text{“} \Vdash_{\mathbb{C}_\kappa} \phi(\check{c})\text{”}$$

Let $[s]$ be a Cohen condition with $c \in [s]$ forcing this statement in V. By Lemma 4.3, first we find a perfect a.d. tree T with $T \in V[c]$, $[T] \subseteq [s]$ and such that all $x \in [T]$ are Cohen over V. Note that this fact remains true in W, since "being a perfect set of Cohen reals" is upwards absolute. Now, for any such $x \in [T]$ (in W), we have that $x \in [s]$, and therefore $V[x]$ satisfies whatever $[s]$ forces, in particular

$$V[x] \models \text{“} \Vdash_{\mathbb{C}_\kappa} \phi(\check{x})\text{”}$$

But, again, the remainder forcing leading from $V[x]$ to W is isomorphic to $\mathbb{C}_\kappa$, and it follows that $W \models \phi(x)$.

Similarly, we also find a perfect a.c. tree S with exactly the same properties. Thus A satisfies both $\mathbb{S}_{\textit{ad-ac}}$ and $\mathbb{S}_{\textit{ac-ad}}$, and the rest follows by Lemma 3.6. $\square$

5 $\aleph_1$-Borel and $\aleph_1$-closed m.i.f's

The question of definable m.i.f's is closely related to questions concerning certain cardinal invariants (compare with [1]).

Definition 5.1.

1. $\mathfrak{i}$ is the least size of a m.i.f.
2. $\mathfrak{i}_{\mathrm{cl}}$ is the least κ such that there exists a collection $\{C_\alpha \mid \alpha < \kappa\}$, where each C_α is a **closed** independent family, and $\bigcup_{\alpha<\kappa} C_\alpha$ is a m.i.f.
3. $\mathfrak{i}_B$ is the least κ such that there exists a collection $\{B_\alpha \mid \alpha < \kappa\}$, where each B_α is a **Borel** independent family, and $\bigcup_{\alpha<\kappa} B_\alpha$ is a m.i.f.

It is clear that $\mathfrak{i}_B \leq \mathfrak{i}_{\text{cl}} \leq \mathfrak{i}$. It is also known that $\mathfrak{r} \leq \mathfrak{i}$ and $\mathfrak{d} \leq \mathfrak{i}$, where $\mathfrak{d}$ and $\mathfrak{r}$ denote the dominating and reaping numbers, respectively. Notice that if $\mathfrak{i}_B > \aleph_1$, then there are no $\mathbf{\Sigma}^1_2$ m.i.f.'s (since $\mathbf{\Sigma}^1_2$-sets are $\aleph_1$-unions of Borel sets).

Theorem 5.2. $\text{cov}(\mathcal{M}) \leq \mathfrak{i}_B$.

Proof. Let $\kappa < \text{cov}(\mathcal{M})$ and let $\{B_\alpha \mid \alpha < \kappa\}$ be a collection of Borel independent families. We need to show that $\mathcal{I} := \bigcup_{\alpha<\kappa} B_\alpha$ is not maximal.

Suppose otherwise, and for every finite $E \subseteq \kappa$ define

$$H_E := \{X \mid \exists F \in [\textstyle\bigcup_{\alpha\in E} B_\alpha]^{<\omega} \; \exists G \in [\bigcup_{\alpha\in E} B_\alpha \setminus F]^{<\omega} \; (\sigma(F;G) \subseteq^* X)\}$$

$$K_E := \{X \mid \exists F \in [\textstyle\bigcup_{\alpha\in E} B_\alpha]^{<\omega} \; \exists G \in [\bigcup_{\alpha\in E} B_\alpha \setminus F]^{<\omega} \; (\sigma(F;G) \cap X =^* \varnothing)\}$$

Notice that by maximality of $\mathcal{I} = \bigcup_{\alpha<\kappa} B_\alpha$, we have

$$\bigcup \{H_E \cup K_E \mid E \in [\kappa]^{<\omega}\} = [\omega]^\omega.$$

Since $\kappa < \mathfrak{d} = \text{cov}(K_\sigma)$, there must exist a finite $E \subseteq \kappa$ such that $H_E \cup K_E \notin \mathcal{M}$. Suppose $H_E \notin \mathcal{M}$: since H_E is analytic, there exists a basic open $[s]$ with $[s] \subseteq^* H_E$. By the argument from Lemma 3.4, there exists a perfect a.d. tree T with $[T] \subseteq H_E$. But then, by the argument from Lemma 3.6, it follows that $\bigcup_{\alpha\in E} B_\alpha$ is not independent, contrary to the assumption. Likewise, if $K_E \notin \mathcal{M}$ then using the argument from Lemma 3.4, there exists a perfect a.c. tree S with $[S] \subseteq H_K$, and the rest is the same. ☐

We end this section with the following open questions:

Question 5.3.

1. *Is it consistent that* $\mathfrak{i}_{\text{cl}} < \mathfrak{d}$ *or* $\mathfrak{i}_B < \mathfrak{d}$*?*
2. *Is it consistent that* $\mathfrak{i}_{\text{cl}} < \mathfrak{r}$ *or* $\mathfrak{i}_B < \mathfrak{r}$*?*
3. *Is it consistent that* $\mathfrak{i}_{\text{cl}} < \mathfrak{i}$ *or* $\mathfrak{i}_B < \mathfrak{i}$*?*
4. *Can we have* $\mathfrak{d} > \aleph_1$ *or* $\mathfrak{r} > \aleph_1$ *together with a* $\mathbf{\Sigma}^1_2$ *m.i.f.?*
5. *Does the existence of a* $\mathbf{\Sigma}^1_{n+1}$ *m.i.f. imply the existence of a* $\mathbf{\Pi}^1_n$ *m.i.f. for* $n > 2$*?*

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