Unbeatable Strategies

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"Stochastic Dynamics in Economics and Finance"

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Part II

1. Lebesgue measure
2. Related properties (no proofs)
3. Flip Sets
4. Wadge reducibility

Recall what we did

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Definition

Let $\mathbf{\Gamma} \subseteq \mathbb{N}^{\mathbb{N}}$ be a (usually topological) class of sets. $\text{Det}(\mathbf{\Gamma})$ abbreviates the statement "for all $A \in \mathbf{\Gamma}$, the infinite game $G(A)$ is determined".

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We have seen:

- Det(Open) and Det(Closed) (Gale-Stewart, 1953).
- $\text{Det}(F_\sigma)$ and $\text{Det}(G_\delta)$ (Wolfe, 1955).
- $\text{Det}(F_{\sigma\delta})$ and $\text{Det}(G_{\delta\sigma})$ (Morton Davis, 1964).
- Det(Borel) (Tony Martin, 1975).
- Assuming "large cardinals", Det(projective) (Martin-Steel, 1989).
- $\text{AD} = \text{Det}(\mathcal{P}(\mathbb{N}^{\mathbb{N}}))$; it is inconsistent with AC.

What we will do today

The results we prove today have the following pattern: if P is some property of sets (subsets of $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{R}$), construct a game G' and prove that **if** $G'(A)$ is determined **then** A satisfies P.

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However, for each such result, there is a corresponding **local** version:

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If $\boldsymbol{\Gamma}$ is a class satisfying certain closure properties, then $\text{Det}(\boldsymbol{\Gamma}) \Longrightarrow$ *all sets $A \in \boldsymbol{\Gamma}$ satisfy P.*

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For the second result, we need to check that the **coding** we use is sufficiently simple (we will skip this).

1. Lebesgue measure

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The local version

Assume $\mathbf{\Gamma}$ *is closed under continuous pre-images, finite unions, intersections and complements, and contains the* F_σ *sets. Then* $\text{Det}(\mathbf{\Gamma}) \Rightarrow$ *all sets in* $\mathbf{\Gamma}$ *are measurable.*

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The original proof is due to Mycielski-Świerczkowski (1964) but we present a proof of Harrington.

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- Fix an enumeration $\{I_n \mid n \in \mathbb{N}\}$ of all possible finite unions of open intervals in $[0,1]$ with rational endpoints (there are only countably many).
- For $x \in 2^{\mathbb{N}}$, let $a : 2^{\mathbb{N}} \longrightarrow [0,1]$ be the function given by

$$a(x) := \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$$

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Easy to see that $a : 2^{\mathbb{N}} \to [0,1]$ is continuous and $\mathrm{ran}(a) = [0,1]$ (think of x as the binary expansion of $a(x)$).

The Covering Game

Given $A \subseteq [0,1]$ and $\epsilon > 0$, we define a game $G_\mu(A,\epsilon)$.

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Definition ($G_\mu(A,\epsilon)$)

$$\begin{array}{r||cccccccc} \text{I}: & x_0 & & x_1 & & x_2 & & \dots \\ \hline \text{II}: & & y_0 & & y_1 & & y_2 & \end{array}$$

- $x_i \in \{0,1\}$ and $y_i \in \mathbb{N}$.
- At every move n, Player II must make sure that
$$\mu(I_{y_n}) < \frac{\epsilon}{2^{2(n+1)}}$$
- Player I wins iff $a(x) \in A \setminus \bigcup_{n=0}^{\infty} I_{y_n}$.

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Intuition: I attempts to play a real number in A, while II attempts to "cover" that real number with the I_n's (of an increasingly smaller measure.)

The main result

Theorem

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ and ϵ be given.

1. *If I has w.s. in $G_\mu(A, \epsilon)$ then there is a measurable $Z \subseteq A$ with $\mu(Z) > 0$.*
2. *If II has w.s. in $G_\mu(A, \epsilon)$ then there is an open O such that $A \subseteq O$ and $\mu(O) < \epsilon$.*

Proof

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1. Let σ be winning for I. Define
   - $f(z)(n) := z(2n)$ and
   - $g(z)(n) := z(2n+1)$.

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It is clear that both f and g are continuous (from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$), and also the mapping $y \mapsto \sigma * y$ is continuous. Hence $y \mapsto a(f(\sigma * y))$ is continuous. Let $Z := \{a(f(\sigma * y)) \mid y \in \mathbb{N}^{\mathbb{N}}\}$. This is an **analytic** set (continuous image of a closed set), hence measurable. As σ was winning, $Z \subseteq A$.

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But if $\mu(Z) = 0$ then Z can be covered by $\{I_{y_n} \mid n \in \mathbb{N}\}$ satisfying $\forall n\, (\mu(I_{y_n}) < \frac{\epsilon}{2^{2(n+1)}})$. Then if II plays $y = \langle y_0, y_1, \ldots \rangle$

$$a(f(\sigma * y)) \in Z \subseteq \bigcup_{n=0}^{\infty} I_{y_n},$$

contradicting that σ is winning for I.

Proof (continued)

2. Now suppose τ is winning for II. For every $s \in \{0,1\}^*$ of length n, define

$$I_s := I_{(s*\rho)(2n-1)}$$

(I_s is the $I_{y_{n-1}}$ where y_{n-1} is the last move of the game in which I played s and II used τ). As τ is winning for II, for every $a \in A$ and every $x \in 2^{\mathbb{N}}$ such that $a(x) = a$, there must be some n such that $a \in I_{x \restriction n}$. In other words, $a \in \bigcup\{I_s \mid s \lhd x\}$ where x is such that $a(x) = a$.

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$$A \subseteq \bigcup_{s \in 2^{\mathbb{N}}} I_s = \bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} I_s.$$

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$$\mu(\bigcup_{s \in \{0,1\}^n} I_s) < \frac{\epsilon}{2^{2n}} \cdot 2^n = \frac{\epsilon}{2^n}.$$

Proof (continued)

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So, indeed, A is contained in an open set of measure $< \epsilon$. □

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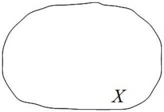
Proof. Let $\mu^*(X) = \delta$.



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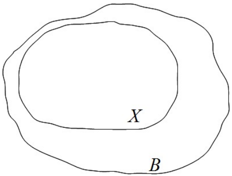

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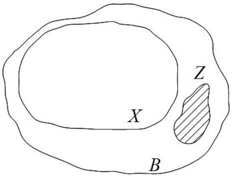
Proof. Let $\mu^*(X) = \delta$. Let B be a G_δ set such that $X \subseteq B$ and $\mu(B) = \delta$.
Now consider the games $G_\mu(B \setminus X, \epsilon)$, for all ϵ. If, for at least one $\epsilon > 0$, I has a w.s., then there is a measurable set $Z \subseteq B \setminus X$ of positive measure, contradicting $\mu^*(X) = \delta$.



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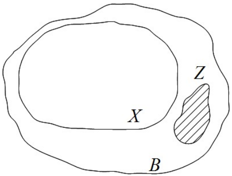
Now consider the games $G_\mu(B \setminus X, \epsilon)$, for all ϵ. If, for at least one $\epsilon > 0$, I has a w.s., then there is a measurable set $Z \subseteq B \setminus X$ of positive measure, contradicting $\mu^*(X) = \delta$.

Hence, by determinacy, II must have a w.s. in $G_\mu(B \setminus X, \epsilon)$ **for every** $\epsilon > 0$. Hence $B \setminus X \subseteq O$ for $\mu(O) < \epsilon$, for every $\epsilon > 0$, therefore $B \setminus X$ has measure 0. So X is measurable. □



2. Related properties

Baire Property

Definition

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Theorem (Banach-Mazur)

AD $\Longrightarrow$ *all sets have the Baire Property.*

The local version

Assume Γ *is closed under continuous pre-images. Then* $\mathrm{Det}(\Gamma) \Rightarrow$ *all sets in* Γ *have the Baire Property.*

Banach-Mazur game

Definition (Banach-Mazur game)

I:	s_0		s_1		$\ldots$	
II:		t_0		t_1		$\ldots$

- $s_i, t_i \in \mathbb{N}^* \setminus \{\langle\rangle\}$.
- Let $z := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \ldots$; Player I wins iff $z \in A$.

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This works on the space $\mathbb{N}^\mathbb{N}$; actually there is a version of the Banach-Mazur game on any Polish space: the players choose basic open sets U_i and V_i such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$ with decreasing diameter. Then $\bigcap_{i=0}^\infty U_i = \bigcap_{i=0}^\infty V_i = \{z\}$ and I wins iff $z \in A$.

Perfect Set Property

Definition

A set $A \subseteq \mathbb{R}$, or $A \subseteq \mathbb{N}^{\mathbb{N}}$, satisfies the **Perfect Set Property** if it is either countable or contains a **perfect set** (a homeomorphic image of the full binary tree $2^{\mathbb{N}}$).

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Note: the Perfect Set Property arose from Cantor's original attempts to prove the Continuum Hypothesis. If all subsets of $\mathbb{R}$ satisfied this property, then all subsets of $\mathbb{R}$ would be either countable or have cardinality $2^{\aleph_0}$ (since $|2^{\mathbb{N}}| = 2^{\aleph_0}$). But using AC one can construct counterexamples.

Perfect Set Property and AD

Theorem (Morton Davis)

AD $\Longrightarrow$ *all sets have the Perfect Set Property.*

The local version

Assume Γ *is closed under continuous pre-images and intersections with closed sets. Then* $\mathrm{Det}(\Gamma) \Rightarrow$ *all sets in* Γ *have the Perfect Set Property.*

The $*$-game

Definition ($*$-game)

I:	s_0		s_1		s_2	
II:		n_1		n_2		$\dots$

- $s_i \in \mathbb{N}^* \setminus \{\langle\rangle\}$.
- $n_i \in \mathbb{N}$.
- I must make sure that, for each $i \geq 1$, $s_i(0) \neq n_i$ (otherwise he loses)
- Let $z := s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots$; Player I wins iff $z \in A$.

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- Let $z := s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots$; Player I wins iff $z \in A$.

Again, this works on $\mathbb{N}^\mathbb{N}$, but there are versions that work on $\mathbb{R}$, $\mathbb{R}^n$ etc.

3. Flip Sets


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Clearly:

- If x and y differ on an **even** number of digits then $x \in X \iff y \in X$.
- If they differ on an **odd** number then $x \in X \iff y \notin X$.
- If they differ on an **infinite** number of digits, we do not know what happens.

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Question: do flip sets exist?

Flip sets and AC

Lemma

Assuming AC, *flip sets exist.*

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Proof.

Let $\sim$ be the equivalent relation on $2^{\mathbb{N}}$ such that $x \sim y$ iff $\{n \mid x(n) \neq y(n)\}$ is finite. For each equivalence class $[x]_\sim$, let $s_{[x]_\sim}$ be some fixed element from that class. Now define X by

$$x \in X \iff |\{n \mid x(n) \neq s_{[x]_\sim}(n)\}| \text{ is even.}$$

This is a flip set: if x, y differ by exactly one digit, then $s_{[x]_\sim} = s_{[y]_\sim}$. But then, by definition, exactly one of x, y is in X. $\square$

Flip sets and AD

Theorem

AD $\Longrightarrow$ *flip sets don't exist.*

The local version

Assume $\mathbf{\Gamma}$ *is closed under continuous pre-images. Then* $\mathrm{Det}(\mathbf{\Gamma}) \Rightarrow$ *there are no flip sets in* $\mathbf{\Gamma}$.

The game

The game is the Banach-Mazur game on $2^{\mathbb{N}}$, we will denote it by $G^{**}(X)$.

Definition ($G^{**}(X)$)

I:	s_0		s_1		$\dots$	
II:		t_0		t_1		$\dots$

- $s_i, t_i \in \{0,1\}^* \setminus \{\langle\rangle\}$.
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Strategy stealing

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Lemma 1

1. *If I has a w.s. in* $G^{**}(X)$ *then I has a w.s. in* $G^{**}(2^{\mathbb{N}} \setminus X)$.
2. *If II has a w.s. in* $G^{**}(X)$ *then II has a w.s. in* $G^{**}(2^{\mathbb{N}} \setminus X)$.

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Proof.

Assume σ is a w.s. for I in $G^{**}(X)$, then define σ':

- The first move $\sigma'(\langle\rangle)$ is a sequence of the same length as $\sigma(\langle\rangle)$ but differs from it at exactly one digit.
- Next, play according to σ, **as if the first move was** $\sigma(\langle\rangle)$.

Clearly, for any sequence y of II's moves, $\sigma * y$ and $\sigma' * y$ differ by exactly one digit. Since $\sigma * y \in X$ and X is a flip set, $\sigma' * y \notin X$, hence σ' is winning for I in $G^{**}(2^{\mathbb{N}} \setminus X)$.

The proof of 2 is analogous. □

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

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If II has a w.s. in $G^{**}(X)$ *then I has a w.s. in the game* $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X) : \quad \begin{array}{r||l} \text{I:} & \\ \hline \text{II:} & \end{array}$$

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{c||l} \text{I:} & s \\ \hline \text{II:} & \end{array}$$

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{c||cc} \text{I:} & s & \\ \hline \text{II:} & & t \end{array}$$

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s	
II:		t

$G^{**}(X)$:

I:	
II:	

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s	
II:		t

$G^{**}(X)$:

I:	$s^\frown t$
II:	

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s	
II:		t

$G^{**}(X)$:

I:	$s^\frown t$	
II:		s_0

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s		s_0
II:		t	

$G^{**}(X)$:

I:	$s^\frown t$	
II:		s_0

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X) : \quad \begin{array}{r||llll} \text{I:} & s & & s_0 & \\ \hline \text{II:} & & t & & t_0 \end{array}$$

$$G^{**}(X) : \quad \begin{array}{r||ll} \text{I:} & s^\frown t & \\ \hline \text{II:} & & s_0 \end{array}$$

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{c||cccc} \text{I:} & s & & s_0 & \\ \hline \text{II:} & & t & & t_0 \end{array}$$

$$G^{**}(X): \quad \begin{array}{c||ccc} \text{I:} & s^\frown t & & t_0 \\ \hline \text{II:} & & s_0 & \end{array}$$

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s		s_0		
II:		t		t_0	

$G^{**}(X)$:

I:	$s^\frown t$		t_0		
II:		s_0		s_1	

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{c||ccccc} \text{I:} & s & & s_0 & & s_1 \\ \hline \text{II:} & & t & & t_0 & \end{array}$$

$$G^{**}(X): \quad \begin{array}{c||ccccc} \text{I:} & s^\frown t & & & t_0 & \\ \hline \text{II:} & & & s_0 & & s_1 \end{array}$$

$\square$

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

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Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s		s_0		s_1	
II:		t		t_0		

$G^{**}(X)$:

I:	$s^\frown t$		t_0		$\dots$	
II:		s_0		s_1		

$\square$

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$$G^{**}(2^{\mathbb{N}} \setminus X) : \quad \begin{array}{c||ccccccc} \text{I:} & s & & s_0 & & s_1 & & \\ \hline \text{II:} & & t & & t_0 & & \dots & \end{array}$$

$$G^{**}(X) : \quad \begin{array}{c||ccccccc} \text{I:} & s^\frown t & & & t_0 & & \dots & \\ \hline \text{II:} & & & s_0 & & s_1 & & \end{array}$$

□

Strategy stealing (continued)

Lemma 2

If II has a w.s. in $G^{**}(X)$ then I has a w.s. in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let τ be winning for II in $G^{**}(X)$. Player I will steal the strategy from II, as follows:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s		s_0		s_1	
II:		t		t_0		$\dots$

$G^{**}(X)$:

I:	$s^\frown t$		t_0		$\dots$
II:		s_0		s_1	

Let $x = s^\frown t^\frown s_0{}^\frown t_0{}^\frown \dots$; then $x \notin X$ since τ was winning in the auxiliary game $G^{**}(X)$. Hence the strategy we just described is winning for I in $G^{**}(2^{\mathbb{N}} \setminus X)$. □

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	
II:	

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0
II:	

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:		s_0
II:		

$G^{**}(X)$:

I:		
II:		

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0
II:	

$G^{**}(X)$:

I:	s
II:	

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0
II:	

$G^{**}(X)$:

I:	s
II:	

- **Case 1.** $|s_0| < |s|$.

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0	
II:		t_0

$G^{**}(X)$:

I:	s
II:	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1
II:		t_0	

$G^{**}(X)$:

I:	s
II:	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0 ^\frown t_0| = |s|$ and $s_0 ^\frown t_0$ differs from s by an **even number of digits**.

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Strategy stealing (continued)

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If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1
II:		t_0	

$G^{**}(X)$:

I:	s	
II:		s_1

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1
II:		t_0	

$G^{**}(X)$:

I:	s		t_1
II:		s_1	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

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Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1	
II:		t_0		t_1

$G^{**}(X)$:

I:	s		t_1
II:		s_1	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0 ^\frown t_0| = |s|$ and $s_0 ^\frown t_0$ differs from s by an **even number of digits**.

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2
II:		t_0		t_1	

$G^{**}(X)$:

I:	s		t_1
II:		s_1	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0 ^\frown t_0| = |s|$ and $s_0 ^\frown t_0$ differs from s by an **even number of digits**.

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2
II:		t_0		t_1	

$G^{**}(X)$:

I:	s			t_1	
II:			s_1		s_2

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2
II:		t_0		t_1	

$G^{**}(X)$:

I:	s		t_1		$\ldots$
II:		s_1		s_2	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2	
II:		t_0		t_1		...

$G^{**}(X)$:

I:	s			t_1		...
II:			s_1		s_2	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2	
II:		t_0		t_1		...

$G^{**}(X)$:

I:	s		t_1		...
II:		s_1		s_2	

- **Case 1.** $|s_0| < |s|$. Play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s by an **even number of digits**.

Let $x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and $y := s^\frown s_1{}^\frown t_1{}^\frown \dots$. Then x and y differ by an even number of digits. Since $y \in X$, also $x \in X$, so the strategy is winning for II. □

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0
II:	

$G^{**}(X)$:

I:	s
II:	

- **Case 2.** $|s_0| \geq |s|$.

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ *then II has w.s. in* $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0
II:	

$G^{**}(X)$:

I:	s	
II:		t

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$.

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		
II:			

$G^{**}(X)$:

I:	s		t'
II:		t	

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$.

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		
II:			t_0

$G^{**}(X)$:

I:	s		t'
II:		t	

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

$\square$

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1	
II:		t_0		

$G^{**}(X)$:

I:	s		t'	
II:		t		

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

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Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1	
II:		t_0		

$G^{**}(X)$:

I:	s		t'	
II:		t		s_1

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1	
II:		t_0		

$G^{**}(X)$:

I:	s		t'		t_1
II:		t		s_1	

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1	
II:		t_0		t_1

$G^{**}(X)$:

I:	s		t'		t_1
II:		t		s_1	

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2
II:		t_0		t_1	

$G^{**}(X)$:

I:	s		t'		t_1
II:		t		s_1	

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2	
II:		t_0		t_1		

$G^{**}(X)$:

I:	s		t'		t_1	
II:		t		s_1		s_2

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2	
II:		t_0		t_1		

$G^{**}(X)$:

I:	s		t'		t_1		$\dots$
II:		t		s_1		s_2	

- **Case 2.** $|s_0| \geq |s|$. Play any t such that $|s^\frown t| > |s_0|$. Play t_0 such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an **even number of digits.**

□

Strategy stealing (continued)

Lemma 3

If I has w.s. in $G^{**}(X)$ then II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Proof.

Let σ be winning for I in $G^{**}(X)$. Player II will do the following:

$G^{**}(2^{\mathbb{N}} \setminus X)$:

I:	s_0		s_1		s_2	
II:		t_0		t_1		...

$G^{**}(X)$:

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Let $x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \ldots$ and $y := s^\frown t^\frown t'{}^\frown s_1{}^\frown t_1{}^\frown \ldots$. Then x and y differ by an even number of digits. Since $y \in X$, also $x \in X$, so the strategy is winning for II. $\square$

Corollary

Combining Lemmas 1, 2 and 3:

Corollary

AD $\Rightarrow$ *flip sets don't exist.*

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Proof.

Suppose X is a flip set. By determinacy I or II has a w.s.

- I has w.s. in $G^{**}(X)$
  $\Longrightarrow$ I has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$
  $\Longrightarrow$ II has w.s. in $G^{**}(X)$.
- II has w.s. in $G^{**}(X)$
  $\Longrightarrow$ II has w.s. in $G^{**}(2^{\mathbb{N}} \setminus X)$
  $\Longrightarrow$ I has w.s. in $G^{**}(X)$.

Both situations are clearly absurd. □

4. Wadge reducibility

Continuous functions on the Baire space

Recall that on the Baire space, $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous at $x \in \mathbb{N}^{\mathbb{N}}$ iff

$$\forall s \lhd f(x) \quad \exists t \lhd x \quad \forall y \, (t \lhd y \to s \lhd f(y))$$

In words: every initial segment of $f(x)$ depends only on an initial segment of x.

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William Wadge (1983) studied continuous functions as a notion of **reducibility** on the Baire space.

Definition

Let $A, B \subseteq \mathbb{N}^\mathbb{N}$. A is **Wadge reducible** to B, notation $A \leq_W B$, iff there is a continuous function $f : \mathbb{N}^\mathbb{N} \longrightarrow \mathbb{N}^\mathbb{N}$ such that for all x:

$$x \in A \iff f(x) \in B$$

For convenience: $\overline{A} := \mathbb{N}^{\mathbb{N}} \setminus A$.

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Properties of $\leq_W$

- $A \leq_W B$ iff $\overline{A} \leq_W \overline{B}$.
- $\leq_W$ is a **pre-wellorder** (transitive and reflexive but not anti-symmetric).
- We can define $A \equiv_W B$ iff $A \leq_W B$ and $B \leq_W A$ and consider $\mathbb{N}^{\mathbb{N}} / \equiv_W$ (the equivalence classes $[A]_W$ are called **Wadge degrees**).

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Remark: The results in this section don't directly apply to $\mathbb{R}$ or $\mathbb{R}^n$ (but they do apply to $\mathbb{R} \setminus \mathbb{Q}$, other product spaces etc.)

Wadge reducibility and AD

Without determinacy, not much can be said about Wadge reducibility. However, under AD we get a very rich structure theory.

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Theorem

AD $\implies$ *for all* $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, *either* $A \leq_W B$ *or* $B \leq_W \overline{A}$.

The local version

Assume $\mathbf{\Gamma}$ *is closed under continuous pre-images, finite unions, intersections and complements, and contains closed sets. Then* $\mathrm{Det}(\mathbf{\Gamma}) \Rightarrow$ *for all* $A, B \in \mathbf{\Gamma}$, *either* $A \leq_W B$ *or* $B \leq_W \overline{A}$.

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Non-trivial corollary

For Borel subsets $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ either $A \leq_W B$ or $B \leq_W \overline{A}$.

The Wadge game

Definition (Wadge game)

Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$. The game $G^W(A, B)$ is played as follows:

$$\begin{array}{c||ccccccc} \text{I:} & x_0 & & x_1 & & \dots & & \\ \hline \text{II:} & & y_0 & & y_1 & & \dots \end{array}$$

- $x_i, y_i \in \mathbb{N}$
- Let $x = \langle x_0, x_1, \dots \rangle$ and $y = \langle y_0, y_1, \dots \rangle$; Player II wins iff

$$x \in A \iff y \in B$$

Main result about Wadge games

Lemma

Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$.

1. *If II has a w.s. in* $G^W(A, B)$ *then* $A \leq_W B$.
2. *If I has a w.s. in* $G^W(A, B)$ *then* $B \leq_W \overline{A}$.

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2. *If I has a w.s. in $G^W(A, B)$ then $B \leq_W \overline{A}$.*

Proof.

As before, fix $f(z)(n) := z(2n)$ and $g(z)(n) := z(2n+1)$. If τ is a winning strategy for II, then for every x played by I

$$x \in A \iff g(x * \tau) \in B.$$

But since g and $x \mapsto x * \tau$ are both continuous, $A \leq_W B$ follows.

Analogously, if σ is winning strategy for I then for every y we have $f(\sigma * y) \in A \iff y \notin B$, so we have $\overline{B} \leq_W A$, or equivalently $B \leq_W \overline{A}$. □

Structure of the Wadge order

Define $A <_W B$ iff $A \leq_W B$ and $B \not\leq_W A$.

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Assuming AD,*if* $A <_W B$ *then I wins both* $G^W(B, A)$ *and* $G^W(B, \overline{A})$.

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If II would win $G^W(B, A)$ we would have $B \leq A$ contrary to assumption.

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Proof.

If II would win $G^W(B, A)$ we would have $B \leq A$ contrary to assumption.

If II would win $G^W(B, \overline{A})$ we would have

$$B \leq_W \overline{A} \leq_W \overline{B} \leq_W A$$

again, contrary to assumption. □

Martin-Monk theorem

Theorem (Martin-Monk)

Assuming AD, *the relation* $<_W$ *is well-founded.*
(*i.e., there are no infinite descending chains*).

The local version

Assume $\mathbf{\Gamma}$ *is closed under continuous pre-images, finite unions, intersections and complements, and contains closed sets. Then* $\text{Det}(\mathbf{\Gamma}) \Rightarrow$ *the relation* $<_W$ *restricted to sets in* $\mathbf{\Gamma}$ *is well-founded.*

Simulateneous Exhibition (Simul)

Proof

Proof: Assume $<_W$ is ill-founded, and let

$$\cdots <_W A_3 <_W A_2 <_W A_1 <_W A_0$$

be an infinite descending chain of subsets of $\mathbb{N}^{\mathbb{N}}$. For every n, by the previous lemma, I has winning strategies in both $G^W(A_n, A_{n+1})$ and $G^W(A_n, \overline{A_{n+1}})$. Call these strategies σ_n^0 and σ_n^1, respectively.

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Abbreviation:

$$G_n^0 := G^W(A_n, A_{n+1})$$
$$G_n^1 := G^W(A_n, \overline{A_{n+1}})$$

Proof (continued)

To any $x \in 2^{\mathbb{N}}$, we can associate an infinite sequence of Wadge games

$$\left\langle G_0^{x(0)}, G_1^{x(1)}, G_2^{x(2)}, \dots \right\rangle$$

played according to I's winning strategies

$$\left\langle \sigma_0^{x(0)}, \sigma_1^{x(1)}, \sigma_2^{x(2)}, \dots \right\rangle.$$

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Fix one particular $x \in 2^{\mathbb{N}}$. Player II will play an **infinitary simul** against all $G_n^{x(n)}$.

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

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Player II

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$G_0^{x(0)}$ I:

II:

$G_1^{x(1)}$ I:

II:

$G_2^{x(2)}$ I:

II:

$G_3^{x(3)}$ I:

II:

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$G_0^{x(0)}$ I: $a_0^x(0)$

II:

$G_1^{x(1)}$ I:

II:

$G_2^{x(2)}$ I:

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$G_3^{x(3)}$ I:

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II: $a_1^x(0)$

$G_1^{x(1)}$ I: $a_1^x(0)$ ⇗

II: $a_2^x(0)$

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Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$
	II:		$a_1^x(0)$		$a_1^x(1)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		
	II:		$a_2^x(0)$			
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	II:					
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	II:					

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Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$
	II:		$a_1^x(0)$		$a_1^x(1)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		
	II:		$a_2^x(0)$			
$G_2^{x(2)}$	I:	$a_2^x(0)$				
	II:					

$G_3^{x(3)}$	I:	$a_3^x(0)$
	II:	

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$ I: $a_0^x(0)$ $a_0^x(1)$ $a_0^x(2)$

II: $a_1^x(0)$ $a_1^x(1)$

$G_1^{x(1)}$ I: $a_1^x(0)$ $a_1^x(1)$

II: $a_2^x(0)$

$G_2^{x(2)}$ I: $a_2^x(0)$

II:

$G_3^{x(3)}$ I: $a_3^x(0)$

II:

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$
	II:		$a_1^x(0)$		$a_1^x(1)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		
	II:		$a_2^x(0)$			
$G_2^{x(2)}$	I:	$a_2^x(0)$				
	II:		$a_3^x(0)$			
$G_3^{x(3)}$	I:	$a_3^x(0)$				
	II:					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$
	II:		$a_1^x(0)$		$a_1^x(1)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		
	II:		$a_2^x(0)$			
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		
	II:		$a_3^x(0)$			
$G_3^{x(3)}$	I:	$a_3^x(0)$				
	II:					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$

I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$
II:		$a_1^x(0)$		$a_1^x(1)$	

$G_1^{x(1)}$

I:	$a_1^x(0)$		$a_1^x(1)$		
II:		$a_2^x(0)$			

$G_2^{x(2)}$

I:	$a_2^x(0)$		$a_2^x(1)$		
II:		$a_3^x(0)$			

$G_3^{x(3)}$

I:	$a_3^x(0)$				
II:					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$ I: $a_0^x(0)$ $a_0^x(1)$ $a_0^x(2)$

II: $a_1^x(0)$ $a_1^x(1)$

$G_1^{x(1)}$ I: $a_1^x(0)$ $a_1^x(1)$

II: $a_2^x(0)$ $a_2^x(1)$

$G_2^{x(2)}$ I: $a_2^x(0)$ $a_2^x(1)$

II: $a_3^x(0)$

$G_3^{x(3)}$ I: $a_3^x(0)$

II:

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$
	II:		$a_1^x(0)$		$a_1^x(1)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$
	II:		$a_2^x(0)$		$a_2^x(1)$	
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		
	II:		$a_3^x(0)$			
$G_3^{x(3)}$	I:	$a_3^x(0)$				
	II:					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$ I: $a_0^x(0)$ $a_0^x(1)$ $a_0^x(2)$

II: $a_1^x(0)$ $a_1^x(1)$

$G_1^{x(1)}$ I: $a_1^x(0)$ $a_1^x(1)$ $a_1^x(2)$

II: $a_2^x(0)$ $a_2^x(1)$

$G_2^{x(2)}$ I: $a_2^x(0)$ $a_2^x(1)$

II: $a_3^x(0)$

$G_3^{x(3)}$ I: $a_3^x(0)$

II:

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$ I: $a_0^x(0)$ $a_0^x(1)$ $a_0^x(2)$

II: $a_1^x(0)$ $a_1^x(1)$ $a_1^x(2)$

$G_1^{x(1)}$ I: $a_1^x(0)$ $a_1^x(1)$ $a_1^x(2)$

II: $a_2^x(0)$ $a_2^x(1)$

$G_2^{x(2)}$ I: $a_2^x(0)$ $a_2^x(1)$

II: $a_3^x(0)$

$G_3^{x(3)}$ I: $a_3^x(0)$

II:

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$	
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$			
	II:		$a_2^x(0)$		$a_2^x(1)$				
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$					
	II:		$a_3^x(0)$						
$G_3^{x(3)}$	I:	$a_3^x(0)$							
	II:								

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
		$\nearrow$		$\nearrow$		$\nearrow$		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
	II:		$a_2^x(0)$		$a_2^x(1)$			
		$\nearrow$		$\nearrow$				
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$				
	II:		$a_3^x(0)$					
		$\nearrow$						
$G_3^{x(3)}$	I:	$a_3^x(0)$						
	II:		$\cdots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
	II:		$a_2^x(0)$		$a_2^x(1)$			
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$				
	II:		$a_3^x(0)$					
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\dots$				
	II:		$\dots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
		$\nearrow$		$\nearrow$		$\nearrow$		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
	II:		$a_2^x(0)$		$a_2^x(1)$			
		$\nearrow$		$\nearrow$				
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$				
	II:		$a_3^x(0)$		$\dots$			
		$\nearrow$						
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\dots$				
	II:		$\dots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
		⇗		⇗		⇗		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
	II:		$a_2^x(0)$		$a_2^x(1)$			
		⇗		⇗				
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		
	II:		$a_3^x(0)$		$\cdots$			
		⇗						
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$				
	II:		$\cdots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$	
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		
	II:		$a_3^x(0)$		$\cdots$			
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$				
	II:		$\cdots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$	
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		
	II:		$a_3^x(0)$		$\cdots$			
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$				
	II:		$\cdots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
			$\Nearrow$		$\Nearrow$		$\Nearrow$	
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$	
			$\Nearrow$		$\Nearrow$			
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		
	II:		$a_3^x(0)$		$\cdots$			
			$\Nearrow$					
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$				
	II:		$\cdots$		$\cdots$			

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$	
		$\Nearrow$		$\Nearrow$		$\Nearrow$		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\dots$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\dots$	
		$\Nearrow$		$\Nearrow$				
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\dots$		
	II:		$a_3^x(0)$		$\dots$			
		$\Nearrow$						
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\dots$		$\dots$		
	II:		$\dots$		$\dots$			

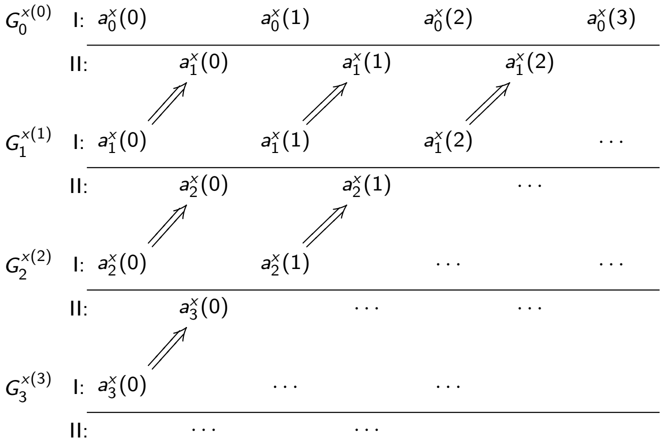
Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$	
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\dots$	
	II:		$a_2^x(0)$		$a_2^x(1)$		$\dots$		
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\dots$			
	II:		$a_3^x(0)$		$\dots$		$\dots$		
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\dots$		$\dots$			
	II:		$\dots$		$\dots$				

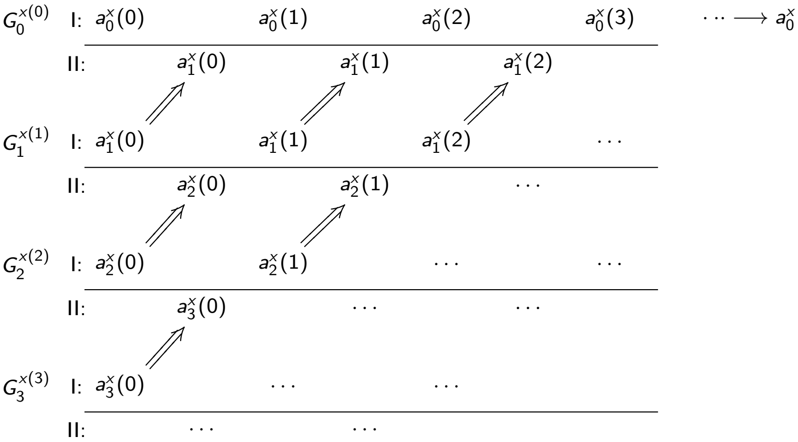
Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.



Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.



Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$	$\cdots \longrightarrow a_0^x$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots \longrightarrow a_1^x$
			⇗		⇗		⇗		
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$	
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$		
			⇗		⇗				
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots$	
	II:		$a_3^x(0)$		$\cdots$		$\cdots$		
			⇗						
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$		$\cdots$			
	II:		$\cdots$		$\cdots$				

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$		$\cdots \longrightarrow a_0^x$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$			$\cdots \longrightarrow a_1^x$
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$		$\cdots \longrightarrow a_1^x$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$			
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots$		
	II:		$a_3^x(0)$		$\cdots$		$\cdots$			
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$		$\cdots$				
	II:		$\cdots$		$\cdots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$		$\cdots \longrightarrow a_0^x$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$			$\cdots \longrightarrow a_1^x$
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$		$\cdots \longrightarrow a_1^x$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$			$\cdots \longrightarrow a_2^x$
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots$		
	II:		$a_3^x(0)$		$\cdots$		$\cdots$			
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$		$\cdots$				
	II:		$\cdots$		$\cdots$					

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$	$\cdots \longrightarrow a_0^x$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots \longrightarrow a_1^x$
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$	$\cdots \longrightarrow a_1^x$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots \longrightarrow a_2^x$
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots$	$\cdots \longrightarrow a_2^x$
	II:		$a_3^x(0)$		$\cdots$		$\cdots$		
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$		$\cdots$			
	II:		$\cdots$		$\cdots$				

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$$
\begin{array}{llccccccccl}
G_0^{x(0)} & \text{I:} & a_0^x(0) & & a_0^x(1) & & a_0^x(2) & & a_0^x(3) & & \cdots \longrightarrow a_0^x \\
\hline
 & \text{II:} & & a_1^x(0) & & a_1^x(1) & & a_1^x(2) & & & \cdots \longrightarrow a_1^x \\
 & & \nearrow & & \nearrow & & \nearrow & & & & \\
G_1^{x(1)} & \text{I:} & a_1^x(0) & & a_1^x(1) & & a_1^x(2) & & \cdots & & \cdots \longrightarrow a_1^x \\
\hline
 & \text{II:} & & a_2^x(0) & & a_2^x(1) & & \cdots & & & \cdots \longrightarrow a_2^x \\
 & & \nearrow & & \nearrow & & & & & & \\
G_2^{x(2)} & \text{I:} & a_2^x(0) & & a_2^x(1) & & \cdots & & \cdots & & \cdots \longrightarrow a_2^x \\
\hline
 & \text{II:} & & a_3^x(0) & & \cdots & & \cdots & & & \cdots \longrightarrow a_3^x \\
 & & \nearrow & & & & & & & & \\
G_3^{x(3)} & \text{I:} & a_3^x(0) & & \cdots & & \cdots & & & & \\
\hline
 & \text{II:} & & \cdots & & \cdots & & & & &
\end{array}
$$

Infinitary Simul

Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$G_0^{x(0)}$	I:	$a_0^x(0)$		$a_0^x(1)$		$a_0^x(2)$		$a_0^x(3)$	$\cdots \longrightarrow a_0^x$
	II:		$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots \longrightarrow a_1^x$
$G_1^{x(1)}$	I:	$a_1^x(0)$		$a_1^x(1)$		$a_1^x(2)$		$\cdots$	$\cdots \longrightarrow a_1^x$
	II:		$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots \longrightarrow a_2^x$
$G_2^{x(2)}$	I:	$a_2^x(0)$		$a_2^x(1)$		$\cdots$		$\cdots$	$\cdots \longrightarrow a_2^x$
	II:		$a_3^x(0)$		$\cdots$		$\cdots$		$\cdots \longrightarrow a_3^x$
$G_3^{x(3)}$	I:	$a_3^x(0)$		$\cdots$		$\cdots$			$\cdots \longrightarrow a_3^x$
	II:		$\cdots$		$\cdots$				

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Let $x \in 2^{\mathbb{N}}$ be fixed. I has winning strategy $\sigma_n^{x(n)}$ in every $G_n^{x(n)}$.

$$
\begin{array}{llcccccccl}
G_0^{x(0)} & \text{I:} & a_0^x(0) & & a_0^x(1) & & a_0^x(2) & & a_0^x(3) & \cdots \longrightarrow a_0^x \\
\hline
 & \text{II:} & & a_1^x(0) & & a_1^x(1) & & a_1^x(2) & & \cdots \longrightarrow a_1^x \\
 & & & \nearrow & & \nearrow & & \nearrow & & \\
G_1^{x(1)} & \text{I:} & a_1^x(0) & & a_1^x(1) & & a_1^x(2) & & \cdots & \cdots \longrightarrow a_1^x \\
\hline
 & \text{II:} & & a_2^x(0) & & a_2^x(1) & & \cdots & & \cdots \longrightarrow a_2^x \\
 & & & \nearrow & & \nearrow & & & & \\
G_2^{x(2)} & \text{I:} & a_2^x(0) & & a_2^x(1) & & \cdots & & \cdots & \cdots \longrightarrow a_2^x \\
\hline
 & \text{II:} & & a_3^x(0) & & \cdots & & \cdots & & \cdots \longrightarrow a_3^x \\
 & & & \nearrow & & & & & & \\
G_3^{x(3)} & \text{I:} & a_3^x(0) & & \cdots & & \cdots & & & \cdots \longrightarrow a_3^x \\
\hline
 & \text{II:} & & \cdots & & \cdots & & & & \cdots \longrightarrow a_4^x
\end{array}
$$

The result

For a fixed x, we have produced a sequence $\langle a_n^x \mid n \in \mathbb{N} \rangle$ of elements of $\mathbb{N}^{\mathbb{N}}$ with the following property:

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- For $n \geq 1$, a_n^x is the sequence of I's moves in $G_n^{x(n)}$, and also the sequence of II's moves in $G_{n-1}^{x(n-1)}$.

The result

For a fixed x, we have produced a sequence $\langle a_n^x \mid n \in \mathbb{N}\rangle$ of elements of $\mathbb{N}^{\mathbb{N}}$ with the following property:

- For $n \geq 1$, a_n^x is the sequence of I's moves in $G_n^{x(n)}$, and also the sequence of II's moves in $G_{n-1}^{x(n-1)}$.
- Since I wins each game $G_n^{x(n)}$, the definition implies

$$x(n) = 0 \implies (a_n^x \in A_n \leftrightarrow a_{n+1}^x \notin A_{n+1})$$
$$x(n) = 1 \implies (a_n^x \in A_n \leftrightarrow a_{n+1}^x \in A_{n+1})$$

(Recall that $G_n^0 = G^W(A_n, A_{n+1})$ and $G_n^1 = G^W(A_n, \overline{A_{n+1}})$).

Comparing different x

To each $x \in 2^{\mathbb{N}}$ corresponds a unique "simul game". Now let's compare different x:

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Claim 1

If $\forall m \geq n\ (x(m) = y(m))$ *then* $\forall m \geq n\ (a^x_m = a^y_m)$.

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Claim 1

If $\forall m \geq n\ (x(m) = y(m))$ *then* $\forall m \geq n\ (a^x_m = a^y_m)$.

Proof.

Note that the values of a^x_m and a^y_m depend only on games $G^{x(m')}_{m'}$ and $G^{y(m')}_{m'}$ for $m' \geq m$. □

Comparing different x (continued)

Claim 2

Let n be such that $x(n) \neq y(n)$ but $\forall m > n\ (x(m) = y(m))$. Then $a_n^x \in A_n \ \leftrightarrow \ a_n^y \notin A_n$.

Comparing different x (continued)

Claim 2

Let n be such that $x(n) \neq y(n)$ but $\forall m > n\ (x(m) = y(m))$. Then $a_n^x \in A_n \ \leftrightarrow\ a_n^y \notin A_n$.

Proof.

Since $x(n) \neq y(n)$ we have two cases:

1. $x(n) = 1$ and $y(n) = 0$. Then

$$\begin{array}{lll} a_n^x \in A_n & \leftrightarrow & a_{n+1}^x \in A_{n+1} \\ a_n^y \in A_n & \leftrightarrow & a_{n+1}^y \notin A_{n+1} \end{array}.$$

   By Claim 1 $a_n^x \in A_n \ \leftrightarrow\ a_{n+1}^x \in A_{n+1} \ \leftrightarrow\ a_{n+1}^y \in A_{n+1} \ \leftrightarrow\ a_n^y \notin A_n$.

2. $x(n) = 0$ and $y(n) = 1$. Then

$$\begin{array}{lll} a_n^x \in A_n & \leftrightarrow & a_{n+1}^x \notin A_{n+1} \\ a_n^y \in A_n & \leftrightarrow & a_{n+1}^y \in A_{n+1} \end{array}.$$

   By Claim 1 $a_n^x \in A_n \ \leftrightarrow\ a_{n+1}^x \notin A_{n+1} \ \leftrightarrow\ a_{n+1}^y \notin A_{n+1} \ \leftrightarrow\ a_n^y \notin A_n$. □

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

Comparing different x (continued)

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Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

Proof.

By Claim 2 $a_n^x \in A_n \leftrightarrow a_n^y \notin A_n$. Since $x(n-1) = y(n-1)$ we have two cases:

1. $x(n-1) = y(n-1) = 0$. Then

$$a_{n-1}^x \in A_{n-1} \leftrightarrow a_n^x \notin A_n$$
$$a_{n-1}^y \in A_{n-1} \leftrightarrow a_n^y \notin A_n.$$

   and therefore $a_{n-1}^x \in A_{n-1} \leftrightarrow a_{n-1}^y \notin A_{n-1}$.

2. $x(n-1) = y(n-1) = 1$. Similar.

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

Proof.

By Claim 2 $a_n^x \in A_n \leftrightarrow a_n^y \notin A_n$. Since $x(n-1) = y(n-1)$ we have two cases:

1. $x(n-1) = y(n-1) = 0$. Then

$$\begin{array}{c} a_{n-1}^x \in A_{n-1} \leftrightarrow a_n^x \notin A_n \\ a_{n-1}^y \in A_{n-1} \leftrightarrow a_n^y \notin A_n. \end{array}$$

   and therefore $a_{n-1}^x \in A_{n-1} \leftrightarrow a_{n-1}^y \notin A_{n-1}$.

2. $x(n-1) = y(n-1) = 1$. Similar.

Now go to the $(n-2)$-th level. Since again $x(n-2) = y(n-2)$ we get, by a similar argument as before, $a_{n-2}^x \in A_{n-2} \leftrightarrow a_{n-2}^y \notin A_{n-2}$.

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

Proof.

By Claim 2 $a_n^x \in A_n \leftrightarrow a_n^y \notin A_n$. Since $x(n-1) = y(n-1)$ we have two cases:

1. $x(n-1) = y(n-1) = 0$. Then

$$a_{n-1}^x \in A_{n-1} \leftrightarrow a_n^x \notin A_n$$
$$a_{n-1}^y \in A_{n-1} \leftrightarrow a_n^y \notin A_n.$$

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Now go to the $(n-2)$-th level. Since again $x(n-2) = y(n-2)$ we get, by a similar argument as before, $a_{n-2}^x \in A_{n-2} \leftrightarrow a_{n-2}^y \notin A_{n-2}$.

We go on like this until we reach level 0, and there we get $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$. □

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n?$	$a_n^y \in A_n?$
	0	0		
	1	1		
	0	0		
	0	0		
	1	1		
$\rightarrow$	**1**	**0**		
	1	1		
	0	0		
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0		
	1	1		
	0	0		
	0	0		
	1	1		
$\rightarrow$	**1**	**0**		
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0		
	1	1		
	0	0		
	0	0		
	1	1		
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0		
	1	1		
	0	0		
	0	0		
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0		
	1	1		
	0	0		
	0	0	no	yes
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0		
	1	1		
	0	0	yes	no
	0	0	no	yes
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0		
	1	1	yes	no
	0	0	yes	no
	0	0	no	yes
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

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Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0	no	yes
	1	1	yes	no
	0	0	yes	no
	0	0	no	yes
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \leftrightarrow a_0^y \notin A_0$.

	x	y	$a_n^x \in A_n$?	$a_n^y \in A_n$?
	0	0	no	yes
	1	1	yes	no
	0	0	yes	no
	0	0	no	yes
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Let $X := \{x \in 2^{\mathbb{N}} \mid a_0^x \in A_0\}$.

Comparing different x (continued)

Claim 3

Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a_0^x \in A_0 \;\leftrightarrow\; a_0^y \notin A_0$.

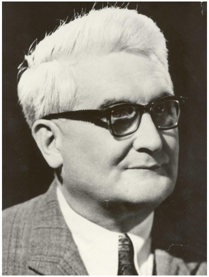
	x	y	$a_n^x \in A_n?$	$a_n^y \in A_n?$
	0	0	no	yes
	1	1	yes	no
	0	0	yes	no
	0	0	no	yes
	1	1	yes	no
$\rightarrow$	**1**	**0**	**yes**	**no**
	1	1	yes	yes
	0	0	yes	yes
	...	...	...	...

Let $X := \{x \in 2^{\mathbb{N}} \mid a_0^x \in A_0\}$.

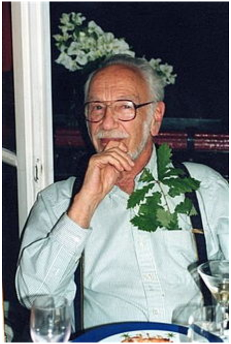
By Claim 3, X is a **flip set**. By AD, this is impossible! □

Ernst Zermelo (1871-1953)

Dénes König (1884–1944)



László Kalmár (1905–1976)



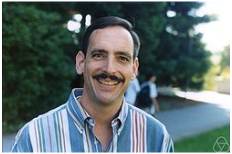
David Gale (1921–2008)

Frank Stewart (Brown U)

William W. Wadge (U Victoria)

Donald A. Martin (UCLA)

John Steel (UC Berkeley)



Hugh Woodin (UC Berkeley)

Thank you!

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