COMPARISON OF HOPF ALGEBRAS ON TREES

RALF HOLTKAMP

Abstract. Several Hopf algebra structures on vector spaces of trees can be found in the literature (cf. [10], [8], [2]). In this paper, we compare the corresponding notions of trees, the multiplications and comultiplications. The Hopf algebras are connected graded or, equivalently, complete Hopf algebras. The Hopf algebra structure on planar binary trees introduced by Loday and Ronco [10] is noncommutative and not cocommutative. We show that this Hopf algebra is isomorphic to the noncommutative version of the Hopf algebra of Connes and Kreimer [3]. We compute its first Lie algebra structure constants in the sense of [7], and show that there is no cogroup structure compatible with the Hopf algebra on planar binary trees.

0. Introduction. Let W_r be the set of planar rooted trees with r internal vertices. The tree $\vee$ is an element of W_1. Let $(\mathbb{Q}W_\infty, \cdot)$ be the free associative algebra with unit $|$ generated over $\mathbb{Q}$ by $\bigcup_{r=1}^{\infty} W_r$. We provide $\mathbb{Q}W_\infty$ with a degree function such that the degree of a tree $T \in W_r$ is r. Every tree T is the grafting $\vee(T_1 ... T_n)$ of n trees of lower degree, where n is the number of outgoing branches at the root $|$. A simple recursive formula $\underline{\Delta}(T) = T \otimes | + (\cdot, \vee)\big(\underline{\Delta}(T_1), ..., \underline{\Delta}(T_n)\big)$, see section 2, defines a coassociative comultiplication on $(\mathbb{Q}W_\infty, \cdot)$ turning it into a graded Hopf algebra. A similar formula occurs in the definition of the Hopf algebra of renormalization introduced by Connes and Kreimer (cf. [8] and [3]) and also in the definition of the Hopf algebra of planar binary rooted trees $(\mathbb{Q}Y_\infty, *, \Delta)$ of Loday and Ronco [10]. While the Hopf algebra of renormalization regarded by Connes and Kreimer is commutative, the Hopf algebra $(\mathbb{Q}Y_\infty, *, \Delta)$ is a noncommutative and not cocommutative Hopf algebra. In [1], [2], Brouder and Frabetti use a noncommutative and not cocommutative Hopf algebra structure on planar binary trees for renormalization of massless quantum electro dynamics. The counit is in all cases the augmentation map.

If we replace abstract trees by planar trees and use the free associative multiplication, we get a noncommutative version $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$ of the Hopf algebra of Connes and Kreimer. Although looking different from $(\mathbb{Q}Y_\infty, *, \Delta)$ at first sight, we construct an isomorphism between those Hopf algebras. As a consequence, the description of the primitive elements given by M. Ronco in [15] can be applied to the Connes-Kreimer Hopf algebra.

In [4], L. Foissy compares various Hopf algebras on trees by different methods, obtaining an identification of $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$ with $(\mathbb{Q}Y_\infty, *, \Delta)$ and the Hopf algebra of Brouder and Frabetti.

As an algebra, the Hopf algebra of Loday and Ronco is isomorphic to the free associative algebra generated by all planar binary trees. Completion with respect to

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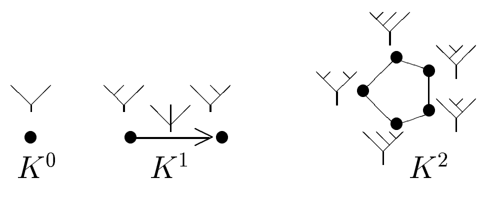
the augmentation ideal yields a similarly looking Hopf algebra structure on the free complete associative algebra of planar binary trees. It is a nontrivial example of an infinite-dimensional formal group law of type $\otimes\mathcal{A}s$, i.e. a formal quantum group law (cf. [9], [7]). The subalgebras generated by trees of a certain maximum number of vertices yield examples of finite-dimensional formal group laws.

We compute the first structure constants of the complete Hopf algebra structure on planar binary trees. Motivated by the question if this complete Hopf algebra falls into the class of complete Hopf algebras classified in [7], corollary 6.6, we show that a complete cogroup structure that "lifts" the Hopf algebra structure of Loday and Ronco does not exist.

In section 1 we recall some notions for abstract, planar and planar binary trees. We define the algebras of trees which are regarded in [10], [8], [2], and [4]. Labeled trees are introduced to distinguish between trees with labeled internal vertices, trees with labeled internal and external vertices, and trees with labeled leaves. We regard planar trees, planar binary trees, and abstract trees. Section 2 contains the definition of various Hopf algebra structures. They are quotient Hopf algebras of the Hopf algebra $(\mathbb{Q}W_\infty, \cdot, \underline{\Delta})$ presented at the beginning of section 2. We first show, that the computation of the comultiplication with the help of admissible cuts (see [8]) can also be used for the description of the comultiplication of $(\mathbb{Q}Y_\infty, *, \Delta)$. Then we construct an isomorphism between $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$, i.e. the noncommutative Connes-Kreimer Hopf algebra, and the Hopf algebra structure of Loday and Ronco. After computing the Lie structure constants for the sub-Hopf algebras generated by trees of a certain maximum number of vertices we show in section 3 that there exists a cogroup structure with these associated Lie structure constants, but that there exists no cogroup structure which has the complete Hopf algebra structure of Loday and Ronco as associated $\otimes\mathcal{A}s$-formal group law.

1. Multiplications of trees.

1.1 *Trees and the associahedron.* Let $n \in \mathbb{N}$, $n \geq 2$. For the definition of the $(n-2)$-dimensional associahedron or Stasheff polytope K^{n-2}, we refer to [17] or [16]: K^{n-2} is a convex polytope of dimension $n-2$ with one vertex for each way of "associating" n ordered variables, i. e. the possible ways of inserting brackets in a word of n letters. By definition these vertices correspond to planar binary (rooted) trees with n leaves and their number is given by the Catalan number $c_n = \frac{(2(n-1))!}{n!(n-1)!} = \sum_{l=1}^{n-1} c_l c_{n-l}$. The number of planar (not necessarily binary) trees with n vertices is also c_n. We note that the cells of the associahedron are labeled by planar trees with n leaves, e.g. the top dimensional cell of K^{n-2} is labeled by the corolla (which has only one internal vertex) with n leaves.

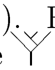


The edges of the polytope correspond to an application of the associativity relation. It is possible to give an orientation to all the edges of the polytope. We let the arrows point from one vertex to the other such that the arrow corresponds to shifting a bracket from left ((xx)x) to right (x(xx)).

We draw the root at the bottom. The tree with exactly one leaf (which is also the root) is denoted by $|$. By definition, $|$ has one external vertex and no internal vertices. By an abstract tree (not necessarily binary), we mean a tree which is not embedded in the plane, i.e. there is no distinction between left and right.

1.2 *The free non-associative algebra.* Let $F_{\mathcal{K}}(x)$ be the free non-associative algebra (without unit) over $\mathbb{Q}$ generated by one variable x. It is a free algebra over the quadratic operad $\mathcal{K}$ freely generated by two binary operations which are permuted by the action of the symmetric group (c.f. [7]).

As a vector space, $F_{\mathcal{K}}(x)$ has the set $Y_\infty := \bigcup_{n=0}^{\infty} Y_n$ of all planar binary trees (with at least one leaf) as a basis, where we denote by Y_n the set of planar binary trees with $n+1$ leaves or, equivalently, n internal vertices (and $2n+1$ overall vertices). The variable x itself corresponds to $| \in Y_0$. We can adjoin a unit to $F_{\mathcal{K}}(x)$ and call it the empty tree, denoted by e.

It is possible to express $F_{\mathcal{K}}(x)$ as a $\circ$-subalgebra of the free associative algebra $F_{\mathcal{A}s}(c,x)$ generated by two variables c, x, where $a \circ b$ is given by cab (Malcev representation). For example, $(x(^2x^3 := cxc^2x^3$ is identified with $(x((xx)x))$, i.e. the planar tree . A representation with respect to $a \circ b := abd$ in $F_{\mathcal{A}s}(x,d)$ is also possible.

Let $F_{\mathcal{H}}(x)$ be the free commutative non-associative algebra (without unit) over $\mathbb{Q}$ generated by one variable x. As in [7], here $\mathcal{H}$ is the quadratic operad freely generated by one commutative binary operation. A vector space basis is given by abstract - i.e. non-planar - binary trees. The set of these trees will be denoted by Y^a_∞.

Similarly the free commutative non-associative algebra $F_{\mathcal{H}}(x_1, ..., x_m)$ and the free non-associative algebra $F_{\mathcal{K}}(x_1, ..., x_m) \subset F_{\mathcal{A}s}(c, x_1, ..., x_m)$ can be treated. Now we get as a vector space basis abstract or planar binary trees with decorated leaves, labeled with letters from $\{x_1, ..., x_m\}$.

1.3 *Labeling vertices.* Instead of labeling the leaves of a tree (planar or abstract), which makes sense in the non-associative algebra point of view, we will often label the vertices, which correspond to pairs of brackets. In the representation of trees without labels as elements of $F_{\mathcal{A}s}(c,x)$, we have to replace c by a set of labels $\{\lambda_1, ..., \lambda_m\}$.

Then labeled trees can be represented by irreducible parenthesized words, in letters λ_i say, cf. [8]. Such a word consists of a string of letters and balanced brackets such that the leftmost opening bracket is matched by the rightmost closing bracket (irreducibility). Our convention will be, that each letter has exactly one opening bracket on its lefthandside, e.g., $(\lambda_1), (\lambda_1(\lambda_2)), (\lambda_1(\lambda_2)(\lambda_3)), (\lambda_1(\lambda_2(\lambda_3))), (\lambda_1(\lambda_2)(\lambda_3)(\lambda_4))$.

An empty pair of brackets without label is also allowed.

The planar binary trees $T \in Y_n$ of [10] have the degree n. Although the trees are not labeled, we should consider them as labeled at all internal vertices in a trivial way, i.e. the vertices are always labeled with the same letter "x", say. By $(\mathbb{Q}Y_\infty, \cdot)$ we denote the free associative algebra of planar binary trees with unit $|$.

For planar trees (not necessarily binary), we will often label all vertices. We denote the set of all (trivially labeled) planar trees with n vertices by X_n and define $X_\infty := \bigcup_{n=0}^{\infty} X_n$, where $X_0 := \{e\}$. For $T \in X_n$, $\deg(T) := n$.

When we exclude the external vertices (as for Y_∞), we define the degree of a general planar tree to be the number of internal vertices. Let W_n denote the set of (trivially labeled) planar trees of degree n. We define $W_\infty := \bigcup_{n=0}^{\infty} W_n$, where $W_0 := \{|\}$.

Let $(\mathbb{Q}X_\infty, \cdot)$ be the free associative algebra $F_{\mathcal{A}s}\big(\bigcup_{n=1}^{\infty} X_n\big)$ with an adjoined unit e and let $(\mathbb{Q}W_\infty, \cdot)$ be the free associative algebra $F_{\mathcal{A}s}\big(\bigcup_{n=1}^{\infty} W_n\big)$ with an adjoined unit $|$. We will see that the subtle distinction between $\mathbb{Q}X_\infty$ and $\mathbb{Q}W_\infty$ is rather important.

Labeled abstract trees without external vertices can be similarly represented by parenthesized words. When we consider abstract trees, the order of subwords on the same level is not determined, e.g., $(\lambda_1(\lambda_2)(\lambda_3)) = (\lambda_1(\lambda_3)(\lambda_2))$. In the trivial case, we use a one letter alphabet.

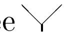
Let $(\mathbb{Q}X^a_\infty, \cdot)$ be the free commutative associative algebra $F_{\mathcal{C}om}\big(\bigcup_{n=1}^{\infty} X^a_n\big)$ generated by all abstract trees, with an adjoined unit e. The multiplication used is just the (commutative) concatenation, e.g., $(\lambda(\lambda)) \cdot (\lambda(\lambda)(\lambda)) = (\lambda(\lambda))(\lambda(\lambda)(\lambda))$. The result is a parenthesized reducible word. Thus $(\mathbb{Q}X^a_\infty, \cdot)$ is the vector space generated by all, not necessarily irreducible, parenthesized words, see [8].

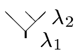
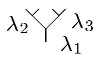
1.4 *Grafting and the operation* Γ. Let abstract or planar trees S and T be given. Let A be the set of leaves of S and B the set of leaves of T. Let a leaf b of T be given. Then the grafting of S to T along b is defined as follows. We identify the root of S with the leaf b. Thus the new set of leaves is $A \cup (B - \{b\})$.

Let $T \in W_\infty$ be a planar tree. Let $\Gamma(T)$ denote the corresponding abstract tree.

There exists a planar tree $\bar{T}$ with the same image under Γ which results from mirroring T along the root axis. Let us denote by $\tilde{\Gamma}(T)$ the abstract tree in X^a_∞ with external vertices, which can be obtained from $\Gamma(T)$ by deleting the leaves.

The operations $\tilde{\Gamma}$, Γ and $T \mapsto \bar{T}$ are canonically extended $\mathbb{Q}$-linearly to the vector space $\mathbb{Q}W_\infty$ generated by all planar trees. Similarly defined are these operations for labeled trees.

Now $\tilde{\Gamma}$ maps $|$ onto e. The tree $_\lambda$, i.e. the tree  with internal vertex labeled by λ, is mapped onto (λ).

Furthermore  $\mapsto (\lambda_1(\lambda_2))$,  $\mapsto (\lambda_1(\lambda_2)(\lambda_3))$.

1.5 *Concatenation of trees.* Let S and T be two planar binary trees in Y_∞ or, more generally, with labels. Define $S\backslash T$ (S under T) by the grafting of T to S along the rightmost leaf of S. Here $S\backslash| = S = |\backslash S$. We note that the degree of the result is the sum of the corresponding degrees for S and T.

Define T/S (T over S) by the grafting of T to S along the leftmost leaf of S, i.e. by $\overline{\bar{S}\backslash\bar{T}}$.

If we graft a planar binary tree S to ${}_\lambda$ (λ a label) along the left leaf of ${}_\lambda$, and a tree T to the result along the right leaf, we get a tree $S \vee_\lambda T$ of degree $1 + \deg(S) + \deg(T)$.

Each nontrivial planar binary tree is uniquely given by $T^l \vee_\lambda T^r$, where T^l and T^r have lower degrees and λ is a label.

A recursive formula for $S\backslash T$ is given by $S^l \vee_\lambda (S^r\backslash T)$ if $S = S^l \vee_\lambda S^r$ (where S has at least one label).

It holds that ${}_\lambda \backslash {}_\rho = {}_\lambda{}^\rho$, ${}_\lambda / {}_\rho = {}^\lambda{}_\rho$.

To simplify notation, we collect all labels from left to right and use the corresponding string as an index: ${}_\lambda{}^\rho = {}_{\lambda\rho}$. To get a unique ordering from left to right, we have to expand some edges: The nearer they are to the root, the more we expand them.

Then ${}_\lambda / {}_{\mu\rho} = {}_{\lambda\mu\rho}$, ${}_\lambda \backslash {}_{\mu\rho} = {}_{\lambda\mu\rho}$ or, with edges expanded, $= {}_{\lambda\mu\rho}$.

It is easy to see, that $/$ and $\backslash$ are associative, i.e. $S/T/Z$ and $S\backslash T\backslash Z$ are well-defined, cf. [10]. They can be extended to associative multiplications $\mathbb{Q}Y_\infty \times \mathbb{Q}Y_\infty \to \mathbb{Q}Y_\infty$.

Let $/^o$ be the multiplication on $\mathbb{Q}Y_\infty$ given by $S/^oT := T/S$ for trees S, T. This multiplication is used in [2] (for massless renormalization, cf. also [1]).

For planar (not necessarily binary) trees, there is a generalization of the operation $\vee_\lambda$. The operation $\vee_\lambda$ can be applied to n arguments $(T_1, ..., T_n)$ or the "word" $T_1 \cdot ... \cdot T_n$. It connects the n trees (in this order) with a new root labeled by λ. The operation $\vee_\lambda$ is canonically extended $\mathbb{Q}$-linearly to the vector space generated by all planar trees.

The analogous operation for abstract trees connects n abstract trees (in any order) with a new root. It is denoted by $B_+(T_1...T_n)$ in [8] and, with chosen T_i' such that $\tilde{\Gamma}(T_i') = T_i$, is simply given by $(\tilde{\Gamma} \circ \vee)(T_1'...T_n')$.

The degrafting $\neg(T)$ of a planar tree $T = \vee(T_1...T_n)$ is given by $T_1...T_n$. The corresponding operator is denoted by B_- in [8].

For T^l, T^r two leaf-labeled planar binary trees, $T^l \vee T^r$ clearly has to be defined as the grafting of T^l and T^r to the left and right leaf of . Then $\vee$ corresponds to the concatenation in the non-associative algebra. Especially $T \vee e = T = e \vee T$. Each planar binary tree (with at least two labels) is uniquely given by $T^l \vee T^r$ with T^l and T^r of degrees lower than $\deg(T)$.

1.6 *Multiplications.* Let trees S and T in Y_∞ be given. S/T, $S\backslash T$ denote vertices of the same associahedron. It is shown in [11], that the orientation of edges in the associahedron induces a partial ordering on the planar binary trees which label its vertices, such that $S/T \leq S\backslash T$. We define $S * T$ to be the sum $\sum_{S/T \leq Z \leq S\backslash T} Z$, i.e. the sum of all trees Z such that there exists a directed path passing Z in the associahedron from S/T to $S\backslash T$.

We extend $*$ linearly to the $\mathbb{Q}$-vector space generated by all planar binary trees (with labels). By construction, c.f. [11], we get an associative (graded) algebra. The map $T \mapsto \bar{T}$ is an involution, $\overline{S * T} = \bar{T} * \bar{S}$.

For every planar binary tree T it holds that $T * | = | * T = T$. Following the path of oriented edges of K^2, we get that $\ _\lambda * \ _\rho = {}^\lambda\ _\rho + {}_\lambda\ ^\rho = (\ + \ )_{\lambda\rho}$ and that $\ _\lambda * \ _{\mu\rho} = (\ + \ + \ )_{\lambda\mu\rho}$.

Analogously, $\ _\lambda * \ _{\mu\rho} = (\ + \ )_{\lambda\mu\rho}$ and $\ _{\lambda_1\lambda_2} * \ _{\lambda_3\lambda_4} =$
$(\ \vee \ + | \vee \ + | \vee \ )_{\lambda_1\ldots\lambda_4}$.

1.7 *Remark.* In [10] it is shown, that $(\mathbb{Q}Y_\infty, *)$ is a free (graded) associative algebra (with unit $|$) with algebra basis in degree $n \geq 1$ given by all trees of the form $T = \mathbb{V}(T') := | \vee T', T'$ of degree $n-1$.

Similarly, the trees $\mathbb{V}(T')$ form an algebra basis of the algebra $(\mathbb{Q}Y_\infty, /^o)$, see [2]. Clearly $\mathbb{V}(S')/^oT = T \vee S'$. Let $\mathbb{V}^n(|) := \mathbb{V}(\mathbb{V}^{n-1}(|))$.

Thus the number of (free) generators in each degree n, $n \geq 1$, is given by c_n for the graded algebras $(\mathbb{Q}Y_\infty, *)$ and $(\mathbb{Q}Y_\infty, /^o)$ as for $(\mathbb{Q}X_\infty, \cdot)$.

The vector space generating function is $\sum_{n=0}^\infty c_{n+1}t^n = \left(1 - \frac{1-\sqrt{1-4t}}{2}\right)^{-1} = \frac{1-\sqrt{1-4t}}{2t}$, as $\sum_{n=1}^\infty c_n t^n = \frac{1-\sqrt{1-4t}}{2}$.

In [10], [11] it is shown, that the associative multiplication $*$ can be written as the sum of two K-linear operations $\prec$ and $\succ$, where $a \succ b := \overline{\bar{b} \prec \bar{a}}$.

While $| \prec |$ and $| \succ |$ are not defined, one defines for $T \neq |$, $| \prec T := 0$ and $T \succ | := 0$.

For T, Z planar binary trees with $| \neq T = T^l \vee_\lambda T^r$, recursively $T \prec Z = T^l \vee_\lambda (T^r * Z)$, where $T * Z = T \prec Z + T \succ Z$ and (for $| \neq Z$) $T \succ Z := \overline{\bar{Z} \prec \bar{T}}$.

It holds that $T \vee Z = (T \succ \ ) \prec Z = T \succ (\ \prec Z)$.

1.8 *Remark.* In the leaf-labeled setting, we would have to define the multiplication differently.

We should define $S\backslash\backslash e := S$ and $e\backslash\backslash T := T$, $|^\lambda\backslash\backslash T = |^\lambda \vee T$. For $S \neq |^\lambda, e$ and T arbitrary we define $S\backslash\backslash T = S^l \vee (S^r\backslash\backslash T)$. Again, we define $T//S$ by $\overline{\bar{S}\backslash\backslash\bar{T}}$, and it is easy to see that $//$ and $\backslash\backslash$ are associative. Now we can define $S * T$ for leaf-labeled trees analogously to our definition of the operation $*$ above, using $//$ and $\backslash\backslash$ instead of $/$ and $\backslash$. It holds that $T*e = e*T = T$ (T arbitrary). For leaf-labeled trees, we collect the labels and write them as an exponent, e.g. we have $\ ^{\lambda_1\lambda_2} \vee \ ^{\lambda_3\lambda_4} = \ ^{\lambda_1\ldots\lambda_4}$. Now

$|^\lambda//|^\rho = |^\lambda\backslash\backslash|^\rho = |^\lambda * |^\rho = {}^\lambda\ ^\rho = \ ^{\lambda\rho}$.

$|^\lambda\backslash\backslash\ ^{\mu\rho} = \ ^{\lambda\mu\rho}$, $|^\lambda//\ ^{\mu\rho} = \ ^{\lambda\mu\rho}$, thus $|^\lambda * |^\mu * |^\rho = |^\lambda * \ ^{\mu\rho} = (\ + \ )^{\lambda\mu\rho}$.

$|^\gamma//\ _{\lambda\mu\rho} = \ _{\gamma\lambda\mu\rho}$, $|^\gamma\backslash\backslash\ _{\lambda\mu\rho} = \ _{\gamma\lambda\mu\rho}$, thus $|^\gamma * \ _{\lambda\mu\rho} = (\ + \ + \ )_{\gamma\lambda\mu\rho}$.
$|^\gamma * \ _{\lambda\mu\rho} = (\ + \ )_{\gamma\lambda\mu\rho}$.

$(\ \vee \ + | \vee \ + | \vee \ )^{\lambda_1\ldots\lambda_5}$ is not the product of two leaf-labeled trees. Its support is the intersection of the supports of $\ ^{\lambda_1\lambda_2} * \ ^{\lambda_3\lambda_4\lambda_5}$ and $\ ^{\lambda_1\lambda_2\lambda_3} * \ ^{\lambda_4\lambda_5}$, though.

2. Hopf algebras of trees.

2.1 *Proposition.*

(i) *Let* $\underline{\Delta} : (\mathbb{Q}W_\infty, \cdot) \to (\mathbb{Q}W_\infty, \cdot) \otimes (\mathbb{Q}W_\infty, \cdot)$ *be the unitary algebra homomorphism which maps any planar tree* $T = \vee(T_1, ..., T_n)$ *with trivially labeled (internal) vertices to*

$$\underline{\Delta}(T) = T \otimes | + (\cdot, \vee)\big(\underline{\Delta}(T_1) \otimes ... \otimes \underline{\Delta}(T_n)\big)$$

where $(\cdot, \vee)$ *is given by* $a_1 \otimes b_1 \otimes ... \otimes a_n \otimes b_n \mapsto a_1 \cdot ... \cdot a_n \otimes \vee(b_1 ... b_n)$. *Then* $(\mathbb{Q}W_\infty, \cdot, \underline{\Delta})$, *together with the augmentation map as counit is a (graded) Hopf algebra.*

(ii) *The same formula, with* $|$ *replaced by* e, *defines a (graded) Hopf algebra structure* $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$ *on planar trees with external vertices.*

Proof: It is easy to check, that $(\cdot, \vee)\big(\underline{\Delta}(T_1) \otimes ... \otimes \underline{\Delta}(T_n)\big)$ is given by $| \otimes T$ plus terms of higher order, thus the augmentation map is a counit. One only has to show that $\underline{\Delta}$ is coassociative. As $\underline{\Delta}(|) = | \otimes |$, $\big((\underline{\Delta} \otimes \mathrm{id})\underline{\Delta} - (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}\big)(|) = 0$.

Let T be a tree. We can assume that $\big((\underline{\Delta} \otimes \mathrm{id})\underline{\Delta} - (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}\big)(Z) = 0$ for all trees Z of a degree smaller than $\deg(T)$.

Now $\underline{\Delta} \otimes \mathrm{id}$ maps $T \otimes | + (\cdot, \vee)\big(\underline{\Delta}(T_1) \otimes ... \otimes \underline{\Delta}(T_n)\big)$ onto

$\Big(T \otimes | + (\cdot, \vee)\big(\underline{\Delta}(T_1) \otimes ... \otimes \underline{\Delta}(T_n)\big)\Big) \otimes | + (\cdot, \cdot, \vee)\big((\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(T_1), ..., (\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(T_n)\big)$.

On the other hand, $\mathrm{id} \otimes \underline{\Delta}$ maps the same element onto

$T \otimes | \otimes | + (\cdot, \underline{\Delta} \circ \vee)\big(\underline{\Delta}(T_1) \otimes ... \otimes \underline{\Delta}(T_n)\big)$

$= T \otimes | \otimes | + (\cdot, \vee)\big(\underline{\Delta}(T_1) \otimes ... \otimes \underline{\Delta}(T_n)\big) \otimes | + (\cdot, \cdot, \vee)\big((\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}(T_1), ..., (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}(T_n)\big)$.

By induction, it follows that $\underline{\Delta}$ is coassociative.

The proof of (ii), where X_∞ is regarded, is literally the same. It is also given in [4] (chap. 2 in the current version).

2.2 *Remark.* In our setting, Hopf algebras will be either graded or complete. The counit is always the augmentation map and the antipode exists (and can be computed iteratively).

The image of $\vee(|,|)$ under $\underline{\Delta}$ is $\vee(|,|) \otimes | + | \otimes \vee(|,|)$, i.e. $\vee(|,|)$ is primitive.

Let us write $\underline{\Delta}(T) = T \otimes 1 + 1 \otimes T + \underline{\Delta}_0(T)$. Then $\underline{\Delta}_0(\vee(\vee(|,|),|)) = \vee(|,|) \otimes \vee(|,|) = \underline{\Delta}_0(\vee(|,\vee(|,|)))$.

In this section, without loss of generality, all labels are omitted.

2.3 *Hopf algebra structure of Connes and Kreimer* [8], [3].

The Connes-Kreimer Hopf algebra is obtained from $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$ by regarding the quotient Hopf algebra with respect to the ideal generated by all commutators and forgetting the planar structure of the trees. Its comultiplication $\tilde{\Delta}$ is the unitary algebra homomorphism defined on $(\mathbb{Q}X^a_\infty, \cdot)$ by

$$B_+(T_1 ... T_n) \mapsto B_+(T_1 ... T_n) \otimes e + (\cdot, B_+)\big(\tilde{\Delta}(T_1) \otimes ... \otimes \tilde{\Delta}(T_n)\big).$$

E.g., it follows that $(x) \mapsto (x) \otimes e + e \otimes (x)$,

$\vee(|,|,|) = (x(x)(x)(x)) \mapsto \vee(|,|,|) \otimes e + e \otimes \vee(|,|,|) + (x)(x)(x) \otimes (x) + 3(x) \otimes (x(x)(x)) + 3(x)(x) \otimes (x(x))$.

2.4 *Hopf algebra structure of Loday and Ronco* [10]. The comultiplication Δ defined on $\big(\mathbb{Q}Y_\infty, *\big)$ is (as vector space homomorphism) given by $\Delta(|) = | \otimes |$ and, for T and Z planar binary trees,

$$\Delta(T \vee Z) = (T \vee Z) \otimes | + (*, \vee)\big(\Delta(T) \otimes \Delta(Z)\big)$$

where $(*, \vee) := (* \otimes \vee) \circ \tau_{23}$, ($\tau_{23}$ is the permutation of the second and third tensor factor).

Let us write $\Delta(T) = T \otimes 1 + 1 \otimes T + \Delta_0(T)$.

It was shown in [10] that the comultiplication is an algebra homomorphism and coassociative.

2.5 *Corollary. There exists an algebra homomorphism $pr : \big(\mathbb{Q}W_\infty, \cdot\big) \to \big(\mathbb{Q}Y_\infty, *\big)$ which is the identity map on (non-trivial) binary trees and the zero map on all non-binary trees. Then the noncommutative Hopf algebra $\big(\mathbb{Q}Y_\infty, *, \Delta\big)$ is the Hopf algebra quotient of $\big(\mathbb{Q}W_\infty, \cdot\big)$ with respect to the projection pr.*

2.6 *Hopf algebra structure of Brouder and Frabetti* [2].

The comultiplication Δ^γ defined on $\big(\mathbb{Q}Y_\infty, /^o\big)$ is the algebra homomorphism given by $\Delta^\gamma(|) = | \otimes |$, $\Delta^\gamma(\ ) = \ \otimes | + | \otimes$, and, for T and Z planar binary trees,

$$\Delta^\gamma\big(\mathbb{V}(T \vee Z)\big) = \mathbb{V}\big(T/\mathbb{V}(Z)\big) \otimes | + (\mathrm{id} \otimes \mathbb{V})\big(\Delta^\gamma(T)/(\Delta^\gamma(\mathbb{V}(Z)) - \mathbb{V}(Z) \otimes |)\big).$$

E.g., $\ \mapsto \ \otimes | + | \otimes \ + \ \otimes \ $.

It was shown in [2] that the comultiplication is coassociative.

2.7 *Cutting trees.* For a planar tree $T \in W_\infty$ (abstract tree $T \in X^a_\infty$, respectively) let $M(T)$ (or $\tilde{M}(T)$, respectively) be the following set: We take all $\{\lambda, \rho\}$-labeled trees $(T)_w$, w a word of length $\deg(T)$, such that the label of the root is ρ and such that for each vertex with label ρ all the vertices on the path to the root are labeled with ρ.

E.g., $M(\ ) = \{\ _{\rho\rho\rho}, \ _{\rho\lambda\rho}, \ _{\rho\lambda\lambda}\}$.

$\tilde{M}(\ ) = \tilde{M}\big(\ (x(x)(x)(x))\ \big) = \big\{(\rho\ (\gamma_1)(\gamma_2)(\gamma_3)) : \gamma_i \in \{\lambda, \rho\}\big\}$.

Then, for $S \in M(T)$, $P_\rho(T)$ is given by S with all λ-labeled vertices deleted, i.e. by the tree which remains after cutting all branches from ρ- to λ-labeled vertices. Let $P_\lambda(T)$ be the tuple of λ-subtrees which fall down (ordered from left to right).

2.8 *Proposition.*

For the comultiplication $\Delta : \big(\mathbb{Q}Y_\infty, *\big) \to \big(\mathbb{Q}Y_\infty, *\big) \otimes \big(\mathbb{Q}Y_\infty, *\big)$ it holds that $\Delta(T)$, T a planar binary tree, is given by

$$T \otimes | + \sum_{S \in M(T)} (* \circ P_\lambda)(S) \otimes P_\rho(S),$$

where $(* \circ P_\lambda)(S)$ is the product of all components of $P_\lambda(S)$.

Proof: If we define a vector space homomorphism by the asserted formula, the recursion for the image of $T \vee Z$ is easily detected to be given by the formula in (2.4).

2.9 *Remark.* The analogous proposition (with M replaced by $\tilde{M}$) holds for the Hopf algebra $\mathbb{Q}X^a_\infty$, see [3].

Let $\mathbb{Q}Y'_n$ be the subvector space of $\mathbb{Q}X^a_n$ generated by all abstract trees that can be obtained by applying $\tilde{\Gamma}$ to a planar binary tree of degree n, and let $\mathbb{Q}Y'_\infty$ be the union of all $\mathbb{Q}Y'_n$. The abstract trees (x), $(x(x))$, $(x(x(x)))$ and $(x(x)(x))$ are the images of , , , under $\tilde{\Gamma}$. An element of $\mathbb{Q}Y_\infty$ which is not in $\mathbb{Q}Y'_\infty$ is for example $(x(x)(x)(x))$. The image of $\big(\mathbb{Q}Y_\infty, *, \Delta\big)$ under $\tilde{\Gamma}$ is naturally a quotient Hopf algebra of $(\mathbb{Q}X^a_\infty, \cdot, \tilde{\Delta})$, which we denote by $(\mathbb{Q}Y'_\infty, \tilde{*}, \tilde{\Delta})$.

For example, $(x(x)(x))$ is mapped to $(x(x)(x)) \otimes e + e \otimes (x(x)(x)) + (x)(x) \otimes (x) + (x) \otimes (x)(x)$ by the comultiplication $\tilde{\Delta}$ of $\big(\mathbb{Q}X^a_\infty, \cdot\big)$.

On the other hand $(x(x)(x))$ is mapped to $(x(x)(x)) \otimes e + e \otimes (x(x)(x)) + 2(x(x)) \otimes (x) + 2(x) \otimes (x(x))$ by the comultiplication $\tilde{\Delta}$ of $\big(\mathbb{Q}Y'_\infty, \tilde{*}\big)$.

It is easy to see that both Hopf algebra structures are commutative but not cocommutative.

Let us now compare the noncommutative version of the Connes-Kreimer Hopf algebra with the Hopf algebra $\big(\mathbb{Q}Y_\infty, *, \Delta\big)$.

Let $\Theta : (\mathbb{Q}Y_\infty, *) \to (\mathbb{Q}X_\infty, \cdot)$ be the (graded) algebra homomorphism which is uniquely defined by $\Theta(|) = e$, $\Theta(\mathbb{V}(Z)) = (\vee \circ \Theta)(Z), Z \in Y_\infty$.

For example, $\mapsto | = (x)$, $\mapsto (x(x))$,

$= \mathbb{V}(* -) \mapsto \vee\big((x)(x)\big) - \vee\big((x(x))\big) = (x\ (x)(x)) - (x(x(x)))$.

2.10 *Theorem.*

The graded algebra homomorphism Θ provides a Hopf algebra isomorphism between the Hopf algebra $\big(\mathbb{Q}Y_\infty, *, \Delta\big)$ of [10] *and the noncommutative version $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$ of the Hopf algebra of* [3].

Proof:

Let us first show that Θ is a bijection. We note that the number of trees of the form $\mathbb{V}(T) \in Y_n$ is c_n and that this is also the number of trees in X_n. Via Θ, $Y_1 = \{\mathbb{V}(|)\}$ corresponds to $X_1 = \{|\}$, and $Y_2 = \{\mathbb{V}( )\}$ corresponds to $X_2 = \{\vee(|)\}$.

Let $\xi : (\mathbb{Q}X_\infty, \cdot) \to (\mathbb{Q}Y_\infty, *)$ be the algebra homomorphism given by $\xi(e) = |$, $\xi\big(T\big) = \vee\big(\xi(\neg(T))\big)$, for planar trees T. Then $\Theta\big(\xi(T)\big) = \big(\vee \circ \Theta \circ \xi \circ \neg\big)(T) = T$ by induction. Similarly $(\xi \circ \Theta)\big(\mathbb{V}(Z)\big) = \mathbb{V}(Z)$, for all Z in Y_∞.

It is left to show that $(\Theta \otimes \Theta) \circ \Delta = \underline{\Delta} \circ \Theta$. Clearly $(\Theta \otimes \Theta)(\Delta(|)) = e \otimes e = \underline{\Delta}(e)$. Again, we use induction over the degree and assume that $(\Theta \otimes \Theta)(\Delta(S)) = \underline{\Delta}\big(\Theta(S)\big)$ for all $S \in \mathbb{Q}Y_\infty$ of degree $< r$.

Let $\mathbb{V}(Z) \in Y_\infty$ be a tree of degree r. Thus Z is a tree of degree $r - 1$.

$\mathbb{V}(Z)$ is mapped by $(\Theta \otimes \Theta) \circ \Delta$ onto $\Theta(\mathbb{V}(Z)) \otimes e + \big(\Theta \otimes (\Theta \circ \mathbb{V})\big)(\Delta(Z))$
$= \vee(\Theta(Z)) \otimes e + (id, \vee)\big((\Theta \otimes \Theta) \circ \Delta(Z)\big) = \vee(\Theta(Z)) \otimes e + (id, \vee)\big(\underline{\Delta}(\Theta(Z))\big)$
$= \underline{\Delta}\big(\vee(\Theta(Z))\big) = (\underline{\Delta} \circ \Theta)\big(\mathbb{V}(Z)\big)$.

The $\mathbb{V}(Z)$ of degree r together with lower degree elements generate all elements of degree r in $\mathbb{Q}Y_\infty$. Thus the equation holds, and Θ is a (graded) Hopf algebra isomorphism.

2.11 *Remark.*

(i) An algebra isomorphism from the algebra $(\mathbb{Q}Y_\infty, /^o)$ of [2] to $(\mathbb{Q}X_\infty, \cdot)$ similar to Θ is used in [4]. Here $\mapsto | = (x)$, $\mapsto (x(x))$, $\mapsto (x(x)(x))$.
The induced Hopf algebra structure on $(\mathbb{Q}X_\infty, \cdot)$ is given by cuts with a special property: Cutting "left branches" is forbidden, i.e. $M(T)$ has to be replaced by the subset $M^l(T)$ of $M(T)$ determined by the condition, that no λ appears which is connected to a vertex labeled by ρ via a branch which is the leftmost outgoing branch of the latter vertex. E.g. $M^l(\ ) = \{(\rho\ (\rho)(\gamma_1)(\gamma_2)) : \gamma_i \in \{\lambda, \rho\}\}$.
In [4] it is also shown, that this Hopf algebra is isomorphic to the noncommutative Connes-Kreimer Hopf algebra. Using this together with Theorem (2.10) we can conclude that the Hopf algebra of [2] is also isomorphic to the Hopf algebra of [10].

(ii) In view of Theorem (2.10), the noncommutative Hopf algebra of [3] is provided with the operations $\prec$ and $T \mapsto \bar{T}$.
For example, $\overline{(x)\cdot(x(x))} = \overline{\Theta(\ ) * \Theta(\ )} = -(x(x))\cdot(x) + (x(x(x)))$ and $(x(x)) \prec (x) = \Theta(\mathbb{V}(\ ) \prec \mathbb{V}(|)) = \Theta(\mathbb{V}(\ * \ )) = (x(x)(x))$.

(iii) Theorem (2.10) also allows to use the description of primitive elements and the connection with brace algebras, see [15](4.7), for the Hopf algebras of renormalization. For example, in degree 2, $\mathrm{Prim}_2(\mathbb{Q}Y_\infty)$ is the vector space generated by the element $\ \prec\ -\ \succ\ $. It is given by the brace algebra element $\langle\ ,\ \rangle$. As $\ -\ = 2\mathbb{V}(\ ) - \ *\ $, the associated primitive element of $(\mathbb{Q}X_\infty, \cdot, \underline{\Delta})$ is $2(x(x)) - (x)(x)$. In degree 3, a vector space basis of $\mathrm{Prim}_3(\mathbb{Q}Y_\infty)$ is given by $\ +\ -\ $ and $\ +\ -\ $. The corresponding primitive elements of the noncommutative Connes-Kreimer Hopf algebra are $(x(x))(x) - (x)(x(x))$ and $(x)(x)(x) + 3(x(x(x))) - 2(x)(x(x)) - (x(x))(x)$.

3. Lie structure constants and cogroups for planar binary trees.

3.1 *Remark.* It has been shown in [13], [15] that in each degree n, the vector space Prim_n of primitive elements of $(\mathbb{Q}Y_\infty, *, \Delta)$ has dimension c_n. A similar result for the noncommutative version of the Connes-Kreimer Hopf algebra has been shown in [4].

We recall, that c_n is also the number of free algebra generators in each degree n. For an algebra basis up to degree 4 we can use the $9 = c_1 + c_2 + c_3 + c_4$ trees $U_1, ..., U_4, U_6, ..., U_{10}$, where

$U_1 := \ = \mathbb{V}(|), U_2 := \ = \mathbb{V}^2(|), U_3 := \ = \mathbb{V}^3(|), U_4 := \ = \mathbb{V}(\ )$,
$U_6 := \mathbb{V}(\ ), U_7 := \mathbb{V}(\ ), U_8 := \mathbb{V}(\ ), U_9 := \mathbb{V}(\ ), U_{10} := \mathbb{V}(\ )$.

We denote by U_5 the tree .

A sub-Hopf algebra of $(\mathbb{Q}W_\infty, \cdot, \underline{\Delta})$ is given by $(\mathbb{Q}Y_\infty, \cdot, \underline{\Delta})$; and $(\mathbb{Q}Y_\infty, *, \Delta)$ is a quotient Hopf algebra of $(\mathbb{Q}Y_\infty, \cdot, \underline{\Delta})$, cf. Corollary (2.5).

If $U_{11} := \ \vee\ $ and $U_{12} := \ \vee\ $, we recall that a basis $\{x_i : i = 1, ..., 22\}$ of $(\mathbb{Q}Y_\infty, \cdot)$ is up to degree 4 given by all nontrivial trees of degree ≤ 4, i.e. by $U_1, U_2, \bar{U}_2, U_3, U_4, U_5, \bar{U}_4, \bar{U}_3, U_6, ..., U_{12}, \bar{U}_{12}, ..., \bar{U}_6$.

When $\{x_n : n = 1, 2, ...\}$ denotes a basis of a complete Hopf algebra (H, Δ), then we call the differences $\psi^{(n)}_{a,b}$ between the coefficients of $x_a \otimes x_b$ and $x_b \otimes x_a$ of $\Delta(x_n)$ the Lie structure constants of the (complete) Hopf algebra, cf. [6]. For primitive x_n, all $\psi^{(n)}_{a,b} = 0$.

3.2 *The first Lie structure constants of* $(\mathbb{Q}Y_\infty, \cdot)$.
The images of the first elements under $\underline{\Delta}_0$, Δ_0 respectively, are:
$U_1 \mapsto 0, U_2 \mapsto \ \otimes\ $, $U_3 \mapsto \ \otimes\ + \ \otimes\ $, $U_4 \mapsto \ \otimes\ + \ \otimes\ $.
$\underline{\Delta}_0(U_5)$ is given by $\ \otimes (\ + \ ) + \ \cdot \ \otimes \ =: \ \otimes \ * \ + \ \cdot \ \otimes \ $ (e.g., for $S = (\ )_{\lambda\rho\lambda}$, $P_\lambda(S) = \ \cdot\ $, $P_\rho(S) = \ $). We note that $\Delta_0(U_5)$ is given by $\ \otimes \ * \ + \ * \ \otimes \ $.

Clearly, for $(\mathbb{Q}Y_\infty, \cdot)$ all $\psi^{(n)}_{a,b}$ are zero up to $n = 4$ and also for $n = 8$ (where $x_8 = \bar{U}_3$). We read from $\underline{\Delta}_0(U_4)$ that $\psi^{(5)}_{1,2} = 1 = -\psi^{(5)}_{2,1}$, $\psi^{(5)}_{1,3} = -1 = -\psi^{(5)}_{3,1}$ and that all other $\psi^{(5)}_{a,b}$ are zero. Similarly, the nontrivial $\psi^{(7)}_{a,b}$ are given by $\psi^{(7)}_{1,2} = -1, \psi^{(7)}_{1,3} = 1$.

We read from $\underline{\Delta}_0(U_5)$ that $\psi^{(6)}_{1,2} = 1, \psi^{(6)}_{1,3} = 1$ and that all other $\psi^{(6)}_{a,b}$ (with $a < b$) are zero.

For $n = 9$ (and $n = 22$) we have trivial $\psi^{(n)}_{a,b}$ as $U_6 \mapsto \ \otimes \ + \ \otimes \ + \ \otimes \ $.
$U_7 \mapsto \ \otimes \ + \ \otimes \ + \ \otimes \ $
and $\psi^{(10)}_{1,4} = -\psi^{(10)}_{1,5} = -\psi^{(10)}_{2,3} = -\psi^{(21)}_{1,7} = \psi^{(21)}_{1,8} = \psi^{(21)}_{2,3} = 1$.
$U_8 \mapsto \ \otimes \ + \ \otimes (\ + \ ) + \ \cdot \ \otimes \ $
and $\psi^{(11)}_{1,4} = \psi^{(11)}_{1,5} = -\psi^{(11)}_{1,6} = -\psi^{(20)}_{1,6} = \psi^{(20)}_{1,7} = \psi^{(20)}_{1,8} = 1$
$U_9 \mapsto \ \otimes \ + \ \otimes \ + \ \otimes \ $ and $\psi^{(12)}_{1,5} = -\psi^{(12)}_{1,7} = -\psi^{(19)}_{1,5} = \psi^{(19)}_{1,7} = 1$.
$U_{10} \mapsto \ \otimes \ + \ \otimes \ + \ \otimes \ $
and $\psi^{(13)}_{1,5} = -\psi^{(13)}_{1,8} = -\psi^{(13)}_{2,3} = -\psi^{(18)}_{1,4} = \psi^{(18)}_{1,7} = \psi^{(18)}_{2,3} = 1$.
$U_{11} \mapsto \ \otimes \ + \ \otimes \ * \ + \ \cdot \ \otimes \ + \ \cdot \ \otimes \ $
and $= \psi^{(14)}_{1,4} = \psi^{(14)}_{1,6} = \psi^{(14)}_{2,3} = \psi^{(17)}_{1,6} = \psi^{(17)}_{1,8} = -\psi^{(17)}_{2,3} = 1$.
$U_{12} \mapsto \ \otimes (\ + \ ) + \ \otimes \ + \ \cdot \ \otimes \ + \ \cdot \ \otimes \ $
and $\psi^{(15)}_{1,5} = \psi^{(15)}_{1,6} = \psi^{(16)}_{1,6} = \psi^{(16)}_{1,7} = 1$.
The remaining $\psi^{(n)}_{a,b}$ for $9 \leq n \leq 22$ (and $a < b$) are zero.
The corresponding Lie algebra is a nilpotent Lie algebra.

3.3 *Lie structure constants of* $(\mathbb{Q}Y_\infty, *)$ *up to tree-degree 4.*
To deal with as many primitive elements as possible, we are going to modify the basis given in [10], starting with the following four elements.
Let $Z_1 := \ $. We set $Z_2 := 2\mathbb{V}(\ ) - \ * \ = \ - \ $, $Z_3 := \ - \ + \ $, and $Z_4 := \ - \ $.
Thus $\ = \frac{1}{2}(Z_1 * Z_1 + Z_2), \ = \frac{1}{2}(Z_1 * Z_1 - Z_2)$ as $Z_1 * Z_1 = \ + \ $.
Furthermore: $\ = \frac{1}{3}(Z_1 * Z_1 * Z_1 + \frac{1}{2}[Z_1, Z_2] - Z_3)$,
$\ = \frac{1}{6}(Z_1 * Z_1 * Z_1 + 2Z_1 * Z_2 + Z_2 * Z_1 + 2Z_3)$,
$\ = \frac{1}{6}(Z_1 * Z_1 * Z_1 - Z_1 * Z_2 - 2Z_2 * Z_1 + 2Z_3)$, $\ + \ = \frac{1}{3}(Z_1 * Z_1 * Z_1 - [Z_1, Z_2] - Z_3)$.

The elements Z_1, Z_2, and Z_3 of the Hopf algebra $(\mathbb{Q}Y_\infty, *)$ are primitive, cf. Remark (2.11). We compute that $\Delta_0(Z_4) = \curlyvee_{\backslash} \otimes \curlyvee - \curlyvee \otimes \curlyvee_{\backslash} - \curlyvee_{/} \otimes \curlyvee + \curlyvee \otimes \curlyvee_{/} = Z_1 \otimes Z_2 - Z_2 \otimes Z_1$. The difference $\psi^{(4)}_{1,2}$ between the coefficients of $Z_1 \otimes Z_2$ and $Z_2 \otimes Z_1$ is 2.

The following elements are primitive:
$Z_5 := U_6 - \bar{U}_6 + \bar{U}_8 - U_8 + U_{10} - U_{12}$, $Z_6 := U_{12} - U_{10} + \bar{U}_{10} - \bar{U}_{12}$,
$Z_7 := \bar{U}_7 - U_7 + U_8 - \bar{U}_8 + \bar{U}_9 - U_9$, $Z_8 := \bar{U}_9 - \bar{U}_{10} - U_{12} + U_{10} + U_9 + U_{11} - U_8$ and $\bar{U}_8 - \bar{U}_{11} - \bar{U}_9 - \bar{U}_{10} + \bar{U}_{12} + U_{10} - U_9$ (which is a linear combination of the above and products of Z_1,...,Z_4).

This can be concluded from [13](4.7) or by a direct computation.

Together with $Z_9 := U_{10} + \bar{U}_{12} + U_{12} + \bar{U}_{10} - U_7 - \bar{U}_7 - U_8 - \bar{U}_{11}$, the $Z_i, i \geq 1$ form a basis of the free algebra $(\oplus_{n=0}^4 Y_n, *)$.

It holds that $\Delta_0(Z_9) =$
$\frac{1}{2} Z_2 \otimes Z_2 + \frac{1}{3}\big(Z_1 \otimes (Z_1 * Z_1 * Z_1 - [Z_1, Z_2] - 7Z_3) + (Z_1 * Z_1 * Z_1 - [Z_1, Z_2] + 5Z_3) \otimes Z_1\big)$
$+ \frac{1}{2}(Z_1 * Z_1 \otimes Z_1 * Z_1)$.

Thus $\psi^{(9)}_{1,3} = \frac{1}{3}(-12) = -4$ and all remaining $\psi^{(n)}_{ab} = 0$ $(a < b)$.

Structure constants for degree 3 can be chosen to be zero (for a suitable basis). The given 9-dimensional Lie algebra is nilpotent and has a 4-dimensional commutative Lie algebra as a direct summand.

3.4 *Remark.* In [10] it is shown, that $(\mathbb{Q}Y_\infty, \prec, \succ)$ is the free dendriform algebra – also called dual dialgebra – on one generator $\curlyvee$.
The categorical coproduct $\mathbb{Q}Y_\infty \sqcup \mathbb{Q}Y_\infty$ is given by the free dendriform algebra $\mathbb{Q}Y_{\infty,2}$ on two generators $\curlyvee_\lambda := \iota_1(\curlyvee)$, $\curlyvee_\rho := \iota_2(\curlyvee)$.

In the category of associative algebras we form the coproduct $(\mathbb{Q}Y_\infty, *) \sqcup_{\mathcal{A}s} (\mathbb{Q}Y_\infty, *)$, which is canonically identified with a subspace of $(\mathbb{Q}Y_{\infty,2}, *, \prec, \succ)$.

Complete cogroups (i.e. a structure similar to a complete Hopf algebra, but with the completed coproduct used instead of the tensor product) in the category of associative algebras are classified by the not necessarily unitary associative algebras, as the operad $\mathcal{A}s$ is selfdual, cf. [5]. The equations for structure constants of associative algebras are $\sum_l \phi^{(n)}_{al} \phi^{(l)}_{bc} = \sum_l \phi^{(n)}_{lc} \phi^{(l)}_{ab}$.

The class of complete Hopf algebras that are induced by complete $\mathcal{A}s$-cogroups is regarded in [7] (6.6). The structure constants for the associated associative algebra determine the cogroup and the Hopf algebra structure up to isomorphism.

Let $\phi^{(n)}_{hi} = 0$ for all $n \leq 3$, all h, i. Let $\phi^{(4)}_{12} = 1, \phi^{(4)}_{21} = -1$ (and all other be 0). One checks that the equations above are fulfilled. Thus there exists a cogroup in the category of complete associative algebras with these structure constants, e.g. we let the first three basis elements be primitive and define the image of Z_4 to be $\iota_1(Z_4) + \iota_2(Z_4) + \iota_1(\curlyvee)\iota_2(Z_2) - \iota_1(Z_2)\iota_1(\curlyvee)$. The associated complete Hopf algebra has the same Lie structure constants as $(\oplus_{n=0}^3 \mathbb{Q}Y_n, *, \Delta)$.

3.5 *Cogroup structure.* Let $\mathbb{Q}Y^u_n$ be the subvector space of $\mathbb{Q}Y^u_n$ generated by all trees T that have the following property: all internal vertices of T lie on the outermost branches, e.g. $T =$. We define a homomorphism $\nabla : \mathbb{Q}Y^u_\infty \to \mathbb{Q}Y^u_\infty \sqcup \mathbb{Q}Y^u_\infty$ by

$$\iota_1(T) + \sum_{S \in M(T)} \iota_1(P^l_\lambda(S)) * \iota_2(P_\rho(S)) * \iota_1(P^r_\lambda(S))$$

where $P_\rho(S)$ is the subtree of S given by all vertices with label ρ and $P^l_\lambda(S)$ is the λ-subtree on the left of $P_\rho(S)$, $P^r_\lambda(S)$ the λ-subtree on the right.

E.g. ∇ fulfills $\mapsto {}_\lambda + {}_\rho$, $\mapsto \iota_1() + \iota_2() + \iota_2() * \iota_1()$. The image of is $\iota_1() + \iota_2() + \iota_1() * \iota_2() + \iota_2() * \iota_1() + \iota_1() * \iota_2() * \iota_1()$. It holds that ∇ is coassociative and that $\nabla(\bar{T}) = \overline{\nabla(T)}$. ∇ defines an $\mathcal{A}s$-group law.

3.6 *Extension.*

Is there an extension to a coassociative homomorphism $\nabla : \mathbb{Q}Y_\infty \to \mathbb{Q}Y_\infty \sqcup \mathbb{Q}Y_\infty$ which extends $\nabla : \mathbb{Q}Y^u_\infty \to \mathbb{Q}Y^u_\infty \sqcup \mathbb{Q}Y^u_\infty$?

For $+$, the image must be $\iota_2() * \iota_1() + \iota_1() * \iota_2() + \iota_2() * \iota_1() * \iota_2()$ to be compatible with $\nabla(* *) = \nabla() * \nabla() * \nabla()$. Using the coassociativity condition a short computation shows that the image of $T =$ (or $T =$) under ∇ must be of the form

$$\iota_1(T) + \iota_2(T) + \frac{1}{2}\big(\iota_1() * \iota_2() + \iota_2() * \iota_1() + \iota_2() * \iota_1() * \iota_2()\big)$$
$$+ c\big(\iota_1() * \iota_2() - \iota_1() * \iota_2() - \iota_2() * \iota_1() + \iota_2() * \iota_1()\big),$$

or, equivalently, the image of $Z_4 = -$ is of the form

$2c\big(\iota_1() * \iota_2() + \iota_1() * \iota_2() - \iota_2() * \iota_1() - \iota_2() * \iota_1()\big)$.

The associated Hopf algebra comultiplication sends Z_4 to $2c((-) \otimes + \otimes (-))$. If we compare with $\Delta(Z_4) = \otimes Z_2 - Z_2 \otimes$ we can conclude, that the extension is not compatible with the given Hopf algebra structure of [10].

3.7 *Concluding remarks.*

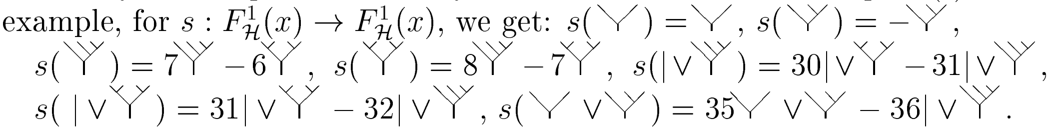
(i) We have seen that the several infinite-dimensional Hopf algebra structures on trees are related by isomorphisms or forming quotients and subalgebras. In [12] a Hopf algebra of planar trees is introduced, which extends the Hopf algebra of [10]. It should also be related to the Hopf algebra structures regarded in this article.

(ii) Not only should the results about the dendriform Hopf algebra $\mathbb{Q}Y_\infty$ be translated to the Connes-Kreimer Hopf algebra, it is also interesting to transfer techniques in the other direction. For example, the pairing between the (non-commutative) Connes-Kreimer Hopf algebra and itself constructed in [4] can be regarded for the dendriform Hopf algebra $\mathbb{Q}Y_\infty$.

(iii) On the free unitary algebras $F^1_{\mathcal{K}}(x)$ and $F^1_{\mathcal{H}}(x)$ the homomorphism Δ defined by $x \mapsto x \otimes 1 + 1 \otimes x$ into $F^1_{\mathcal{K}}(x) \otimes F^1_{\mathcal{K}}(x)$, $F^1_{\mathcal{H}}(x) \otimes F^1_{\mathcal{H}}(x)$ respectively, is coassociative: $x \otimes 1 + 1 \otimes x$ is mapped onto $x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$. We get that $\Delta() = \Delta(xx) = \otimes 1 + 1 \otimes + 2| \otimes |$, where we use the (trivial) labeling of leaves.

While the (complete or graded) Hopf algebras regarded in this article were

associative, $F^1_{\mathcal{K}}(x)$ together with Δ and the augmentation map ε is a non-associative bialgebra. Again we get the existence and uniqueness of a left antipode and a right antipode. For the commutative non-associative bialgebra $F^1_{\mathcal{H}}(x)$ we get a unique antipode, which is a linear map $F^1_{\mathcal{H}}(x) \to F^1_{\mathcal{H}}(x)$ with $\sum s(a_i)b_i = 0$ (where $\sum a_i \otimes b_i = \Delta(f)$) for all f.
It is easy to compute recursively the first values of the antipode(s). For example, for $s : F^1_{\mathcal{H}}(x) \to F^1_{\mathcal{H}}(x)$, we get:



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Ralf Holtkamp, Bochum
E-mail address: ralf.holtkamp@ruhr-uni-bochum.de