Lecture Notes on

Minimal Surfaces and Plateau's Problem

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If you find mistakes in these notes and/or have any other comments, please communicate them to the author, either in person or at schmidt@math.fau.de.

Chapter 1

Basic concepts and examples

The **(classical) Plateau problem** consists in finding a surface of least area among the two-dimensional surfaces with a given closed boundary curve in $\mathbb{R}^3$. Implementations of this problem describe the behavior of soap films which are spanned by a wire, and the problem is named after the physicist Plateau, who experimentally investigated this aspect in the 19th century. The mathematical problem, however, has already been raised by Lagrange in the 18th century. It can also be posed in higher dimensions, and thus the classical problem is in fact the special case $n = 2$, $X = \mathbb{R}^3$ of the **(generalized) Plateau problem**

$$\inf\{n\text{-area}(M) : M \; n\text{-dimensional surface in } X, \; \text{boundary}(M) = R\}\,, \tag{1.1}$$

where R is a given $(n-1)$-dimensional closed surface in an ambient space X of dimension $\geq n$. In order to turn (1.1) into a meaningful problem it remains, however, to specify suitable notions of 'n-dimensional surface', 'n-area', and 'boundary', and in fact this can be achieved by quite different approaches, whose availability depends also on the ambient space X; compare the outline in Chapter 2. Though it is possible to admit, as ambient spaces X, abstract manifolds or, in some more recent developments, even possibly ∞-dimensional metric spaces, in these notes we restrict the attention to Euclidean spaces $X = \mathbb{R}^\ell$, $\ell \geq n$. In the Euclidean case we call $n \in \mathbb{N}$ the **dimension** and $N := \ell - n \in \mathbb{N}_0$ the **codimension** of the problem. Even though there are some exceptions, the dimension-one case $n = 1$ and the codimension-zero case $N = 0$ are *typically* of limited interest. The reason for this is that, for $n = 1$, straight lines are potentially extremely simple solutions, while, for $N = 0$ there is normally just one admissible competitor. Therefore, the following exposition focuses on the first interesting cases $n = 2$ and $N = 1$ (but occasionally and towards the end we even deal with arbitrary dimensions $n, N \in \mathbb{N}$).

Next, in order to concretize the Plateau problem, we introduce *the* modern notion of n-dimensional surfaces (which, anyway, turns out to be too restrictive for some of our purposes).

1.1 Submanifolds and their curvature

Definitions (submanifold, tangent space).

(1) *A subset M of $\mathbb{R}^n$ is called an n-dimensional C^k* ***submanifold with boundary*** *in $\mathbb{R}^\ell$ (with $n, \ell \in \mathbb{N}$, $\ell \geq n$, $k \in \mathbb{N} \cup \{\infty, \omega\}$) if the following holds true for every $y_0 \in M$: there exist an open neighborhood V of y_0 in $\mathbb{R}^\ell$, an open neighborhood U of 0 in $\mathbb{R}^n$, and a*

C^k *mapping* $Y: U \to \mathbb{R}^\ell$ *of maximal rank* n (*that means* $\operatorname{rank}(\mathrm{D}Y(x)) = n$ *or equivalently* $\det(\mathrm{D}Y(x)^*\mathrm{D}Y(x)) > 0$ *for all* $x \in U$) *such that there holds* $Y(0) = y_0$ *and*

either Y *maps* U *homeomorphicly*[1] *onto* $M \cap V$
or Y *maps* $\{x \in U : x_1 \geq 0\}$ *homeomorphicly onto* $M \cap V$.

The mappings Y *in this definition are known as* ***regular $\mathbf{C}^k$ parametrizations***; *between possibly decreased neighborhoods* U *and* V, *they are always biLipschitz maps.*

(2) *The offset alternative in the preceding definition is actually strict in the sense that its two cases cannot both occur for a fixed point* $y_0 \in M$, *not even with different parametrizations. Thus, it makes sense to define the* ***geometric interior*** $M \setminus \partial M$ *of* M *as the set of points* $y_0 \in M$ *for which the first alternative occurs. Correspondingly, the* ***geometric boundary*** ∂M *is the set of points of* M *for which the second alternative occurs. In the case* $\partial M = \emptyset$, *we call* M *a* ***submanifold without boundary***, *and we remark that, whenever* M *is an* n*-dimensional submanifold with boundary, then* $M \setminus \partial M$ *is an* n*-dimensional submanifold without boundary, and* ∂M *is an* $(n-1)$*-dimensional submanifold without boundary.*

(3) *The* ***tangent space*** $\mathrm{T}_{y_0}M$ *to* M *in* $y_0 \in M \setminus \partial M$ *is, for a regular parametrization* Y *as above, given as the* n*-dimensional subspace*

$$T_{y_0}M := \{\mathrm{D}Y(0)v : v \in \mathbb{R}^n\}$$

of $\mathbb{R}^\ell$. *One can show the characterization*

$$T_{y_0}M := \{c'(0) : c \text{ is a } \mathrm{C}^1 \text{ curve with } c(t) \in M \text{ for } |t| \ll 1\},$$

and, consequently, $T_{y_0}M$ *does not depend on the choice of a parametrization* Y.

Remark (alternative definitions of a submanifold). *Alternatively, one can define the notion of an* n*-dimensional* C^k *submanifold* M *in* $\mathbb{R}^\ell$ *by the*

- ***implicit representation***: M *is locally the set of solutions* x *of a possibly non-linear system of equations* $g(x) = 0$ *with a* C^k *function* $g: \mathbb{R}^\ell \supset\!\to \mathbb{R}^{\ell-n}$ *of maximal rank* $(\ell-n)$;

or by the

- ***explicit representation***: M *is locally a rotated graph of a* C^k *function* $u: \mathbb{R}^n \supset\!\to \mathbb{R}^{\ell-n}$.

It follows from the inverse and implicit function theorems that these alternative definitions yield the same notion as the above approach of ***parametric representation***.

Definitions (curvature of hypersurfaces). *Consider an* n*-dimensional* C^2 *submanifold* M *with boundary in* $\mathbb{R}^{n+1}$.

(1) *The* ***Gauss map*** *of* M *maps points* $x \in M \setminus \partial M$ *to a (unique up to its sign) unit normal vector* $\nu(x)$ *to* M *in* x, *that is, to a unit vector* $\nu(x)$ *which is orthogonal to* T_xM *in* $\mathbb{R}^{n+1}$. *Since* M *is* C^2, *every* $x \in M \setminus \partial M$ *has a neighborhood* U_x *in* M *such that the Gauss map has a realization* $\nu \in \mathrm{C}^1(U_x, \mathrm{S}_1)$ (*while a global realization* $\nu \in \mathrm{C}^1(M \setminus \partial M, \mathrm{S}_1)$ *exists only for orientable* M, *but not, for instance, in case of the Möbius strip*).

[1] Here, the weaker requirement that the mapping is merely one-to-one does not yield the standard notion of submanifolds. Indeed, for $n = 1$ and $\ell = 2$, it is possible that a smooth $\mathbb{R}^2$-valued one-to-one map Y on an open interval (a, b) wraps the interval around such that the potential endpoint $\lim_{s \nearrow b} Y(s)$ comes to lie on an another point $Y(t)$ with $t \in (a, b)$. Homeomorphisms, however, cannot create this type of almost self-intersection.

(2) *The* ***Weingarten map*** $S(x)$*, also know as the* ***shape operator****, of* M *in* $x \in M \setminus \partial M$ *is the (negated) derivative of the Gauss map* ν *in a point* $x \in M \setminus \partial M$*, that means*

$$S(x) := -\mathrm{D}\nu(x)\colon \mathrm{T}_x M \to \mathrm{T}_x M,\ v \mapsto -\mathrm{D}\nu(x)v\,.$$

Here, it can be justified in two different ways that $S(x)$ *is well-defined and* $\mathrm{T}_x M$*-valued. Relying only on standard Euclidean differential calculus, one may understand that* $\nu \in \mathrm{C}^1(U_x, \mathrm{S}_1)$ *indicates the existence of a* C^1 *extension* $\widetilde{\nu}\colon \widetilde{U}_x \to \mathrm{S}_1 \subset \mathbb{R}^{n+1}$ *to an open set* $\widetilde{U}_x$ *in* $\mathbb{R}^{n+1}$*. Then, by the chain rule for differentiation along curves,* $\mathrm{D}\widetilde{\nu}(x)v$ *with* $v \in \mathrm{T}_x M$ *does not depend on the choice of the extension (though* $\mathrm{D}\widetilde{\nu}(x)$*, in general, does) and moreover the computation* $\nu(x)\cdot \mathrm{D}\widetilde{\nu}(x)v = \frac{1}{2}v\cdot\nabla|\widetilde{\nu}|^2(x) = 0$ *shows* $\mathrm{D}\widetilde{\nu}(x)v \in \mathrm{T}_x M$*. This justifies that* $\mathrm{D}\nu(x)v \in \mathrm{T}_x M$ *is well-defined. Alternatively, one may view and differentiate* ν *as a* C^1 *mapping between manifolds. Then it is immediate that the derivative* $\mathrm{D}\nu(x)\colon \mathrm{T}_x M \to \mathrm{T}_{\nu(x)}\mathrm{S}_1 = \{\nu(x)\}^\perp = \mathrm{T}_x M$ *makes sense.*

(3) ***The Weingarten map*** $\boldsymbol{S(x)}$ ***is always self-adjoint*** *with respect to the Euclidean inner product on* $\mathrm{T}_x M \subset \mathbb{R}^{n+1}$*. To verify this claim, it suffices to check it for some basis of* $\mathrm{T}_x M$*, which can be taken equal to* $\partial_1 Y(0)$*,* $\partial_2 Y(0)$*, …,* $\partial_n Y(0)$ *for a regular* C^2 *parametrization* Y *of* M *near* x *with* $Y(0) = x$*. Differentiating the identity* $\partial_i Y \cdot \nu(Y) \equiv 0$ *near* 0 *with respect to the* j*-th variable one finds* $\partial_j\partial_i Y(0)\cdot\nu(x) + \partial_i Y(0)\cdot \mathrm{D}\nu(x)\partial_j Y(0) = 0$ *for arbitrary* $i, j \in \{1, 2, \ldots, n\}$*. Using the last identity twice, one deduces*

$$\partial_i Y(0)\cdot S(x)\partial_j Y(0) = \partial_j\partial_i Y(0)\cdot\nu(x) = \partial_j Y(0)\cdot S(x)\partial_i Y(0)$$

and thus arrives at the claim.

(4) *The self-adjointness of* $S(x)$ *implies that eigenvectors of* $S(x)$ *form an orthonormal base of* $T_x M$*. The corresponding* n *eigenvalues* $\kappa_1(x)$*,* $\kappa_2(x)$*, …,* $\kappa_n(x)$ *(where evidently multiplicity* m *eigenvalues are listed* m *times) are called the principal curvatures of* M *in* x*. One further defines the* ***mean curvature*** $\mathrm{H}(x)$ *(or* $\mathrm{H}_M(x)$*) of* M *at* x *as*

$$\mathrm{H}(x) := \frac{1}{n}\mathrm{trace}\, S(x) = \frac{1}{n}\sum_{i=1}^{n}\kappa_i(x) \qquad \text{for } x \in M \setminus \partial M\,.$$

Since ν*,* S*, and* H *are only unique up to change of sign, one preferably works with the fully determined* ***mean curvature vector*** $\vec{\mathrm{H}} := \mathrm{H}\nu\colon M \setminus \partial M \to \mathbb{R}^{n+1}$ *(also denoted by* $\vec{\mathrm{H}}_M$*). Another function* $\mathrm{K}_M\colon M \setminus \partial M \to \mathbb{R}$ *of the principal curvatures is given by*

$$\mathrm{K}_M(x) := \det S(x) = \prod_{i=1}^{n}\kappa_i(x) \qquad \text{for } x \in M \setminus \partial M\,.$$

and is known, in the case $n = 2$*, as the* ***Gauss curvature****. The 'Theorema Egregium' of Gauss asserts that the Gauss curvature in dimension* $n = 2$ *is an* intrinsic *geometric quantity of* M*, that is, it can determined solely by intrinsic measurements of lengths and angles in* M*.*

(5) *The* ***first, second, and third fundamental form*** *of* M *in a point* $x \in M \setminus \partial M$ *are the symmetric bilinear forms* I_x*,* II_x*,* III_x *on* $\mathrm{T}_x M$ *which are given on* $v, w \in \mathrm{T}_x M$ *by*

$$\mathrm{I}_x(v,w) := v\cdot w\,,\qquad \mathrm{II}_x(v,w) := v\cdot S(x)w = -v\cdot\mathrm{D}\nu(x)w\,,\qquad \mathrm{III}_x(v,w) := S(x)v\cdot S(x)w\,.$$

The first fundamental form I_x is nothing but the Euclidean inner product on T_xM and can be seen as a Riemannian metric. For our purposes, mainly the second fundamental form II_x is relevant, and we remark that its symmetry follows from the self-adjointness of the Weingarten map $S(x)$ and that II_x contains the same information as $S(x)$ on the curvature of M at x.

Examples. *Here are some very simple cases, where the previous quantities are quickly computed*:

(1) *For the n-dimensional **sphere** $M = \mathrm{S}_r \subset \mathbb{R}^{n+1}$ with radius r, starting from the unit normal field $\nu(x) = \pm x/r$ and its derivative $\mathrm{D}\nu(x)v = \pm v/r$ for $v \in \mathrm{T}_x\mathrm{S}_r = \{x\}^\perp$, one computes*

$$\kappa_1 = \kappa_2 = \ldots = \kappa_n \equiv \mp\frac{1}{r}\,, \qquad \mathrm{H} \equiv \mp\frac{1}{r}\,, \qquad \vec{\mathrm{H}}(x) = -\frac{x}{r^2}\,, \qquad K \equiv \frac{(\mp 1)^n}{r^n}\,.$$

(2) *Similarly, (the lateral surface of) the **cylinder** with radius r in $\mathbb{R}^3$ has the principal curvatures $\mp 1/r$ and 0, and consequently one has $H \equiv \mp 1/(2r)$ and $K \equiv 0$ in this case.*

(3) *Finally, for (a piece of) an **hyperplane**, all curvature quantities vanish.*

Definitions (curvature of higher-codimension surfaces). *Consider an n-dimensional C^2 submanifold M with boundary in $\mathbb{R}^\ell$ with arbitrary $\ell > n$.*

(1) *Locally near every $x \in M \setminus \partial M$, one can find $(\ell{-}n)$ mutually orthogonal C^1 unit normal vector fields $\nu_1, \nu_2, \ldots, \nu_{\ell-n}$. Motivated by the codimension-one considerations, one then defines the **second fundamental form** of M in a point $x \in M \setminus \partial M$ as the bilinear map $\mathrm{II}_x \colon \mathrm{T}_xM \times \mathrm{T}_xM \to (\mathrm{T}_xM)^\perp$ with values in the orthogonal complement $(\mathrm{T}_xM)^\perp$ of T_xM, given by*

$$\mathrm{II}_x(v,w) := -\sum_{k=1}^{\ell-n} \big(v \cdot \mathrm{D}\nu_k(x)w\big)\nu_k(x) \qquad \text{for } v,w \in \mathrm{T}_xM\,.$$

In the hypersurface case, this definition is actually not completely consistent with the previous one: it differs through an additional multiplication with the unit normal ν. In the sequel, we will mostly rely on the new definition, which has the advantage that it eliminates the uncertainty of the sign of II_x.

(2) *Also in higher codimension, **II_x is symmetric, and it does not depend on the choice of $\nu_1, \nu_2, \ldots, \nu_{\ell-n}$**. To verify this, as in the codimension-one case one takes a regular C^2 parametrization Y of M near x with $Y(0) = x$. Then from $\partial_iY \cdot \nu_k(Y) \equiv 0$ near 0 one deduces, for all $i,j \in \{1,2,\ldots,n\}$, first $\partial_j\partial_iY(0) \cdot \nu_k(x) + \partial_iY(0) \cdot \mathrm{D}\nu_k(x)\partial_jY(0) = 0$ and then*

$$\mathrm{II}_x(\partial_iY(0), \partial_jY(0)) = \sum_{k=1}^{\ell-n} \big(\partial_j\partial_iY(0) \cdot \nu_k(x)\big)\nu_k(x) = \partial_j\partial_iY(0)^\perp\,.$$

Here, $\partial_j\partial_iY(0)^\perp$ stands for the orthogonal projection of $\partial_j\partial_iY(0) \in \mathbb{R}^\ell$ on $(\mathrm{T}_xM)^\perp$.

(3) *Taking the trace of II_x, one obtains the concept of the **mean curvature vector** $\vec{\mathrm{H}} = \vec{\mathrm{H}}_M$ in the higher-codimension case: in fact, this vector is given by*

$$\vec{\mathrm{H}}(x) := \frac{1}{n}\sum_{i=1}^{n} \mathrm{II}_x(v_i, v_i) \in (\mathrm{T}_xM)^\perp \qquad \text{for } x \in M \setminus \partial M\,,$$

where $v_1, v_2, \ldots, v_n$ is an Euclidean orthonormal basis of T_xM. This definition of $\vec{\mathrm{H}}$ is consistent with the previously given one for the hypersurface case and does not depend on the choice of $v_1, v_2, \ldots, v_n$.

1.2 The first variation of area and minimal submanifolds

Relying on the concepts of Section 1.1, we can concretize the Plateau problem in the form

$$\inf\{\mathcal{H}^n(M) : M \text{ complete } n\text{-dimensional } \mathrm{C}^k \text{ submanifold in } \mathbb{R}^{n+N} \text{ with boundary } \partial M = R\}\,,$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^{n+N}$ and R is a given complete $(n-1)$-dimensional C^k submanifold without boundary in $\mathbb{R}^{n+N}$. Here, completeness of M is meant in the sense[2] that (the trace on M of) the metric of the ambient space $\mathbb{R}^{n+N}$ turns M into a complete metric space or, equivalently, but in simpler words, in the sense that M is a closed subset of $\mathbb{R}^{n+N}$. This requirement prevents one from 'cheating' with the geometric boundary ∂M by not including potential boundary points in M. The requirement that R is complete with $\partial R = \emptyset$ is obviously necessary for the existence of an admissible M.

Next we compute the Euler equation and thus a necessary condition for solutions of the concretized problem. This is in fact achieved in the next statements, which apply to solutions M_* of Plateau's problem in the interior (i.e. with $M_* \setminus \partial M_*$ in place of M):

Theorem 1.1 (first variation of area). *Consider an n-dimensional C^2 submanifold M without boundary in $\mathbb{R}^{n+N}$ and a vector field $\Phi \in \mathrm{C}^1_{\mathrm{cpt}}(M, \mathbb{R}^{n+N})$. Setting $\Phi_t(x) := x + t\Phi(x)$ for $x \in M$ and $t \in \mathbb{R}$, we then have*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{H}^n(\Phi_t(M)) = \int_M \mathrm{div}_M \Phi \,\mathrm{d}\mathcal{H}^n = -n \int_M \Phi \cdot \vec{\mathrm{H}} \,\mathrm{d}\mathcal{H}^n} \tag{1.2}$$

with the mean curvature vector $\vec{\mathrm{H}}$.

For a proof of Theorem 1.1 see the end of this section.

Remarks.

(1) *Here, $\Phi \in \mathrm{C}^1_{\mathrm{cpt}}(M, \mathbb{R}^{n+N})$ indicates that Φ has a C^1 extension on a neighborhood of M and that[3] $\{x \in M : \Phi(x) \neq 0\} \Subset M$. The latter condition is a (strong) way of expressing that Φ has zero boundary values.*

(2) *The **tangential divergence** $\mathrm{div}_M \Phi$ of Φ along M is explained by*

$$\mathrm{div}_M \Phi(x) := \sum_{i=1}^n v_i \cdot \mathrm{D}\Phi(x) v_i \qquad \text{for } x \in M\,,$$

where $v_1, v_2, \ldots, v_n$ is an orthonormal basis of T_xM. As a matter of fact, $\mathrm{div}_M \Phi$ does not depend on the choice of $v_1, v_2, \ldots, v_n$ (and the extension of Φ outside M).

[2] One may also understand completeness of M in the alternative sense that each connected component of M is a complete metric space with respect to the *inner* metric of M. This is, in fact, a slightly weaker notion of completeness, since it allows that M resembles a thin spiral which has infinite length (but possibly finite $\mathcal{H}^n$-measure) and whose limit point does not belong to M.

[3] The notation $A \Subset B$ for sets A and B (in some topological space) signifies that A is relatively compact in B, that means A is contained in a compact subset of B.

(3) *A version of the* ***divergence theorem*** *holds* ***for tangential vector fields****: namely, if $\Phi \in \mathrm{C}^1_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})$ satisfies $\Phi(x) \in \mathrm{T}_xM$ for all $x \in M$, then one has*

$$\int_M \mathrm{div}_M \Phi \,\mathrm{d}\mathcal{H}^n = 0\,. \tag{1.3}$$

We stress, however, that the vector fields Φ in Theorem 1.1 need not be tangential.

Proof of the divergence theorem in (1.3). Possibly decomposing Φ into a finite sum, we can reduce to the case that $M = Y(\Omega)$ is covered by a single regular C^2 parametrization Y on an open subset Ω of $\mathbb{R}^n$. Since Φ is tangential, we can write

$$\Phi(Y) = (\mathrm{D}Y)V \qquad \text{on } \Omega$$

for some vector field $V = (V_1, V_2, \ldots, V_n) \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega,\mathbb{R}^n)$. By the chain and product rules we then find $\mathrm{D}\Phi(Y)\mathrm{D}Y = (\mathrm{D}Y)\mathrm{D}V + \sum_{k=1}^n (\partial_k \mathrm{D}Y)V_k$ on Ω, and a computation[4] in a suitable basis yields

$$(\mathrm{div}_M \Phi)(Y) = \frac{\mathrm{div}(\sqrt{\gamma}\,V)}{\sqrt{\gamma}} \qquad \text{on } \Omega \tag{1.4}$$

with the abbreviation $\gamma := \det(\mathrm{D}Y^*\mathrm{D}Y) > 0$. Combining the area formula, the preceding transformation rule (1.4) for divergences, and the standard divergence theorem, we arrive at

$$\int_M \mathrm{div}_M \Phi \,\mathrm{d}\mathcal{H}^n = \int_\Omega (\mathrm{div}_M \Phi)(Y)\sqrt{\gamma}\,\mathrm{d}x = \int_\Omega \mathrm{div}(\sqrt{\gamma}\,V)\,\mathrm{d}x = 0\,.$$

(4) *For submanifolds M with boundary, there are variants of Theorem 1.1 and the divergence theorem in* (1.3) *which involve an additional boundary integral over ∂M. Anyway, the given statements suffice for our momentary purposes.*

[4] *Elementary proof of the transformation rule in* (1.4). We fix an arbitrary $x \in \Omega$ and an orthonormal basis $b_1, b_2, \ldots, b_n$ of $\mathbb{R}^n$ such that each b_i is an eigenvector of the symmetric matrix $\mathrm{D}Y(x)^*\mathrm{D}Y(x)$ with corresponding eigenvalue $\lambda_i^2 = |\mathrm{D}Y(x)b_i|^2 > 0$. Then we compute $(\mathrm{div}_M\Phi)(Y(x))$ in the orthonormal basis $\frac{1}{\lambda_1}\mathrm{D}Y(x)b_1$, $\frac{1}{\lambda_2}\mathrm{D}Y(x)b_2, \ldots, \frac{1}{\lambda_n}\mathrm{D}Y(x)b_n$ of $\mathrm{T}_{Y(x)}M$ and use the above connection between $\mathrm{D}\Phi$ and $\mathrm{D}V$ as follows:

$$\begin{aligned}
(\mathrm{div}_M\Phi)(Y(x)) &= \sum_{i=1}^n \frac{1}{\lambda_i^2}\mathrm{D}Y(x)b_i \cdot \mathrm{D}\Phi(Y(x))\mathrm{D}Y(x)b_i \\
&= \sum_{i=1}^n \frac{1}{\lambda_i^2}\mathrm{D}Y(x)^*\mathrm{D}Y(x)b_i \cdot \mathrm{D}V(x)b_i + \sum_{k=1}^n \left(\sum_{i=1}^n \frac{1}{\lambda_i^2}\mathrm{D}Y(x)b_i \cdot \partial_k \mathrm{D}Y(x)b_i\right)V_k(x) \\
&= \sum_{i=1}^n b_i \cdot \mathrm{D}V(x)b_i + \sum_{k=1}^n \frac{1}{2}\left(\sum_{i=1}^n \frac{\partial_k\big[|\mathrm{D}Yb_i|^2\big](x)}{|\mathrm{D}Y(x)b_i|^2}\right)V_k(x)\,.
\end{aligned}$$

To rewrite the last term on the right-hand side we exploit that, thanks to the above choice of the b_i, the matrix $(\mathrm{D}Yb_i \cdot \mathrm{D}Yb_j)_{i,j=1,2,\ldots n}$ is diagonal, *when evaluated at the point x*. Exploiting this twice, we get

$$\partial_k\gamma(x) = \partial_k\big[\det(\mathrm{D}Yb_i \cdot \mathrm{D}Yb_j)_{i,j=1,2,\ldots n}\big](x) = \sum_{i=1}^n \partial_k\big[|\mathrm{D}Yb_i|^2\big](x)\prod_{\substack{j=1\\ j\neq i}}^n |\mathrm{D}Y(x)b_j|^2 = \gamma(x)\sum_{i=1}^n \frac{\partial_k\big[|\mathrm{D}Yb_i|^2\big](x)}{|\mathrm{D}Y(x)b_i|^2}\,.$$

Combining the last two computations, we arrive at

$$(\mathrm{div}_M\Phi)(Y(x)) = (\mathrm{div}\,V)(x) + \sum_{k=1}^n \frac{\partial_k\gamma(x)}{2\gamma(x)}V_k(x) = \frac{\mathrm{div}(\sqrt{\gamma}\,V)}{\sqrt{\gamma}}(x)\,. \qquad \square$$

Corollary 1.2. *Suppose that M is an n-dimensional C^2 submanifold without boundary in $\mathbb{R}^{n+N}$. Then we have:*

$$\mathcal{H}^n(M) \le \mathcal{H}^n(S) \quad \begin{cases} \text{for every open set } U \Subset \mathbb{R}^{n+N} \text{ with } M\cap U \Subset M \text{ and} \\ \text{all } n\text{-dimensional } \mathrm{C}^2 \text{ submanifolds } S \text{ without boundary in } \mathbb{R}^{n+N} \\ \text{such that } S\cap U \Subset S \text{ and } S\setminus U = M\setminus U \end{cases}$$

$$\implies \int_M \mathrm{div}_M \Phi \,\mathrm{d}\mathcal{H}^n = 0 \quad \text{for all } \Phi \in \mathrm{C}^2_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})$$

$$\iff \vec{\mathrm{H}} \equiv 0 \text{ on } M\,.$$

Remark. *The assertions of the corollary are a manifestation of the following general scheme in the calculus of variations:*

local minimality property

$\implies$ *weak formulation of the Euler equation*

$\iff$ *strong formulation of the Euler equation.*

Motivated by the corollary we now define minimal surfaces as surfaces with vanishing mean curvature, that is, essentially, as the critical points of the Plateau problem.

Definition 1.3 (minimal submanifold). *A C^2 submanifold M in $\mathbb{R}^\ell$ with boundary is called a minimal surface or a minimal submanifold, if there holds $\vec{\mathrm{H}}_M(x) = 0$ for all $x \in M\setminus \partial M$.*

Remark. *We emphasize that — in spite of their name —* ***minimal surfaces need not be minimizers, but are merely critical points of the area functional.***

Proof of Corollary 1.2. We have the following chain of implications and equivalences, where the first implication is justified below, where the second implication is just the first-order calculus criterion for minima, and where the equivalences result from Theorem 1.1 and (a simple version of) the fundamental lemma of the calculus of variations:

$$\begin{aligned}
&\mathcal{H}^n(M) \le \mathcal{H}^n(S) \text{ for all } U,S \text{ as above}\\
&\quad\implies t \mapsto \mathcal{H}^n(\Phi_t(M)) \text{ has a local minimum at 0, for all } \Phi \in \mathrm{C}^2_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})\\
&\quad\implies \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{H}^n(\Phi_t(M)) = 0 \text{ for all } \Phi \in \mathrm{C}^2_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})\\
&\quad\iff \int_M (\mathrm{div}_M\Phi)\,\mathrm{d}\mathcal{H}^n = 0 \text{ for all } \Phi \in \mathrm{C}^2_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})\\
&\quad\iff \int_M \Phi\cdot\vec{\mathrm{H}}\,\mathrm{d}\mathcal{H}^n = 0 \text{ for all } \Phi \in \mathrm{C}^2_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})\\
&\quad\iff \vec{\mathrm{H}} \equiv 0 \text{ on } M\,.
\end{aligned}$$

These observations finish the reasoning once we fully justify the first implication. To this end, we extend a fixed $\Phi \in \mathrm{C}^2_{\mathrm{cpt}}(M,\mathbb{R}^{n+N})$ as a $\mathrm{C}^2_{\mathrm{cpt}}$ vector field to $\mathbb{R}^{n+N}$. Then, Φ_t is one-to-one and $\mathrm{D}\Phi_t = \mathrm{Id} + t\mathrm{D}\Phi(x)$ has non-zero determinant whenever $|t|\max_{\mathbb{R}^{n+N}}\|\mathrm{D}\Phi\| < 1$ holds, with the operator norm $\|\mathrm{D}\Phi\|(x)$ of $\mathrm{D}\Phi(x)$. Hence, Φ_t is a C^2 diffeomorphism of $\mathbb{R}^{n+N}$ and local regular C^2 parametrizations Y of M yield, by composition, local regular C^2 parametrizations $\Phi_t\circ Y$ of $\Phi_t(M)$, so that $\Phi_t(M)$ is a C^2 submanifold without boundary for $|t| \ll 1$. Moreover, choosing U as a small open neighborhood of $\operatorname{spt}\Phi$, we have $U \Subset \mathbb{R}^{n+N}$ and $M \cap U \Subset M$, and we get $\Phi_t(M) \cap U \Subset \Phi_t(M)$ and $\Phi_t(M) \setminus U = M \setminus U$ for $|t| \ll 1$. In view of $\Phi_0(M) = M$, the assumed minimality thus implies $\mathcal{H}^n(\Phi_0(M)) \leq \mathcal{H}^n(\Phi_t(M))$ for $|t| \ll 1$, and the proof is complete. □

In order to prove Theorem 1.1 we next record a lemma.

Lemma 1.4 (area formula for submanifolds). *Consider an n-dimensional C^1 submanifold without boundary in $\mathbb{R}^\ell$ and a one-to-one mapping $\Psi \in \mathrm{C}^1(M, \mathbb{R}^m)$. Then there holds*

$$\int_{\Psi(M)} f \,\mathrm{d}\mathcal{H}^n = \int_M f(\Psi(x)) \sqrt{\det(\mathrm{D}_M\Psi(x)^*\mathrm{D}_M\Psi(x))}\,\mathrm{d}\mathcal{H}^n(x)$$

for all Borel functions $f\colon \Psi(M) \to \mathbb{R}$ (with the understanding that either both integrals exist or both do not exist). Here, $\mathrm{D}_M\Psi(x)\colon \mathrm{T}_xM \to \mathbb{R}^m$ is the derivative tangentially to M and $\mathrm{D}_M\Psi(x)^*\colon \mathbb{R}^m \to \mathrm{T}_xM$ is the adjoint with respect to the Euclidean inner products, so that the Jacobian is given by the formula

$$\det(\mathrm{D}_M\Psi(x)^*\mathrm{D}_M\Psi(x)) = \det(\mathrm{D}\Psi(x)v_i \cdot \mathrm{D}\Psi(x)v_j)_{i,j=1,2,\ldots,n}$$

for every orthonormal basis v_1, v_2, $\ldots$, v_n of T_xM.

Proof. One can easily reduce to the case that $M = Y(\Omega)$ is given by a *single* regular C^1 parametrization Y on an open subset Ω of $\mathbb{R}^n$, and for notational simplicity we only treat this simplified case here. Since $\Psi \circ Y$ maps Ω one-to-one and C^1 onto $\Psi(M)$, the standard version of the area formula gives

$$\int_{\Psi(M)} f \,\mathrm{d}\mathcal{H}^n = \int_\Omega f(\Psi(Y)) \sqrt{\det(\mathrm{D}(\Psi \circ Y)^*\mathrm{D}(\Psi \circ Y))}\,\mathrm{d}x\,.$$

For the determinant, we have

$$\begin{aligned}\det(\mathrm{D}(\Psi \circ Y)^*\mathrm{D}(\Psi \circ Y)) &= \det(\mathrm{D}Y^*\mathrm{D}_M\Psi(Y)^*\mathrm{D}_M\Psi(Y)\mathrm{D}Y)\\ &= \det(\mathrm{D}_M\Psi(Y)^*\mathrm{D}_M\Psi(Y))\det(\mathrm{D}Y^*\mathrm{D}Y)\,,\end{aligned}$$

where the latter identity is readily checked by a computation in an orthonormal basis of $\mathbb{R}^n$ which consists of eigenvectors of $\mathrm{D}Y^*\mathrm{D}Y$. Finally, a second application of the area formula yields

$$\int_\Omega f(\Psi(Y))\sqrt{\det(\mathrm{D}_M\Psi(Y)^*\mathrm{D}_M\Psi(Y))}\sqrt{\det(\mathrm{D}Y^*\mathrm{D}Y)}\,\mathrm{d}x = \int_M f(\Psi)\sqrt{\det(\mathrm{D}_M\Psi^*\mathrm{D}_M\Psi)}\,\mathrm{d}\mathcal{H}^n\,,$$

and combining the last three displayed formulas, we arrive at the claim. □

Proof of Theorem 1.1. As noticed in the proof of Corollary 1.2, Φ_t is one-to-one for $|t| \ll 1$. Thus, exploiting Lemma 1.4 and interchanging integration and differentiation (here unproblematic, since $\mathrm{D}\Phi$ is bounded), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{H}^n(\Phi_t(M)) = \int_M \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\sqrt{\det(\mathrm{D}_M\Phi_t(x)^*\mathrm{D}_M\Phi_t(x))}\,\mathrm{d}\mathcal{H}^n(x)\,. \tag{1.5}$$

Computing the determinant in an orthonormal basis $v_1, v_2, \ldots, v_n$ of T_xM, we further get

$$\begin{aligned}
&\det(\mathrm{D}_M\Phi_t(x)^*\mathrm{D}_M\Phi_t(x)) \\
&\qquad = \det([v_i{+}t\mathrm{D}\Phi(x)v_i]\cdot[v_j{+}t\mathrm{D}\Phi(x)v_j])_{i,j=1,2,\ldots,n} \\
&\qquad = \det(\delta_{ij}+t[\mathrm{D}\Phi(x)v_i\cdot v_j+v_i\cdot\mathrm{D}\Phi(x)v_j]+t^2\mathrm{D}\Phi(x)v_i\cdot\mathrm{D}\Phi(x)v_j)_{i,j=1,2,\ldots,n} \\
&\qquad = 1+2\bigg(\sum_{i=1}^n v_i\cdot\mathrm{D}\Phi(x)v_i\bigg)t+a_2t^2+a_3t^3+\ldots+a_{2n}t^{2n}
\end{aligned}$$

with some t-independent coefficients $a_2, a_3, \ldots a_{2n} \in \mathbb{R}$. Therefore, differentiation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\sqrt{\det(\mathrm{D}_M\Phi_t(x)^*\mathrm{D}_M\Phi_t(x))}=\sum_{i=1}^n v_i\cdot\mathrm{D}\Phi(x)v_i=(\mathrm{div}_M\Phi)(x)\,,$$

and plugging this result into (1.5), we arrive at the first equality in (1.2).

Next, we decompose Φ into its normal part $\Phi^\perp$ and its tangential part $\Phi-\Phi^\perp$ such that we have $\Phi^\perp(x)\in(\mathrm{T}_xM)^\perp$ and $\Phi(x)-\Phi^\perp(x)\in\mathrm{T}_xM$ for all $x\in M$. Since $\Phi-\Phi^\perp$ is a tangential field and div_M is a linear operator, the divergence theorem in (1.3) above implies

$$\int_M \mathrm{div}_M\Phi\,\mathrm{d}\mathcal{H}^n=\int_M \mathrm{div}_M\Phi^\perp\,\mathrm{d}\mathcal{H}^n\,.$$

Finally, we compute $\big(\mathrm{div}_M\Phi^\perp\big)(x)$ by using an orthonormal basis $v_1, v_2, \ldots, v_n$ as above and mutually orthogonal unit normal vector fields $\nu_1, \nu_2, \ldots, \nu_N$ to M (locally near x defined). We find

$$\begin{aligned}
\big(\mathrm{div}_M\Phi^\perp\big)(x)&=\bigg[\mathrm{div}_M\sum_{k=1}^N(\Phi\cdot\nu_k)\nu_k\bigg](x)\\
&=\sum_{i=1}^n\sum_{k=1}^N v_i\cdot\mathrm{D}[(\Phi\cdot\nu_k)\nu_k](x)v_i\\
&=\sum_{i=1}^n\sum_{k=1}^N v_i\cdot\underbrace{(\Phi\cdot\nu_k)(x)}_{\text{scalar}}\mathrm{D}\nu_k(x)v_i+\sum_{i=1}^n\sum_{k=1}^N\underbrace{v_i\cdot\nu_k(x)}_{=0}\underbrace{\mathrm{D}(\Phi\cdot\nu_k)(x)v_i}_{\text{scalar}}\\
&=\sum_{i=1}^n\underbrace{\bigg[\sum_{k=1}^N(v_i\cdot\mathrm{D}\nu_k(x)v_i)\nu_k(x)\bigg]}_{=\,-\mathrm{II}_x(v_i,v_i)}\cdot\Phi(x)\\
&=-n\vec{\mathrm{H}}(x)\cdot\Phi(x)\,,
\end{aligned}\tag{1.6}$$

and combining the last two displayed formulas, we deduce the second equality in (1.2). □

1.3 Parametric minimal surfaces

Definition 1.5 (**parametric surfaces**). *Consider parameters $n,\ell\in\mathbb{N}$, $\ell\ge n$, $k\in\mathbb{N}\cup\{\infty,\omega\}$, and an open set Ω in $\mathbb{R}^n$. A mapping $Y\in\mathrm{C}^k(\Omega,\mathbb{R}^\ell)$ of maximal rank n is called a C^k* ***immersion*** *of Ω in $\mathbb{R}^\ell$ or — thinking of $Y(\Omega)$ — a* ***$\mathbf{C}^k$-immersed n-dimensional surface*** *in $\mathbb{R}^\ell$. If Y is additionally an homeomorphism, then it is called a C^k* ***embedding*** *of Ω in $\mathbb{R}^\ell$ or a* ***$\mathbf{C}^k$-embedded n-dimensional surface*** *in $\mathbb{R}^\ell$.*

Remarks.

(1) *In the embedded case, the image $Y(\Omega)$ is an n-dimensional C^k submanifold without boundary in $\mathbb{R}^\ell$.* ***In the immersed case,*** *$Y(\Omega)$* ***can exhibit self-intersections*** *and need not be a submanifold, but at least every $x \in \Omega$ has a open neighborhood U_x such that $Y|_{U_x}$ is an embedding and $Y(U_x)$ is a submanifold.*

(2) ***Curvatures notions and the notion of minimal surfaces can be extended to the parametric context.*** *To this end, we set $\vec{\mathrm{H}}(x) := \vec{\mathrm{H}}_{Y(U_x)}(Y(x))$ for $x \in \Omega$ and observe that the result of this specification does not depend on the choice of U_x (and that, in the embedded case, we can take $U_x = \Omega$, anyway). Furthermore, we call Y minimal or a parametric minimal surface if $\vec{H}(x) = 0$ holds for all $x \in \Omega$.*

(3) *Clearly, one can also define the notion of a parametric surface with boundary. To this end, one admits n-dimensional submanifolds with boundary in $\mathbb{R}^n$ as the domain of definition Ω.*

Next we aim at rewriting the equation $\vec{H} \equiv 0$ for minimal surfaces in the parametric context. To this end, we recall that a C^2 immersion Y as above induces a **Riemannian metric** on Ω, that is the family $(g_x)_{x\in\Omega}$ of x-dependent inner products on $\mathbb{R}^n$ with $g_{ij}(x) := g_x(\mathrm{e}_i, \mathrm{e}_j) := \partial_i Y(x) \cdot \partial_j Y(x)$ (where $\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n$ denotes the standard basis of $\mathbb{R}^n$). The metric $g = (g_x)_{x\in\Omega}$ on Ω simulates the intrinsic geometry of $Y(\Omega)$ in the sense that g-lengths and g-angles in Ω equal the corresponding Euclidean lengths and angles in $Y(\Omega)$. Moreover, g also comes with an associated natural Laplace operator, which is known as the Riemannian Laplace operator or the **Laplace-Beltrami operator** Δ_g. This operator is given by

$$\Delta_g u := \frac{1}{\sqrt{\gamma}} \sum_{i,j=1}^{n} \partial_i\big(\sqrt{\gamma}\, g^{ij} \partial_j u\big) \qquad \text{for } u \in \mathrm{C}^2(\Omega)\,,$$

where we abbreviated $\gamma := \det(g_{ij})_{i,j=1,2,\ldots,n}$ and where $(g^{ij})_{i,j=1,2,\ldots,n}$ stands for the inverse of the metric $(g_{ij})_{i,j=1,2,\ldots,n}$. In the embedded case, it is a consequence of (1.4) that $\boldsymbol{\Delta_g}$ **is directly related to the intrinsic Laplace-Operator** $\boldsymbol{\Delta_M := \mathrm{div}_M \nabla_M}$ of the submanifold $M := Y(\Omega)$ by

$$\Delta_g(w \circ Y) = (\Delta_M w) \circ Y \qquad \text{for } w \in \mathrm{C}^2(M)\,. \tag{1.7}$$

With these concepts at hand, we next state the main result of this section.

Theorem 1.6. *For every C^2-immersed surface $Y \colon \Omega \to \mathbb{R}^\ell$ (with Ω open in $\mathbb{R}^n$ and $\ell \geq n$), one has*

$$\Delta_g Y = n\vec{\mathrm{H}} \qquad \text{on } \Omega\,,$$

where Δ_g acts component-wise on Y.

Corollary 1.7. *For a C^2-immersed surface $Y \colon \Omega \to \mathbb{R}^\ell$ (with Ω open in $\mathbb{R}^n$ and $\ell \geq n$), there holds*

$$\boxed{Y \text{ parametric minimal surface} \iff \Delta_g Y \equiv 0 \text{ on } \Omega\,.} \qquad \square$$

Proof of Theorem 1.6. Since the statement is local, we can assume that Y is an embedding and thus $M := Y(\Omega)$ is a submanifold. For fixed $i \in \{1, 2, \ldots, \ell\}$, we consider the component Y_i of Y and the corresponding component $w_i \in \mathrm{C}^2(\mathbb{R}^\ell)$ of the position vector, that is $w_i(x) := x_i$.

Then ∇w_i is constant on $\mathbb{R}^\ell$ and equals the ith vector in the canonical basis of $\mathbb{R}^\ell$. Using this along with (1.7), the decomposition $\nabla w_i = \nabla_M w_i + [\nabla w_i]^\perp$, and (1.6), we arrive at

$$\begin{aligned}\Delta_g Y_i = (\Delta_M w_i) \circ Y &= (\mathrm{div}_M \nabla_M w_i) \circ Y \\ &= (\underbrace{\mathrm{div}_M \nabla w_i}_{=0} - \mathrm{div}_M [\nabla w_i]^\perp) \circ Y = n\vec{\mathrm{H}} \cdot \nabla w_i = n\vec{\mathrm{H}}_i \qquad \square\end{aligned}$$

Next we discuss that, in some cases, the usage of specific parametrizations Y leads to a simplification.

Definition 1.8 (conformal mapping). *Consider an open set Ω in $\mathbb{R}^n$ and $Y \in \mathrm{C}^1(\Omega, \mathbb{R}^\ell)$. Then Y is called conformal if we have*

$$|\partial_i Y| = |\partial_j Y| \qquad \text{and} \qquad \partial_i Y \cdot \partial_j Y \equiv 0 \qquad \text{on } \Omega\,, \qquad \text{for all } i,j \in \{1,2,\ldots,n\} \text{ with } i \neq j\,.$$

For later purposes we also extend the definition to merely weakly differentiable Y, simply by imposing the same conditions $\mathcal{L}^n$-a.e. on Ω on the weak derivatives.

Remarks.

(1) *It is equivalent to require that $\mathrm{D}Y(x)^*\mathrm{D}Y(x)$ is represented by a multiple of the $(n{\times}n)$ unit matrix and thus corresponds to a scaling of $\mathbb{R}^n$.*

(2) *A conformal mapping Y preserves angles (away from the zeros of $\mathrm{D}Y$), but in general changes lengths.*

In the **case $n = 2$**, we can now record:

Corollary 1.9. *Consider an open set Ω* ***in*** $\mathbb{R}^2$ *and a* ***conformally parametrized*** C^2*-immersed surface $Y \colon \Omega \to \mathbb{R}^\ell$ (with $\ell \geq 2$). Then one has*

$$\Delta Y = 2\sqrt{\gamma}\,\vec{\mathrm{H}} \qquad \text{on } \Omega\,,$$

where Δ stands for the ordinary Laplace operator. In particular, under the preceding assumptions, one has the equivalence

$$\boxed{Y \text{ parametric minimal surface} \iff \Delta Y \equiv 0 \text{ on } \Omega\,.}$$

Proof. Conformality of Y means $g_{ij} = \lambda\delta_{ij}$ on Ω for some function $\lambda \colon \Omega \to [0,\infty)$, and since Y is an immersion, we even have $\lambda > 0$ on Ω. Relying on the assumption $n = 2$, the determinant and the inverse of $(g_{ij})_{i,j=1,2,\ldots,n}$ compute as $\gamma = \lambda^2$ and $g^{ij} = \lambda^{-1}\delta_{ij}$, and the definition of Δ_g simplifies to

$$\Delta_g u = \frac{1}{\lambda}\sum_{i=1}^{2} \partial_i\partial_i u = \frac{1}{\sqrt{\gamma}}\Delta u \qquad \text{for every } u \in \mathrm{C}^2(\Omega)\,.$$

Plugging this description of Δ_g into Theorem 1.6 and Corollary 1.7, respectively, the claims are immediate. $\square$

Remarks.

(1) *The existence of a conformal parametrization is a less restrictive hypothesis than it may seem to be. Indeed, two-dimensional surfaces in $\mathbb{R}^3$ generally admit a conformal reparametrization (under mild regularity assumptions), but we do no discuss this aspect in detail, since our later existence theory bypasses this problem anyway.*

(2) *Conformality of Y and the Laplace equation $\Delta Y \equiv 0$ make sense even across possible zeros of $\mathrm{D}Y$. Thus, we can take Corollary 1.9 as a motivation to introduce the following generalized notion of two-dimensional minimal surfaces.*

Definition 1.10 (branched minimal surfaces). *Consider an open set Ω **in** $\mathbb{R}^2$. A non-constant mapping $Y \in \mathrm{C}^2(\Omega, \mathbb{R}^\ell)$ (with $\ell \geq 2$) is called a conformally parametrized (possibly) branched minimal surface, if Y is conformal with $\Delta Y \equiv 0$ on Ω. The branch points are the zeros of $\mathrm{D}Y$.*

Since harmonic functions occur as real and imaginary parts of holomorphic functions, Corollary 1.9 and Definition 1.10 are closely connected to the complex variables theory of minimal surfaces, which we discuss next.

1.4 Holomorphic representations and classical examples of two-dimensional minimal surfaces

Theorem 1.11. *For every open and simply connected set Ω in $\mathbb{R}^2 = \mathbb{C}$ and for every $Y \in \mathrm{C}^2(\Omega, \mathbb{R}^\ell)$ (with $\ell \geq 2$), the following equivalence holds true:*

$$\begin{aligned}&Y \text{ is a conformally parametrized (possibly) branched minimal surface}\\ &\iff Y = \operatorname{Re} H \text{ for a non-constant holomorphic } H\colon \Omega \to \mathbb{C}^\ell \text{ with } \sum_{j=1}^{\ell} \big(H_j'\big)^2 \equiv 0 \text{ on } \Omega\,.\end{aligned}$$

*In this case, also $Y^* := \operatorname{Im} H$ is a conformally parametrized (possibly) branched minimal surface, which is called the **conjugate minimal surface** of Y.*

Remarks.

(1) *Here, a $\mathbb{C}^\ell$-valued function H is called holomorphic if its component functions H_j, $j \in \{1, 2, \ldots, \ell\}$, are all $\mathbb{C}$-valued holomorphic functions in the usual sense with complex derivatives $H_j' \colon \Omega \to \mathbb{C}$. If such an H satisfies the condition of the theorem, it is called an **isotropic curve** in $\mathbb{C}^\ell$.*

(2) *In the situation of the theorem, Y determines the component functions of H and Y^*, respectively, up to addition of a purely imaginary and a real constant. Moreover, there holds $(Y^*)^* = -Y + \mathrm{const}$.*

(3) *As it is generally true for holomorphic (and harmonic) functions, **the functions H, Y, and Y^* in the theorem are automatically of class C^ω**, that is complex- and real-analytic, respectively, on Ω.*

(4) *Branch points of Y correspond to zeros of H' and thus to common zeros of H_1', H_2', …, H_ℓ'. Therefore, the identity theorem implies that the set of branch points of the minimal surface Y is at most countable and has no limit points in Ω. In particular, every single branch point is isolated in the sense that a sufficiently small punctured neighborhood in Ω contains no other branch point. The local behavior near branch points resembles the one of the holomorphic m-fold covering $\mathbb{C} \to \mathbb{C}, z \mapsto z^m$ with $m \in \{2, 3, 4, \ldots\}$. For a minimal surface Y, this essentially means that up to m different 'sheets' of the surface $Y(\Omega)$ can be 'glued together' in a single point of $Y(\Omega)$.*

Proof of Theorem 1.11. We first recall the basic relationship

$$\Delta Y \equiv 0 \text{ on } \Omega \iff Y = \operatorname{Re} H \text{ for a holomorphic } H\colon \Omega \to \mathbb{C}^\ell\,. \tag{1.8}$$

This equivalence follows from the characterization of a holomorphic function $H = Y + \mathbf{i}Y^*$ (in the variables $\mathbb{C} \ni z = x_1 + \mathbf{i}x_2 = (x_1, x_2) \in \mathbb{R}^2$) by the validity of the **Cauchy-Riemann equations**

$$\frac{\partial Y}{\partial x_1} = \frac{\partial Y^*}{\partial x_2}\,, \qquad \frac{\partial Y}{\partial x_2} = -\frac{\partial Y^*}{\partial x_1}\,.$$

Indeed, the implication '$\Longleftarrow$' in (1.8) is immediately verified by computing, with the help of these equations,

$$\Delta Y = \frac{\partial}{\partial x_1}\left(\frac{\partial Y^*}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial Y^*}{\partial x_1}\right) = 0\,.$$

Moreover, in order to prove '$\Longrightarrow$' in (1.8) one needs to find, for given harmonic Y, a solution Y^* of the Cauchy-Riemann equations or, in other words, an indefinite integral of the vector field

$$V := \left(-\frac{\partial Y}{\partial x_2}, \frac{\partial Y}{\partial x_1}\right).$$

However, it is well-known that a necessary and (on our simply connected Ω) also sufficient condition for the existence of the indefinite integral is the vanishing of $\partial_1 V_2 - \partial_2 V_1 = \Delta Y$. This concludes the proof of (1.8).

Next, for holomorphic $H = Y + \mathbf{i}Y^*$, we check

$$\begin{aligned}
\sum_{j=1}^{\ell} \left(H_j'\right)^2 \equiv 0 &\iff \sum_{j=1}^{\ell}\left(\frac{\partial Y_j}{\partial x_1} + \mathbf{i}\frac{\partial Y_j^*}{\partial x_1}\right)^2 \equiv 0\\
&\iff \left|\frac{\partial Y}{\partial x_1}\right|^2 - \left|\frac{\partial Y^*}{\partial x_1}\right|^2 + 2\mathbf{i}\frac{\partial Y}{\partial x_1}\cdot\frac{\partial Y^*}{\partial x_1} \equiv 0\\
&\iff \left|\frac{\partial Y}{\partial x_1}\right| = \left|\frac{\partial Y^*}{\partial x_1}\right| \text{ and } \frac{\partial Y}{\partial x_1}\cdot\frac{\partial Y^*}{\partial x_1} \equiv 0\\
&\iff \left|\frac{\partial Y}{\partial x_1}\right| = \left|\frac{\partial Y}{\partial x_2}\right| \text{ and } \frac{\partial Y}{\partial x_1}\cdot\frac{\partial Y}{\partial x_2} \equiv 0\\
&\iff Y \text{ is conformal,}
\end{aligned}$$

where we have used the second Cauchy-Riemann equation in the penultimate step. This, together with (1.8), proves the equivalence claimed in the theorem. The additional claim that also $Y^* = \operatorname{Im} H$ is a branched minimal surface follows from this equivalence, since we have $Y^* = \operatorname{Re}(-\mathbf{i}H)$ and also $-\mathbf{i}H$ is an isotropic curve in $\mathbb{C}^\ell$. □

Remarks (on holomorphic representation formulas). *In the **case $\ell = 3$**, the preceding theorem is the starting point for several different representations of isotropic curves $H\colon \Omega \to \mathbb{C}^3$ and conformally parametrized (possibly) branched minimal surfaces $Y\colon \Omega \to \mathbb{R}^3$, respectively, over simply connected open sets Ω in $\mathbb{C} = \mathbb{R}^2$.*

(1) ***Locally*** *near every point $z_0 \in \Omega$ such that $H_1'(z_0) \neq 0 \neq H_3'(z_0)$ (for non-branch points z_0, this can always be achieved by permuting the components of H), a **formula of Monge** asserts*

$$H \circ \tau(z) = \left(z,\, h(z),\, H_3(z_0) + \int_{H_1(z_0)}^{z} \mathbf{i}\sqrt{1+h'(\zeta)^2}\,\mathrm{d}\zeta\right) \qquad \textit{for } |z - H_1(z_0)| \ll 1\,,$$

*with the **biholomorphic change of coordinates** $\tau := (H_1)^{-1}$ and some **holomorphic function** h on a neighborhood of $H_1(z_0)$ such that $h'(H_1(z_0)) \neq \pm\mathbf{i}$. Here, the root stands for a suitably chosen complex square root, and the integral is the complex line integral (which is evaluated along some path from $H_1(z_0)$ to z and yields an indefinite integral of $\mathbf{i}\sqrt{1+h'(z)^2}$).*

Proof. Taking $\tau := H_1^{-1}$ (as announced above) and $h := H_2 \circ \tau$, one has

$$\begin{aligned} 1+\big(h'\big)^2+\big((H_3{\circ}\tau)'\big)^2 &= \big((H_1{\circ}\tau)'\big)^2+\big((H_2{\circ}\tau)'\big)^2+\big((H_3{\circ}\tau)'\big)^2 \\ &= \big[\big(\big(H_1'\big)^2+\big(H_2'\big)^2+\big(H_3'\big)^2\big) \circ \tau\big]\big(\tau'\big)^2 \equiv 0\,. \end{aligned}$$

Solving this equality for $(H_3{\circ}\tau)'$ and integrating the result, one finds the claimed representation of $H_3{\circ}\tau$. □

(2) *If Y is free of branch points, one has the **Weierstrass representation formula** (which works **globally** and **without reparametrization**)*

$$H(z) = H(z_0) + \left(\int_{z_0}^{z}(g(\zeta)^2-h(\zeta)^2)\,\mathrm{d}\zeta,\ \int_{z_0}^{z}\mathbf{i}(g(\zeta)^2+h(\zeta)^2)\,\mathrm{d}\zeta,\ \int_{z_0}^{z}2g(\zeta)h(\zeta)\,\mathrm{d}\zeta\right)$$

*for all $z, z_0 \in \Omega$, with some **pair of holomorphic functions** $g, h\colon \Omega \to \mathbb{C}$ without common zeros.*

Proof. Since there are no branch points, from the equality $(H_1'-\mathbf{i}H_2')(H_1'+\mathbf{i}H_2') = -\big(H_3'\big)^2$, one reads off that the functions $(H_1'-\mathbf{i}H_2')$ and $(H_1'+\mathbf{i}H_2')$ have no common zeros. Relying on the same equality once more, it then follows that all zeros of $(H_1'-\mathbf{i}H_2')$ and $(H_1'+\mathbf{i}H_2')$ are of even order. As a consequence, one can define $g := \sqrt{\frac{1}{2}(H_1'-\mathbf{i}H_2')}$ and $h := \sqrt{-\frac{1}{2}(H_1'+\mathbf{i}H_2')}$ as holomorphic functions, and then it is straightforward to check $g^2-h^2 = H_1'$, $\mathbf{i}(g^2+h^2) = H_2'$, and — choosing the right signs for the roots — also $2gh = H_3'$. □

(3) *If the (in branch points suitably extended[5]) Gauss map $\frac{\partial_1 Y}{|\partial_1 Y|} \times \frac{\partial_2 Y}{|\partial_2 Y|}$ is one-to-one and if its image misses the north pole $(0,0,1)$ of the unit sphere in $\mathbb{R}^3$, then a **global formula of (Enneper-)Weierstrass** asserts*

$$H\circ\tau(z) = H\circ\tau(z_0) + \left(\int_{z_0}^{z}(1-\zeta^2)f(\zeta)\,\mathrm{d}\zeta,\ \int_{z_0}^{z}\mathbf{i}(1+\zeta^2)f(\zeta)\,\mathrm{d}\zeta,\ \int_{z_0}^{z}2\zeta f(\zeta)\,\mathrm{d}\zeta\right)$$

*for all $z, z_0 \in \widetilde{\Omega}$, with some **biholomorphic change of coordinates** $\tau\colon \widetilde{\Omega} \to \Omega$ and some non-constant **holomorphic function** $f\colon \widetilde{\Omega} \to \mathbb{C}$ on a open set $\widetilde{\Omega}$ in $\mathbb{C}$.*

On the proof. From the assumptions on the Gauss map, one can deduce that $\frac{H_3'}{H_1'-\mathbf{i}H_2'}$ is biholomorphic from Ω onto its image $\widetilde{\Omega}$. Then one can verify the claim with $\tau := \left(\frac{H_3'}{H_1'-\mathbf{i}H_2'}\right)^{-1}$ and $f := \frac{1}{2}(H_1{\circ}\tau-\mathbf{i}H_2{\circ}\tau)'$. For further details, we refer to [18, Section 3.3]. □

[5] For the existence of such an extension, see [18, Section 3.2].

(4) *The preceding formulas and other related ones can also be used, vice versa, by inserting specific holomorphic functions in order to find minimal surfaces. We briefly remark that, for this type of conclusion, one can also insert g and h* with *common zeros in the formula of (2); then one obtains, however, merely a branched minimal surface $Y = \operatorname{Re} H$ with branch points in the common zeros.*

Examples (of two-dimensional parametric minimal surfaces in $\mathbb{R}^3$).

(0) *The simplest examples are evidently **planes** and parts of planes.*

(1) *For*

$$H(z) = \alpha \begin{pmatrix} \cosh z \\ \mathbf{i} \sinh z \\ z \end{pmatrix}$$

with a non-zero parameter $\alpha \in \mathbb{R}$ on $\Omega = \mathbb{C} = \mathbb{R}^2$, one readily checks

$$\left(H_1'\right)^2 + \left(H_2'\right)^2 + \left(H_3'\right)^2 = \alpha^2 \left[\sinh^2 + \mathbf{i}^2 \cosh^2 + 1\right] \equiv 0\,.$$

Hence, $Y := \operatorname{Re} H$ is a conformally parametrized minimal surface. Relying on the calculus rules

$$\begin{aligned}\cosh(x_1{+}\mathbf{i}x_2) &= \cosh x_1 \cos x_2 + \mathbf{i} \sinh x_1 \sin x_2\,,\\ \sinh(x_1{+}\mathbf{i}x_2) &= \sinh x_1 \cos x_2 + \mathbf{i} \cosh x_1 \sin x_2\,,\end{aligned}$$

we compute the real part of H and arrive at the parametrization

$$Y(x_1, x_2) = \alpha \begin{pmatrix} \cosh x_1 \cos x_2 \\ -\cosh x_1 \sin x_2 \\ x_1 \end{pmatrix}.$$

This shows that $Y(\mathbb{R}^2)$ is a surface of revolution in $\mathbb{R}^3$, which arises by rotation of a catenary (essentially the graph of cosh*) around the y_3-axis. This surface is known as a **catenoid**. Checking $H' \neq 0$, we see that Y has no branch points and is thus an immersion. Moreover, the catenoid $Y(\mathbb{R}^2)$ has no self-intersections and is a smooth submanifold, but nonetheless Y is not an embedding, since it is 2π-periodic in x_2 and thus overlaps itself.*

The case of the catenoid corresponds to the choices $g(z) = \sqrt{\frac{\alpha}{2}}\mathrm{e}^{z/2}$, $h(z) = \sqrt{\frac{\alpha}{2}}\mathrm{e}^{-z/2}$ in Remark 2 above, while Remark 3 applies, after restriction to an injectivity domain of Y, with $\tau = -\operatorname{Log}$ (for a suitable branch Log *of the complex logarithm) and $f(\zeta) = -\frac{\alpha}{2\zeta^2}$.*

(2) *The **conjugate minimal surface of the catenoid*** is *computed, by using the same formulas as above, and reads, still on $\Omega = \mathbb{C} = \mathbb{R}^2$ as*

$$Y^*(x_1, x_2) = \alpha \begin{pmatrix} \sinh x_1 \sin x_2 \\ \sinh x_1 \cos x_2 \\ x_2 \end{pmatrix}.$$

*The surface parametrized by Y^* is known as a **helicoid**. It is the trace of a straight line, which stays parallel to the y_1y_2-plane, moves with constant speed in the y_3-direction, and rotates, at the same time, with constant angular speed around the y_3-axis. The parametrization Y^* of the helicoid is an even embedding of $\mathbb{R}^2$ in $\mathbb{R}^3$.*

(3) *The* ***Enneper surface*** *results from the choice* $f \equiv 1$ *on* $\Omega = \mathbb{C} = \mathbb{R}^2$ *in Remark* 3 *above. One computes*

$$H(z) = \begin{pmatrix} z - \frac{1}{3}z^3 \\ \mathbf{i}\big(z + \frac{1}{3}z^3\big) \\ z^2 \end{pmatrix},$$

$$Y(x_1, x_2) = \begin{pmatrix} x_1 - \frac{1}{3}x_1^3 + x_1 x_2^2 \\ -x_2 - x_1^2 x_2 + \frac{1}{3}x_2^3 \\ x_1^1 - x_2^2 \end{pmatrix}.$$

In this case, Y *has no branch points, but* $Y(\mathbb{R}^2)$ *has* ***self-intersections****. Consequently,* Y *is an immersion, but not an embedding.*

(4) *The* ***Henneberg surface*** *is given on* $\Omega = \mathbb{C} = \mathbb{R}^2$ *by*

$$H(z) = \begin{pmatrix} -1 + \cosh(2z) \\ -\mathbf{i}\big(\cosh z + \frac{1}{3}\cosh(3z)\big) \\ -\sinh z + \frac{1}{3}\sinh(3z) \end{pmatrix},$$

$$Y(x_1, x_2) = \begin{pmatrix} -1 + \cosh(2x_1)\cos(2x_2) \\ \sinh x_1 \sin x_2 + \frac{1}{3}\sinh(3x_1)\sin(3x_2) \\ -\sinh x_1 \cos x_2 + \frac{1}{3}\sinh(3x_1)\cos(3x_2) \end{pmatrix}.$$

In this case, H *and* Y *are* 2π*-periodic in* x_2 *and exhibit countably many branch points at* $k\frac{\mathbf{i}\pi}{2} = (0, k\frac{\pi}{2})$ *with* $k \in \mathbb{Z}$*, which are, however, mapped onto just* ***two branch points*** *in the image* $Y(\mathbb{R}^2)$*. Moreover,* $Y(\mathbb{R}^2)$ *has* ***self-intersections*** *and is a* ***non-orientable*** *surface.*

(5) ***Scherk's first surface*** *is obtained by inserting* $f(\zeta) = \frac{2}{1-\zeta^4}$ *on* $\Omega = \mathbb{C} \setminus \{1, \mathbf{i}, -1, -\mathbf{i}\}$ *into the formula of Remark* 3*. One finds*

$$H(z) = \begin{pmatrix} \mathbf{i}\log\dfrac{z+\mathbf{i}}{z-\mathbf{i}} \\ \mathbf{i}\log\dfrac{z+1}{z-1} \\ \log\dfrac{z^2+1}{z^2-1} \end{pmatrix}.$$

Strictly speaking, $Y = \operatorname{Re} H$ *then parametrizes only* a part *of Scherk's first surface, which depends on the choice of suitable branches of the complex logarithm in the preceding formula. The whole surface is the union over all the branches and looks like a 'doubly-periodic family of infinitely high towers over an infinite chess board'.*

(6) ***For illustrative pictures of these surfaces and an extensive discussion of these and many other examples, we refer to*** **[18, Sections 3.5, 3.8]*****.***

Chapter 2

A rough overview of different approaches to Plateau's problem

We here understand that **Plateau's problem** concerns either true minima or, more generally, critical points for the area. Using the terminology of Definition 1.3, the problem then consists in the following (not yet fully formalized) task:

> Find and study n-dimensional **surfaces of minimal area** or n-dimensional **minimal surfaces** in $\mathbb{R}^{n+N}$ with a prescribed $(n-1)$-dimensional boundary.

As the traditional approach (see point (A1) below) works with parametric surfaces in the sense of Section 1.3, this version of Plateau's problem is generally known as the **parametric Plateau problem** — even when the approach in fact differs from the traditional one and the surfaces need no longer be parametric ones in the original meaning. The sense of this terminology is to distinguish the above-formulated problem from the **non-parametric Plateau problem**, that is the Plateau problem restricted to surfaces which are graphs. More precisely, in the non-parametric problem, the admissible surfaces are only the graphs $\{(x,u(x)) : x \in \Omega\} \subset \mathbb{R}^{n+N}$ of functions $u\colon \Omega \to \mathbb{R}^N$ on a fixed bounded open set $\Omega \subset \mathbb{R}^n$ (so, despite the naming, the non-parametric problem is certainly not one which involves no parametrization, but rather it fixes certain *non-selectable* parametrizations, namely the parametrizations of graphs by their graph mappings). Clearly, the non-parametric problem can be formulated only for boundaries, which on their part are graphs $\{(x,\varphi(x)) : x \in \partial\Omega\}$ of functions $\varphi\colon \partial\Omega \to \mathbb{R}^N$ on the boundary $\partial\Omega$ of Ω, and in this case the problem corresponds to the minimization problem for the **non-parametric area functional**[1]

$$u \mapsto \mathcal{H}^n(\operatorname{Graph} u) = \int_\Omega \sqrt{1+|\mathcal{M}(\nabla u)|^2}\,\mathrm{d}x \tag{2.1}$$

[1]Here we use the shortcut notation $\mathcal{M}(z) \in \mathbb{R}^{\tau(n,N)}$ for the vector of all minors of a matrix $z \in \mathbb{R}^{N\times n}$ of all orders from 1 up to $\min\{n,N\}$, and $\tau(n,N) = \sum_{k=1}^{\min\{n,N\}} \binom{N}{k}\binom{n}{k}$ is the total number of such minors. Then the equality in (2.1) results (for C^1 functions u) from the area formula, used with the graph mapping $x \mapsto (x,u(x))$ as a parametrization of $\operatorname{Graph} u$.

or the solvability problem for the **minimal surface system**[2] (N equations for the N components of u)

$$\sum_{i,j=1}^{n} \partial_i(\sqrt{\gamma}\, g^{ij}\partial_j u) \equiv 0 \quad \text{on } \Omega\,, \qquad \text{where } g_{ij} := \delta_{ij} + \partial_i u \cdot \partial_j u \text{ for } i,j \in \{1,2,\ldots,n\}\,, \tag{2.2}$$

in both cases coupled with the **Dirichlet boundary condition**

$$u = \varphi \qquad \text{on } \partial\Omega\,.$$

We remark that both (2.1) and (2.2) simplify in the codimension-one case $N = 1$ (and also in the trivial case $n = 1$). In fact, the codimension-one non-parametric area is just

$$u \mapsto \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$$

(since there are only minors of order 1), and the minimal surface system reduces[3] for $N = 1$ to the **minimal surface equation**

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \equiv 0 \qquad \text{on } \Omega\,. \tag{2.3}$$

The main advantage of the graph constraint is that these non-parametric problems take the form of standard variational or PDE problems with a standard boundary condition. However, one may say that they are geometrically less significant than the original parametric problems.

In the sequel we list and roughly describe the **most common approaches to** the different versions (minima or critical points, parametric or non-parametric) of **Plateau's problem**.

(A) In two dimensions ($n = 2$):

(A1) The **classical parametric approach** works, for a given closed boundary curve $\Gamma \subset \mathbb{R}^3$, with parametric surfaces $Y : \overline{\mathrm{B}_1} \to \mathbb{R}^3$ such that Y maps $\partial\mathrm{B}_1$ homeomorphicly onto Γ. Here, B_1 stands for the two-dimensional open unit disk in $\mathbb{R}^2$, and the choice of the fixed parameter-domain B_1 restricts the considerations to surfaces of the topological type of the disk. Anyhow, using this type of parametric surfaces, the first general existence theorems for the Plateau problem have been achieved, independently by R. Garnier [28], J. Douglas [21, 22], and T. Radó [51, 52] around 1930. Subsequently, the Douglas-Radó theory has been considerably simplified, again independently, by R. Courant [13, 14] and L. Tonelli [60, 61]. In all instances, one gets, in the terminology of Definition 1.10, the existence of

[2] As introduced in Section 1.3, γ and $(g^{ij})_{i,j=1,2,\ldots,n}$ denote the determinant and the inverse of the metric $(g_{ij})_{i,j=1,2,\ldots,n}$ induced by the graph mapping $x \mapsto (x, u(x))$, and (2.2) is a corresponding rewriting (for C^2 functions u) of the equations for parametric minimal surfaces found in Section 1.3. Indeed, applying Theorem 1.6 with the choice $Y(x) := (x, u(x))$, we see that (2.2) expresses the vanishing of the 'vertical' components $\vec{\mathrm{H}}_{n+1}$, $\vec{\mathrm{H}}_{n+2}, \ldots \vec{\mathrm{H}}_{n+N}$ of the mean curvature vector. However, since non-zero normals to $\operatorname{Graph} u$ can never point into a purely 'horizontal' direction, this is equivalent with the vanishing of the whole vector $\vec{\mathrm{H}}$.

[3] In fact, (2.3) is obtained from (2.2) as follows. For fixed $x \in \Omega$ we write $\mathrm{id}, \mathrm{p}_x, \mathrm{p}_x^\perp : \mathbb{R}^n \to \mathbb{R}^n$ for the identity, the orthogonal projection on the 1-dimensional subspace spanned by $\nabla u(x)$, and the projection on its orthogonal complement, respectively. Then $(g_{ij}(x))_{i,j=1,2,\ldots,n}$ is the matrix of $\mathrm{id}+|\nabla u(x)|^2\mathrm{p}_x$. Thus, we have $\gamma(x) = 1+|\nabla u(x)|^2$, and one checks that $(g^{ij}(x))_{i,j=1,2,\ldots,n}$ is the matrix of $\frac{\mathrm{id}+|\nabla u(x)|^2\mathrm{p}_x^\perp}{1+|\nabla u(x)|^2}$. Consequently, we have $\sum_{j=1}^n \sqrt{\gamma(x)}\, g^{ij}(x)\partial_j u(x) = \frac{\partial_i u(x)}{1+|\nabla u(x)|^2}$, and (2.3) is identified as a codimension-one reformulation of (2.2).

a conformally parametrized *possibly branched* minimal surface bounded by Γ, and we will provide a detailed account of this existence proof in Chapter 3. A quite difficult argument for the de-facto **absence of branch points in the area-minimizing case** has only been found much later and is contained in works by R. Osserman [50], R.D. Gulliver [32], H.W. Alt [6, 7], and Gulliver–Osserman–Royden [33]. Thus, area-minimizing solutions are smooth immersions of the open disk B_1 in $\mathbb{R}^3$. However, simple examples[4] show that parametric solutions of Plateau's problem (even area-minimizing ones) may exhibit self-intersections and need not be embedded.

While the classical works consider only the case $N=1$ of surfaces in $\mathbb{R}^3$, many results (not the absence of branch-points, however) extend without difficulty to surfaces of arbitrary codimension $N \in \mathbb{N}$ in $\mathbb{R}^{2+N}$. Moreover, the parametric approach can be extended to the **Douglas problem**, that is a Plateau type problem for surfaces of higher (but still a-priori prescribed) topological type with two or more curves as the given contour.

For more information, we refer to [18, 19, 20, 49] and [29, Chapter 6.3.1].

(B) In codimension one ($N = 1$), hypersurface case:

(B1) The **(semi)classical theory of the minimal surface equation** comprises different methods for proving the existence of solutions. In the following we roughly describe one such approach, which is also known as **Hilbert-Haar existence theory**. Indeed, for fixed $M > 0$, an application of the Arzelà-Ascoli theorem shows that the area functional

$$u \mapsto \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$$

attains its minimum on the class

$$\left\{u \in \mathrm{C}^{0,1}\big(\overline{\Omega}\big) \,:\, u|_{\partial\Omega} = \varphi,\ \mathrm{Lip}(u) \le M\right\}$$

whenever this class is non-empty. The decisive step in the existence proof is now to verify that a minimizer u in this restricted class satisfies not only $\mathrm{Lip}(u) \le M$ but even the strict inequality

$$\mathrm{Lip}(u) < M\,. \tag{2.4}$$

Once this is at hand, u is, in some sense, an interior minimum point. By convexity, u thus minimizes the area functional even in the unrestricted class

$$\left\{u \in \mathrm{C}^{0,1}\big(\overline{\Omega}\big) \,:\, u|_{\partial\Omega} = \varphi\right\},$$

and then, modulo regularity theory, $u \in \mathrm{C}^\omega(\Omega) \cap \mathrm{C}^{0,1}\big(\overline{\Omega}\big)$ is a classical solution of the Dirichlet problem for the minimal surface equation. The decisive condition (2.4) can be verified ...

- ... either if **(Ω, φ) satisfies the bounded slope condition**, that is, there exists some $L < \infty$ and, for every $x_0 \in \partial\Omega$, there are affine functions $a^\pm_{x_0} \colon \mathbb{R}^n \to \mathbb{R}$ with $a^-_{x_0} \le \varphi \le a^+_{x_0}$ on $\partial\Omega$, with $a^-_{x_0}(x_0) = \varphi(x_0) = a^+_{x_0}(x_0)$, and with $\mathrm{Lip}(a^\pm_{x_0}) \le L$.

  Additional remarks on this condition:

  – With the bounded slope condition at hand, a kind of maximum principle for $|\nabla u|$ gives (2.4) for $M > L$.

[4] One such example is the 'disk with a tongue' discussed in [46, Chapter 8] and [15, Section 3.a].

  – Convexity of Ω is necessary for (Ω,φ) satisfying the bounded slope condition.
  – C^2 regularity of $\partial\Omega$ with strictly positive principal curvatures on all of $\partial\Omega$ is sufficient for (Ω,φ) satisfying the bounded slope condition for all $\varphi\in\mathrm{C}^2(\partial\Omega)$.

- ... or if **$\partial\Omega$ has non-negative mean curvature $\mathrm{H}_{\partial\Omega}\ge 0$ on $\partial\Omega$ and one has a regular boundary datum $\varphi\in\mathrm{C}^2(\partial\Omega)$.**

However, besides the above-described approach also other methods for solving the Dirichlet problem for the minimal surface equation have been developed. In particular, for $n=2$ and convex domains Ω, existence results have been obtained by S. Bernstein [9], A. Haar [34], and T. Radó [51] (where the last-mentioned work proceeds by showing that the parametric approach described under point (A1) yields also a non-parametric solution; compare with Theorem 3.22). Later on, existence results in higher dimensions $n>2$, but still for convex Ω, have been obtained by D. Gilbarg [30], and G. Stampacchia [59], while the relevance of the mean curvature of the boundary has been recognized only eventually in connection with the following **optimal existence result of Jenkins–Serrin** [39]: The Dirichlet problem for the minimal surface equation on a bounded C^2 domain $\Omega\subset\mathbb{R}^n$ has a classical solution $u\in\mathrm{C}^2(\Omega)\cap\mathrm{C}^0\big(\overline{\Omega}\big)$ for every given *continuous* boundary datum $\varphi\in\mathrm{C}^0(\partial\Omega)$ if and only if $\mathrm{H}_{\partial\Omega}\ge 0$ holds on $\partial\Omega$.

For more information, we refer to [31, Chapter 12] and [29, Chapter 6.1.2].

(B2) The **non-parametric BV theory** starts from the observation that area of the graph of $u\in\mathrm{W}^{1,1}(\Omega)$ can be expressed as the total variation $|(\mathcal{L}^n,(\nabla u)\mathcal{L}^n)|(\Omega)$ of the $\mathbb{R}^{1+n}$-valued measure $(\mathcal{L}^n,(\nabla u)\mathcal{L}^n)$. Motivated by this identity, one then **extends the non-parametric area functional to BV(Ω)** by setting[5]

$$\int_\Omega\sqrt{1+|\nabla u|^2}:=|(\mathcal{L}^n,\nabla u)|(\Omega)\qquad\text{for all }u\in\mathrm{BV}(\Omega)\,,$$

where the weak gradient ∇u is now regarded as an $\mathbb{R}^n$-valued measure. Relying on **weak-∗ compactness** in $\mathrm{BV}(\Omega)$ one can then obtain existence results for minimizers of the extended functional. We remark, in this connection, that weak-∗ convergence in BV need not preserve boundary values, so that the proper handling of the Dirichlet boundary condition is not completely straightforward. But still, existence, uniqueness, and regularity results for solutions of the Dirichlet problem can be obtained and are part of well-developed theory, which includes celebrated ideas of E. Bombieri, E. De Giorgi, M. Miranda [11, 43, 44, 45]. This theory allows to recover most results mentioned under point (B1), but also bypasses some limits of the classical theory and allows to obtain and investigate generalized solutions in case of (somewhere) negative mean curvature $\mathrm{H}_{\partial\Omega}$ and for discontinuous boundary values φ.

For more information, we refer to [31, Chapters 14, 15] and [29, Chapter 6.1.2].

[5] Writing out the definitions of the weak gradient and the total variation, one has

$$\int_\Omega\sqrt{1+|\nabla u|^2}=\sup\left\{\int_\Omega\big(V_0-u\,\mathrm{div}\,V\big)\,\mathrm{d}x\,:\,\begin{matrix}(V_0,V)\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}\times\mathbb{R}^n)\\ \sup_\Omega|(V_0,V)|\le 1\end{matrix}\right\}\qquad\text{for all }u\in\mathrm{BV}(\Omega)\,.$$

In principle, this formula may be used to define the area functional even for all $u\in\mathrm{L}^1(\Omega)$ (and without any knowledge about BV functions), but in the end this does not bring a true amplification, since the resulting functional is infinite on $\mathrm{L}^1(\Omega)\setminus\mathrm{BV}(\Omega)$.

(B3) The **parametric BV theory** works with **sets of finite perimeter** in $\mathbb{R}^{n+1}$ in the sense of R. Caccioppoli [12] and E. De Giorgi [16]. A set A of finite perimeter in $\mathbb{R}^{n+1}$ is, in fact, a set with mildly regular boundary, characterized by the conditions $\mathbb{1}_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $\mathrm{P}(A) := |\nabla \mathbb{1}_A|(\mathbb{R}^{n+1}) < \infty$, and in particular every bounded C^1 domain $A \subset \mathbb{R}^{n+1}$ has finite perimeter $\mathrm{P}(A) = \mathcal{H}^{n-1}(\partial A) < \infty$. Given an open set $\Omega \Subset \mathbb{R}^{n+1}$ and a set $A_0 \subset \mathbb{R}^{n+1}$ of finite perimeter, a version of the Plateau problem is then encoded in the minimization problem

$$\inf\{\mathrm{P}(E) \,:\, E \text{ set of finite perimeter in } \mathbb{R}^{n+1} \text{ with } E \setminus \Omega = A_0 \setminus \Omega\}\,,$$

in which the *boundary* of a minimizer A in Ω yields a generalized minimal surface. This geometric version of Plateau's problem has been extensively studied in the literature, for instance by E. Bombieri, E. De Giorgi, H. Federer, E. Giusti, J. Simons [16, 58, 10, 25], but also by many others. In particular, general existence results and a highly developed regularity theory for optimal sets are available, and we will detail some of these achievements in Chapter 4.

For more information, we refer to [31, Part I], [29, Chapter 6.1.1], and [42, Parts II, III].

(C) In arbitrary dimension and codimension ($n, N \in \mathbb{N}$):

The study of Plateau type problems and minimal surfaces in arbitrary dimension and codimension falls into the field of **geometric measure theory**. In the sequel, we briefly touch upon some of the relevant tools and approaches, which have been widely developed in the fundamental works of H. Federer, W.H. Fleming, E.R. Reifenberg, and F.J. Almgren [26, 53, 54, 27, 2, 3, 24, 4, 5] and have eventually been extended by W.K. Allard, E. Bombieri, R. Hardt, F. Morgan, R. Schoen, L. Simon, J.E. Taylor, B. White, and many others. For a more detailed (but still introductory) exposition, we refer to the monograph [46] and the survey [57].

(C1) **Currents.** An n-dimensional current T in $\mathbb{R}^{n+N}$ is a continuous linear functional

$$T\colon \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^{n+N}, \wedge^n \mathbb{R}^{n+N}) \to \mathbb{R}\,,$$

where $\mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^{n+N}, \wedge^n \mathbb{R}^{n+N})$ stands for the space of smooth and compactly supported differential forms of degree n on $\mathbb{R}^{n+N}$. Since the integration of differential forms over every[6] oriented n-dimensional submanifold in $\mathbb{R}^{n+N}$ gives such a functional, one considers **currents as generalized** (possibly very irregular) **oriented submanifolds**. However, the abstract definition of a current T as a linear functional also allows to explain the surface area, known as the **mass** $\mathbf{M}(T)$, and the **boundary** ∂T of a current T in a very natural and simple fashion. In the latter case, for instance, the definition is motivated by Stokes' theorem and the $(n{-}1)$-dimensional current ∂T is, in fact, given by $\partial T(\omega) := T(\mathrm{d}\omega)$, where $\mathrm{d}\omega$ denotes the exterior derivative of a differential form ω of degree n.

An important class of currents with surface-like behavior is the class of n-dimensional **integral currents**, these are currents with finite mass and boundary mass which arise by weighted integration (with measurable, integer-valued weight or multiplicity function) over

[6] To be technically precise, one should consider only submanifolds M which are locally $\mathcal{H}^n$-finite, thus avoiding that M forms a 'spiral of infinite length' around a limit point which is not contained in M.

an oriented countably $\mathcal{H}^n$-rectifiable[7] set in $\mathbb{R}^{n+N}$. A **generalized version of Plateau's problem is now given by the minimization problem for the mass** of integral currents with prescribed boundary, and there are very general existence and regularity results for this reformulation of the problem. We stress that — in contrast to the solutions of the parametric mapping problem described under point (A1) — the current solutions are not subject to any topological restrictions (apart from orientability) and in the case $n \le 6$, $N = 1$ they yield smooth embedded solutions (in particular, free of self-intersections).

Finally, we briefly mention that also the orientability constraint can be removed by working with congruence classes of currents modulo 2 (or modulo some larger integer) and that currents are further generalized by the concept of flat chains, which also goes well together with the introduction of general Abelian coefficient groups.

(C2) Varifolds. An n-dimensional varifold in $\mathbb{R}^{n+N}$ is a Radon measure on the Cartesian product $\mathbb{R}^{n+N} \times \mathrm{Gr}(n, \mathbb{R}^{n+N})$, where the Grassmannian $\mathrm{Gr}(n, \mathbb{R}^{n+N})$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+N}$ (endowed with a suitable metric). An n-dimensional **integral varifold** is an n-dimensional varifold which can be written as the image of a finite measure $\theta\mathcal{H}^n \llcorner S$ on $\mathbb{R}^{n+N}$ under the ($\mathcal{H}^n \llcorner S$-a.e. defined) mapping $x \mapsto (x, \mathrm{T}_x S)$ where S is a countably $\mathcal{H}^n$-rectifiable set in $\mathbb{R}^{n+N}$ and $\theta\colon S \to \mathbb{N}$ is $\mathcal{H}^n$-measurable. Minimal surfaces (and also area-minimizing integral currents) in n dimensions are now generalized by **stationary integral varifolds** $V = (S, \theta)$, the latter are critical points of the area functional $(S, \theta) \mapsto \int_S \theta \,\mathrm{d}\mathcal{H}^n$, which are also characterized by the vanishing of the first variation $\int_S (\mathrm{div}_S \Phi)\theta \,\mathrm{d}\mathcal{H}^n = 0$ for all $\Phi \in \mathrm{C}^1_{\mathrm{cpt}}(S, \mathbb{R}^{n+N})$. We remark that the theory of varifolds (unlike the one of currents) does not provide a natural boundary operator, and thus it becomes difficult to formulate and solve a version of Plateau's problem in this setting. Nevertheless, varifolds are suitable for modeling minimal surfaces, also non-orientable ones, in a large generality, and very general regularity results have been obtained in this framework (while others have remained conjectural).

(C3) $(\mathbf{M}, 0, \delta)$-minimal sets. A bounded set $S \subset \mathbb{R}^{n+N} \setminus R$ is called an $(\mathbf{M}, 0, \delta)$-minimal set (in the sense of Almgren) with respect to a given closed boundary $R \subset \mathbb{R}^{n+N}$ if there hold $\mathcal{H}^n(S) < \infty$ and $\emptyset \ne S = \mathrm{spt}(\mathcal{H}^n \llcorner S) \setminus R$ and if S has the minimality property

$$\mathcal{H}^n(S) \le \mathcal{H}^n(\varphi(S)) \qquad \begin{array}{c}\text{for every Lipschitz mapping } \varphi\colon \mathbb{R}^{n+N} \to \mathbb{R}^{n+N} \text{ such that} \\ \{x \in \mathbb{R}^{n+N} \,:\, \varphi(x) \ne x\} \subset \mathrm{B}_\delta(x_0) \subset \mathbb{R}^{n+N} \setminus R \text{ for some } x_0\,.\end{array}$$

Here, the technical condition $S = \mathrm{spt}(\mathcal{H}^n \llcorner S) \setminus R$ implies that S is closed in $\mathbb{R}^{n+N} \setminus R$, and additionally it roughly expresses the requirement that S does not contain lower-dimensional parts. Moreover, the first parameter $\mathbf{M}$ in the triple $(\mathbf{M}, 0, \delta)$ indicates that we presently consider the mass functional only (while in principle also other functionals could be studied), and the second parameter is meant to allow for a certain type of almost-minimality (where the present choice 0 indicates that we deal with true minimality only). Finally, we emphasize that the mapping φ is not required to be one-to-one and can thus 'pinch together' different parts of the surface S.

[7] Here, $S \subset \mathbb{R}^{n+N}$ is called countably $\mathcal{H}^n$-rectifiable if can be covered up to an $\mathcal{H}^n$-negligible set by countably many biLipschitz images of $\mathbb{R}^n$. The most important characteristic property of such sets S is that the tangent space $\mathrm{T}_x S$ can be explained for $\mathcal{H}^n$-a.e. $x \in S$.

Among all the different approaches to Plateau's problem, $(\mathbf{M}, 0, \delta)$-minimal sets and some variants[8] of this notion yield the best models for physical soap films, and particularly in the classical case $n = 2$, $N = 1$ there are very good regularity results which rule out all but the classical soap film singularities.

(C4) A **further notion of area-minimizing sets** results directly from the minimization of the Hausdorff measure among either closed sets (Reifenberg) or compact n-rectifiable sets (Almgren) in $\mathbb{R}^{n+N}$. Indeed, these area-minimizing sets with given boundary $R \subset \mathbb{R}^{n+N}$ are the solutions of the Plateau problem

$$\inf\{\mathcal{H}^n(S) \,:\, S \text{ is as before and 'spans' } R\}\,,$$

where the 'spanning' of R is defined as an advanced **topological condition in terms of** (relative Čech or Vietoris) **homology**. In this framework, one may also introduce general group coefficients, and very general existence and regularity results have been obtained.

(C5) Finally, a **non-parametric theory in arbitrary dimension and codimension** or, in other words, a successful general approach to the minimal surface system is **currently not available**. While a result of C.B. Morrey guarantees that C^1 solutions are automatically analytic, there is few hope to obtain general existence results for such solutions; compare with the counterexamples in [41, 1], for instance.

[8] For instance, there is the notion of sliding minimizers, for which it is additionally required that the φ in the minimality property is connected to the identity by a suitable continuous 1-parameter family of Lipschitz mappings.

Chapter 3

The classical parametric approach to Plateau's problem

Addendum on traces of $\mathrm{W}^{1,2}$ functions

Generalized boundary values of Sobolev or BV functions are know as **traces** (on the boundary). Such a concept can be defined for $\mathrm{W}^{m,p}$ or BV^m functions on domains with bounded Lipschitz boundary, but here we confine ourselves to the following very simple case.

Proposition. *Consider an open ball $\mathrm{B}_r(x_0) \Subset \mathbb{R}^n$. Then there exists a uniquely determined continuous linear operator, known as the* $(\mathrm{W}^{1,2})$ ***trace operator****,*

$$T\colon\ \mathrm{W}^{1,2}(\mathrm{B}_r(x_0),\mathbb{R}^N) \to \mathrm{L}^2(\partial\mathrm{B}_r(x_0),\mathbb{R}^N;\mathcal{H}^{n-1})\,,$$

such that, for every $u \in \mathrm{W}^{1,2}(\mathrm{B}_r(x_0),\mathbb{R}^N) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_r(x_0)},\mathbb{R}^N\big)$, there holds

$$Tu = u|_{\partial\mathrm{B}_r(x_0)} \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial\mathrm{B}_r(x_0)\,. \tag{3.1}$$

Remark. *Here, $u \in \mathrm{W}^{1,2}(\mathrm{B}_r(x_0),\mathbb{R}^N) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_r(x_0)},\mathbb{R}^N\big)$ means that the $\mathrm{W}^{1,2}$ function u on $\mathrm{B}_r(x_0)$ has a representative which extends continuously to $\overline{\mathrm{B}_r(x_0)}$. The restriction on the right-hand side of* (3.1) *is understood as the restriction of this representative.*

Proof of the proposition. Passing to the components, we assume $N=1$. Then we initially define

$$Tu := u|_{\partial\mathrm{B}_r(x_0)} \qquad \text{for } u \in \mathrm{C}^1\big(\overline{\mathrm{B}_r(x_0)}\big)\,.$$

Via the divergence theorem and Young's inequality we find

$$\begin{aligned}
\int_{\partial\mathrm{B}_r(x_0)} (Tu)^2\,\mathrm{d}\mathcal{H}^{n-1} &= \int_{\partial\mathrm{B}_r(x_0)} u^2(x)\frac{x-x_0}{r}\cdot\frac{x-x_0}{r}\,\mathrm{d}\mathcal{H}^{n-1}(x)\\
&= \int_{\mathrm{B}_r(x_0)} \mathrm{div}_x\Big(u^2(x)\frac{x-x_0}{r}\Big)\,\mathrm{d}x\\
&= \frac{n}{r}\int_{\mathrm{B}_r(x_0)} u^2\,\mathrm{d}x + 2\int_{\mathrm{B}_r(x_0)} u(x)\nabla u(x)\cdot\frac{x-x_0}{r}\,\mathrm{d}x\\
&\le \Big(1+\frac{n}{r}\Big)\int_{\mathrm{B}_r(x_0)} u^2\,\mathrm{d}x + \int_{\mathrm{B}_r(x_0)} |\nabla u|^2\,\mathrm{d}x\,.
\end{aligned}$$

This proves

$$\|Tu\|_{\mathrm{L}^2(\partial\mathrm{B}_r(x_0);\mathcal{H}^{n-1})} \le \sqrt{1+\frac{n}{r}}\,\|u\|_{\mathrm{W}^{1,2}(\mathrm{B}_r(x_0))}$$

for $u \in \mathrm{C}^1\big(\overline{\mathrm{B}_r(x_0)}\big)$, and consequently T extends in a unique way as a continuous linear operator from the dense subset $\mathrm{C}^1\big(\overline{\mathrm{B}_r(x_0)}\big)$ to the whole space $\mathrm{W}^{1,2}(\mathrm{B}_r(x_0))$. Finally, to verify (3.1) for all $u \in \mathrm{W}^{1,2}(\mathrm{B}_r(x_0)) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_r(x_0)}\big)$, we approximate such a u, uniformly on $\overline{\mathrm{B}_r(x_0)}$ and in the norm of $\mathrm{W}^{1,2}(\mathrm{B}_r(x_0))$, from $\mathrm{C}^1\big(\overline{\mathrm{B}_r(x_0)}\big)$; for instance, we can use the approximations $(u_k)_{\varepsilon_k}$, where the functions u_k are defined by $u_k(x) := u(x_0+(x-x_0)/(1+1/k))$ and are continuous and of class $\mathrm{W}^{1,2}$ on enlarged balls, and where the subscript $_{\varepsilon_k}$ stands for mollification with suitably small positive radii ε_k. □

A basic properties of the trace operator is recorded in the following lemma.

Lemma. *Consider a ball* $\mathrm{B}_r(x_0) \Subset \mathbb{R}^n$ *and* $\varrho \in [0,r[$. *Then the* ***Poincaré inequality***

$$\int_{\mathrm{B}_r(x_0)\setminus\mathrm{B}_\varrho(x_0)} |u|^2\,\mathrm{d}x \le 4(r-\varrho)^2 \int_{\mathrm{B}_r(x_0)\setminus\mathrm{B}_\varrho(x_0)} |\mathrm{D}u|^2\,\mathrm{d}x + 2(r-\varrho)\int_{\partial\mathrm{B}_r(x_0)} |Tu|^2\,\mathrm{d}\mathcal{H}^{n-1}$$

holds for every $u \in \mathrm{W}^{1,2}(\mathrm{B}_r(x_0),\mathbb{R}^N)$.

Proof. For simplicity of notation, we assume $N=1$ and $x_0=0$. Moreover, by density, we can restrict our considerations to functions $u \in \mathrm{C}^1\big(\overline{\mathrm{B}_r(x_0)}\big)$. Similar to the previous proof, the divergence theorem then yields

$$\begin{aligned}\int_{\partial\mathrm{B}_r} (Tu)^2\,\mathrm{d}\mathcal{H}^{n-1} &= \int_{\mathrm{B}_r\setminus\mathrm{B}_\varrho} \mathrm{div}_x\Big(u(x)^2\frac{(|x|-\varrho)x}{(r-\varrho)|x|}\Big)\,\mathrm{d}x\\ &\ge \frac{1}{r-\varrho}\int_{\mathrm{B}_r\setminus\mathrm{B}_\varrho} |u|^2\,\mathrm{d}x - 2\int_{\mathrm{B}_r\setminus\mathrm{B}_\varrho} |u|\,|\nabla u|\,\mathrm{d}x\,.\end{aligned}$$

Next we rearrange terms and use Young's inequality to deduce

$$\frac{1}{r-\varrho}\int_{\mathrm{B}_r\setminus\mathrm{B}_\varrho} |u|^2\,\mathrm{d}x \le \frac{1}{2(r-\varrho)}\int_{\mathrm{B}_r\setminus\mathrm{B}_\varrho} |u|^2\,\mathrm{d}x + 2(r-\varrho)\int_{\mathrm{B}_r\setminus\mathrm{B}_\varrho} |\nabla u|^2\,\mathrm{d}x + \int_{\partial\mathrm{B}_r} (Tu)^2\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Finally, absorbing one term on the left-hand side, we arrive at the claim □

The next lemma is not used in the following, but it may still be worth taking note of. It asserts that zero boundary values of $\mathrm{W}^{1,2}$ functions in the sense of trace are equivalent with zero boundary values in the sense of the space $\mathrm{W}^{1,2}_0$ (defined as usual as the closure of $\mathrm{C}^\infty_{\mathrm{cpt}}$ functions).

Lemma. *Consider a ball* $\mathrm{B}_r(x_0) \Subset \mathbb{R}^n$. *Then, for* $u \in \mathrm{W}^{1,2}(\mathrm{B}_r(x_0),\mathbb{R}^N)$, *we have*

$$u \in \mathrm{W}^{1,2}_0(\mathrm{B}_r(x_0),\mathbb{R}^N) \iff Tu = 0 \text{ in } \mathrm{L}^2(\partial\mathrm{B}_r(x_0),\mathbb{R}^N;\mathcal{H}^{n-1})\,.$$

Proof. We assume $x_0=0$. The forward implication is immediate by the definition of the subspace $\mathrm{W}^{1,2}_0(\mathrm{B}_r,\mathbb{R}^N)$, by (3.1), and by the continuity of T. To establish the backward implication, we consider some $u \in \mathrm{W}^{1,2}(\mathrm{B}_r,\mathbb{R}^N)$ with vanishing trace Tu. For $k \in \mathbb{N}$ with $k > r^{-1}$, we then choose a cut-off function $\eta_k \in \mathrm{C}^1_{\mathrm{cpt}}(\mathrm{B}_r)$ with $\mathbb{1}_{\mathrm{B}_{r-k^{-1}}} \le \eta_k \le 1$ and $|\nabla\eta_k| \le 2k$, and we observe that $\eta_k u \in \mathrm{W}^{1,2}_0(\mathrm{B}_r,\mathbb{R}^N)$ converges to u in $\mathrm{L}^2(\mathrm{B}_r,\mathbb{R}^N)$. Moreover, we can estimate

$$\begin{aligned}\int_{\mathrm{B}_r} |\mathrm{D}(\eta_k u)-\mathrm{D}u|^2\,\mathrm{d}x &\le 2\int_{\mathrm{B}_r}(1-\eta_k)^2|\mathrm{D}u|^2\,\mathrm{d}x + 2\int_{\mathrm{B}_r}|\nabla\eta_k|^2|u|^2\,\mathrm{d}x\\ &\le \int_{\mathrm{B}_r\setminus\mathrm{B}_{r-k^{-1}}} |\mathrm{D}u|^2\,\mathrm{d}x + 8k^2\int_{\mathrm{B}_r\setminus\mathrm{B}_{r-k^{-1}}} |u|^2\,\mathrm{d}x \le 33\int_{\mathrm{B}_r\setminus\mathrm{B}_{r-k^{-1}}} |\mathrm{D}u|^2\,\mathrm{d}x\,,\end{aligned}$$

where we used the assumption $Tu \equiv 0$ and the Poincaré inequality of the preceding lemma in the last step. Consequently, $\eta_k u$ converges to u even in $\mathrm{W}^{1,2}(\mathrm{B}_r,\mathbb{R}^N)$, and since the subspace $\mathrm{W}^{1,2}_0(\mathrm{B}_r,\mathbb{R}^N)$ is closed with respect to this convergence, we can conclude $u \in \mathrm{W}^{1,2}_0(\mathrm{B}_r,\mathbb{R}^N)$. □

Remark. *Later on, we often write* $u|_{\partial \mathrm{B}_r(x_0)}$ *instead of* Tu*, even if* u *is just in* $\mathrm{W}^{1,2}(\mathrm{B}_r(x_0), \mathbb{R}^N)$*. However, we keep in mind that, in general, this restriction is only turned into a meaningful notion by relying on the trace operator. Occasionally, we emphasize this fact by saying that certain identities hold* ***in the sense of trace****.*

3.1 Boundary curves; area functional versus Dirichlet integral

In the sequel, $\mathrm{B}_1 := \{z \in \mathbb{C} : |z| < 1\}$ stands for the **two-dimensional unit disk** in the complex plane $\mathbb{C}$, which we identify with the two-dimensional real Euclidean space $\mathbb{R}^2$. Correspondingly, we write $\partial \mathrm{B}_1$ for the **unit circle** in $\mathbb{C} = \mathbb{R}^2$.

Definition 3.1 (**Jordan curve**). *Consider a subset* Γ *of* $\mathbb{R}^\ell$*. If* Γ *is homeomorphic to* $\partial \mathrm{B}_1$*, then* Γ *is called a* ***(closed) Jordan curve*** *in* $\mathbb{R}^\ell$*. If* $c \colon \partial \mathrm{B}_1 \to \Gamma$ *is an homeomorphism, then the* ***length*** $\ell(\Gamma)$ *of* Γ *is defined as*

$$\ell(\Gamma) := \sup\left\{ \sum_{j=1}^{k} \left|c\big(\mathrm{e}^{\mathbf{i}t_j}\big) - c\big(\mathrm{e}^{\mathbf{i}t_{j-1}}\big)\right| \,:\, k \in \mathbb{N}\,,\, 0 = t_0 \le t_1 \le t_2 \le \ldots \le t_{k-1} \le t_k = 2\pi \right\},$$

and Γ *is called* ***rectifiable*** *if its length* $\ell(\Gamma)$ *is finite.*

Remarks.

(1) *It can be shown that the length* $\ell(\Gamma)$ *of* Γ *is well-defined in the sense that it depends only on* Γ *but not on the choice of the homeomorphism* c*.*

(2) *It is easy to construct examples of non-rectifiable Jordan curves. Moreover, while a Jordan curve* Γ *cannot contain an open subset of* $\mathbb{R}^\ell$*, there are Jordan curves* Γ *in* $\mathbb{R}^2$ *with positive 2-dimensional Lebesgue measure* $\mathcal{L}^2(\Gamma) > 0$*. The construction of such curves is closely related to Peano's construction of space-filling (but non-homeomorphic) curves and has first been carried out by W.F. Osgood.*

Definitions (**oriented Jordan curve, weakly monotonous parametrization**). *Consider a fixed closed Jordan curve* Γ *in* $\mathbb{R}^\ell$*.*

(1) *Given an homeomorphism* $c \colon \partial \mathrm{B}_1 \to \Gamma$ *and a continuous mapping* $\varphi \colon \partial \mathrm{B}_1 \to \Gamma$*, we call* φ *an* ***orientation-preserving reparametrization*** *of* c *if there exists a continuous, non-decreasing function* $\chi \colon [0, 2\pi] \to \mathbb{R}$ *with* $\chi(2\pi) = \chi(0) + 2\pi$ *such that*

$$\varphi\big(\mathrm{e}^{\mathbf{i}t}\big) = c\big(\mathrm{e}^{\mathbf{i}\chi(t)}\big) \qquad \text{holds for all } t \in [0, 2\pi]\,.$$

(2) *Orientation-preserving reparametrization yields an equivalence relation on homeomorphisms* $\partial \mathrm{B}_1 \to \Gamma$*, which divides these homeomorphisms into exactly two equivalence classes. We call* Γ *together with the choice of one of these two classes an* ***oriented Jordan curve****, and we say that every homeomorphism in the chosen class is an* ***orienting homeomorphism*** *for* Γ*.*

(3) ***An (oriented) Jordan curve is rectifiable if and only if there exists an (orienting) homeomorphism*** $\boldsymbol{c \colon \partial \mathrm{B}_1 \to \gamma}$ ***which is Lipschitz continuous.*** *Indeed,* c *can even be chosen as a constant speed parametrization of* Γ*, that means*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} c\big(\mathrm{e}^{\mathbf{i}t}\big)\right| = \frac{\ell(\Gamma)}{2\pi} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}\,.$$

In particular, this observation connects rectifiability of curves to the more general concept of rectifiability which has been briefly mentioned in Chapter 2.

(4) *If c is any orienting homeomorphism for Γ, we call the orientation-preserving reparametrizations of Γ also* ***weakly monotonous parametrizations*** *of Γ. Roughly speaking, these mappings run through Γ in a prescribed direction of rotation but are also allowed to stop at a fixed point for while.*

(5) ***Weak monotonicity passes to the limit*** *in the following sense. Whenever φ_k are weakly monotonous parametrizations of the oriented Jordan curve Γ and φ_k converges to φ, uniformly on $\partial \mathrm{B}_1$, then φ is a weakly monotonous parametrization of Γ. We remark that homeomorphisms do* not *have the corresponding closure property and that this is the true reason why we work with weak monotonicity in the following.*

Now we are ready to introduce the class of admissible parametric surfaces, that is the class $\mathscr{C}(\Gamma)$ of surfaces satisfying the **Plateau boundary condition** for a given Jordan curve Γ. In this class we eventually look for a solution of the Plateau problem.

Definition 3.2 (**admissible class $\mathscr{C}(\Gamma)$**). *For an oriented Jordan curve Γ in $\mathbb{R}^{2+N}$, we set*

$$\mathscr{C}(\Gamma) := \{y \in \mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N}) \,:\, Y_{|\partial \mathrm{B}_1} \text{ is a weakly monotonous parametrization of } \Gamma\}\,.$$

Here, the restriction is understood in the sense of trace, and the weak monotonicity requirement is imposed on the Lebesgue representative of the trace.

Moreover, we introduce and discuss the two functionals with which we frequently work in the sequel.

Definition 3.3 (**area functional and Dirichlet integral**). *We introduce the area functional A_Ω and the Dirichlet integral Dir_Ω by setting*[1]

$$\mathrm{A}_\Omega[Y] := \int_\Omega \sqrt{\det(\mathrm{D}Y^*\mathrm{D}Y)}\,\mathrm{d}x = \int_\Omega \sqrt{|\partial_1 Y|^2|\partial_2 Y|^2 - (\partial_1 Y \cdot \partial_2 Y)^2}\,\mathrm{d}x$$

and

$$\mathrm{Dir}_\Omega[Y] := \frac{1}{2}\int_\Omega |\mathrm{D}Y|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega \left(|\partial_1 Y|^2 + |\partial_2 Y|^2\right)\mathrm{d}x\,.$$

for every $Y \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^{2+N})$. We mostly use these definitions for $\Omega = \mathrm{B}_1$, and in this case we abbreviate $\mathrm{A} := \mathrm{A}_{\mathrm{B}_1}$ and $\mathrm{Dir} := \mathrm{Dir}_{\mathrm{B}_1}$.

Discussion (**area functional versus Dirichlet integral**).

(1) *The preceding definition of $\mathrm{A}[Y]$ makes sense for arbitrary $Y \in \mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N})$ but is clearly motivated and connected to our earlier considerations by the equality*

$$\mathrm{A}[Y] = \mathcal{H}^2(Y(\mathrm{B}_1)) \qquad \text{for one-to-one mappings } Y \in \mathrm{C}^1(\mathrm{B}_1, \mathbb{R}^{2+N})\,.$$

In view of this observation, the problem of finding a minimizer of the area functional A in $\mathscr{C}(\Gamma)$ turns out as a version of Plateau's problem.

[1] In the classical case $N = 1$ of the ambient space $\mathbb{R}^3$, one can also write $\mathrm{A}_\Omega[Y] = \int_\Omega |\partial_1 Y \times \partial_2 Y|\,\mathrm{d}x$ with the vector product $\times$ of vectors in $\mathbb{R}^3$.

(2) *The minimization problem for* A *in* $\mathscr{C}(\Gamma)$ *incorporates the problem that minimizers may exhibit lower-dimensional parts. For instance, for the curve* $\Gamma = \partial\mathrm{B}_1{\times}\{0\}$ *in* $\mathbb{R}^3$*, not only the parametrizations of the disk* $\mathrm{B}_1{\times}\{0\}$ *do minimize* A *in* $\mathscr{C}(\Gamma)$*, but also the following parametrization of the* ***'disk*** $\mathbf{B_1{\times}\{0\}}$ ***with the*** $\mathbf{C^\infty}$***-grown hair*** $\mathbf{\{(0,0)\}{\times}[0,1]}$***'*** *does:*

$$Y(x) := \begin{cases} \left(0,0,\exp\!\left(1-\frac{1}{1-4|x|^2}\right)\right) & \text{for } |x| \le \frac{1}{2} \\ \left(\exp\!\left(\frac{1}{3}-\frac{1}{4|x|^2-1}\right)\frac{x}{|x|},0\right) & \text{for } \frac{1}{2} < |x| \le 1 \end{cases}.$$

The point of this example is that the hair $\{(0,0)\}{\times}[0,1]$*, parametrized over* $\mathrm{B}_{1/2}$*, does not contribute to the area* $\mathrm{A}[Y]$*, since* $\mathrm{D}Y$ *has rank* ≤ 1 *on* $\mathrm{B}_{1/2}$*. Analogously, one can produce examples of* $\mathrm{C}^\infty(\overline{\mathrm{B}_1},\mathbb{R}^{2+N})$ *minimizers of* A *with many hairs, which do not correspond to regular surfaces. In order to rule out this type of irregular minimizers one may evidently admit only immersions* Y *as admissible parametric surfaces, but unfortunately the maximal rank condition, which characterizes immersions, is not preserved under limits and cannot be kept upright in the existence proof.*

(3) *We thus use a different approach, which builds on the observation of Corollary 1.9 that conformally parametrized minimal surfaces* Y *are characterized by the Laplace equation* $\Delta Y \equiv 0$*. A variational counterpart of this fact is the equality*

$$\mathrm{A}[Y] = \int_{\mathrm{B}_1} |\partial_1 Y|\,|\partial_2 Y|\,\mathrm{d}x = \mathrm{Dir}[Y] \qquad \text{for } \textbf{\textit{conformal}}\ Y \in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N})\,,$$

which results directly from Definitions 1.8 and 3.3. Therefore, in the next section we first deal with the simpler minimization problem for the Dirichlet integral Dir *in* $\mathscr{C}(\Gamma)$*. We eventually return to the connection between* A *and* Dir *and the original minimization problem for* A *in Section 3.4.*

3.2 Minimizing the Dirichlet integral under a Plateau boundary condition

The first decisive step towards the solution of Plateau's problem is contained in the following theorem.

Theorem 3.4 (**existence of Dirichlet minimizers**)**.** *Consider an oriented Jordan curve* Γ *in* $\mathbb{R}^{2+N}$ *with* $\mathscr{C}(\Gamma) \neq \emptyset$*.*

(I) Then there exists some $X \in \mathscr{C}(\Gamma)$ *with*

$$\mathrm{Dir}[X] \le \mathrm{Dir}[Y] \qquad \text{for all } Y \in \mathscr{C}(\Gamma)\,,$$

(II) and each such X *satisfies* $X \in \mathrm{C}^\infty(\mathrm{B}_1,\mathbb{R}^{2+N}) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^{2+N}\big)$ *and* $\Delta X \equiv 0$ *on* B_1*.*

The proof of Theorem 3.4 is carried out towards the end of this section.

Remarks.

(1) *Eventually it will turn out that Dirichlet minimizers* X *as found in Theorem 3.4 also solve the original Plateau problem.*

(2) *Clearly, the practicability of Theorem 3.4 crucially depends on criteria which ensure the assumption $\mathscr{C}(\Gamma) \neq \emptyset$. Therefore, we next provide such a criterion.*

Proposition 3.5. *For every* rectifiable *oriented Jordan curve Γ in $\mathbb{R}^{2+N}$, we have $\mathscr{C}(\Gamma) \neq \emptyset$.*

Proof of Proposition 3.5. As observed in Section 3.1, there is a Lipschitz continuous orienting homeomorphism $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{2+N}) \colon \partial \mathrm{B}_1 \to \Gamma$. However, here we do not even need to know that φ is an homeomorphism but merely that it is Lipschitz and weakly monotonous. Anyway, denoting the Lipschitz constant of φ by L, we introduce, for every $k \in \{1, 2, \ldots, 2+N\}$, a function $Y_k \colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$Y_k(x) := \inf_{\widetilde{x} \in \partial \mathrm{B}_1} \left[\varphi_k(\widetilde{x}) + L|x - \widetilde{x}|\right] \qquad \text{for } x \in \mathbb{R}^2 .$$

One readily checks that Y_k coincides with φ_k on $\partial \mathrm{B}_1$, and moreover, being a pointwise supremum of functions with Lipschitz constant L, also Y_k is Lipschitz continuous with constant L. It follows that $Y := (Y_1, Y_2, \ldots, Y_{2+N}) \colon \mathbb{R}^2 \to \mathbb{R}^{2+N}$ is a Lipschitz extension of φ with Lipschitz constant at most $L\sqrt{2+N}$. By Rademacher's theorem, this implies in particular $Y \in \mathrm{W}^{1,2}(\mathrm{B}_r(x_0), \mathbb{R}^{2+N})$, and all in all we end up with $Y \in \mathscr{C}(\Gamma)$. $\square$

Remark. *Proposition 3.5 can also be proved via Fourier expansion. From this alternative argument it is clearly visible that the existence of a Lipschitz weakly monotonous parametrization (which is equivalent with rectifiability) of Γ is only sufficient but not necessary for having $\mathscr{C}(\Gamma) \neq \emptyset$. In fact, it turns out that a necessary* and *sufficient condition is the existence of a weakly monotonous parametrization which is in the fractional Sobolev space $\mathrm{W}^{\frac{1}{2},2}(\partial \mathrm{B}_1, \mathbb{R}^{2+N}; \mathcal{H}^1)$.*

Next, in order to approach the proof of Theorem 3.4, we provide some preparatory lemmas.

Lemma 3.6 (conformal invariance of Dir). *For every $Y \in \mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N})$, every open subset Ω of $\mathbb{C}$, and every biholomorphic mapping τ of Ω onto B_1, there holds*

$$\mathrm{Dir}_\Omega[Y \circ \tau] = \mathrm{Dir}[Y] .$$

Remark. *By virtue of the Cauchy-Riemann equations,* ***a biholomorphic mapping and its inverse are both conformal****. This explains why the above property is known as conformal invariance.*

Proof of Lemma 3.6. The proof consists in a simple calculation based on the conformality of τ. Specifically, we exploit the equality $|\det \mathrm{D}\tau| = |\partial_1 \tau|^2 = |\partial_2 \tau|^2$ and the fact that $|\mathrm{D}Y|^2$ can be computed in both the orthonormal basis $\frac{\partial_1 \tau}{|\partial_1 \tau|}, \frac{\partial_2 \tau}{|\partial_2 \tau|}$ and the standard basis $\binom{1}{0}, \binom{0}{1}$ of $\mathbb{R}^2$. Using also the change of variables formula for the diffeomorphism τ, we find indeed

$$\begin{aligned}
\mathrm{Dir}_\Omega[Y \circ \tau] &= \frac{1}{2} \int_\Omega \left(|\partial_1 (Y \circ \tau)|^2 + |\partial_2 (Y \circ \tau)|^2\right) \mathrm{d}x \\
&= \frac{1}{2} \int_\Omega \left(\left|(\mathrm{D}Y \circ \tau) \frac{\partial_1 \tau}{|\partial_1 \tau|}\right|^2 + \left|(\mathrm{D}Y \circ \tau) \frac{\partial_2 \tau}{|\partial_2 \tau|}\right|^2\right) |\det \mathrm{D}\tau| \, \mathrm{d}x \\
&= \frac{1}{2} \int_\Omega \left(|\partial_1 Y \circ \tau|^2 + |\partial_2 Y \circ \tau|^2\right) |\det \mathrm{D}\tau| \, \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathrm{B}_1} \left(|\partial_1 Y|^2 + |\partial_2 Y|^2\right) \mathrm{d}x = \mathrm{Dir}[Y] .
\end{aligned}$$

$\square$

Lemma 3.7 (**reduction to $\mathscr{C}^*(\Gamma)$; three-point condition**). *Consider a Jordan curve Γ in $\mathbb{R}^{2+N}$ which is oriented by an homeomorphism $c\colon \partial \mathrm{B}_1 \to \Gamma$. Moreover, fix three distinct points $x_1, x_2, x_3 \in \partial \mathrm{B}_1$ and their (necessarily distinct) images $y_1 := c(x_1)$, $y_2 := c(x_2)$, $y_3 := c(x_3)$ in Γ. Then, for the class*

$$\mathscr{C}^*(\Gamma) := \{Y \in \mathscr{C}(\Gamma) \,:\, Y(x_1) = y_1\,,\, Y(x_2) = y_2\,,\, Y(x_3) = y_3\}$$

(where the evaluation of Y at x_1, x_2, x_3 is understood as the evaluation of the continuous representative of the trace of Y), we have

$$\mathscr{C}^*(\Gamma) \neq \emptyset \iff \mathscr{C}(\Gamma) \neq \emptyset$$

and

$$\inf\{\mathrm{Dir}[Y] \,:\, Y \in \mathscr{C}^*(\Gamma)\} = \inf\{\mathrm{Dir}[Y] \,:\, Y \in \mathscr{C}(\Gamma)\}\,.$$

Remark. *Later on, the three-point condition is useful in order to rule out certain degenerate minimizing sequences; compare with the remark at the end of this section.*

Proof of Lemma 3.7. Since $\mathscr{C}^*(\Gamma) \subset \mathscr{C}(\Gamma)$ holds by definition, it suffices to show that

$$\text{for every } Y \in \mathscr{C}(\Gamma) \text{ we can find a corresponding } Y^* \in \mathscr{C}^*(\Gamma) \text{ with } \mathrm{Dir}[Y^*] = \mathrm{Dir}[Y]\,. \tag{3.2}$$

To verify this, we work with distinct points $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in \partial \mathrm{B}_1$ such that (evaluating once more the continuous representative of the trace) we have $Y(\widetilde{x}_i) = y_i$ for $i = 1, 2, 3$. Such points $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ exist and lie on $\partial \mathrm{B}_1$ in the same 'order' as x_1, x_2, x_3, since the trace of Y maps $\partial \mathrm{B}_1$ weakly monotonously and surjectively onto Γ. We next claim that

$$\text{there exists a biholomorphic } \tau\colon \overline{\mathrm{B}_1} \to \overline{\mathrm{B}_1} \text{ with } \tau(x_i) = \tau(\widetilde{x}_i) \text{ for } i = 1, 2, 3 \tag{3.3}$$

(where $\tau\colon \overline{\mathrm{B}_1} \to \overline{\mathrm{B}_1}$ is called biholomorphic if $\tau\colon \overline{\mathrm{B}_1} \to \overline{\mathrm{B}_1}$ is one-to-one and if τ and τ^{-1} extend to holomorphic functions on a neighborhood of $\overline{\mathrm{B}_1}$ in $\mathbb{C}$).

Indeed, the claim (3.3) can be checked by writing the set of biholomorphic mappings $\mathrm{B}_1 \to \mathrm{B}_1$ (known as the conformal group of the disk) as an explicit 3-parameter family of Möbius transformations. However, we here prefer to first solve a corresponding problem on the extended upper half-plane $\overline{\mathrm{H}} := \{z \in \mathbb{C} \,:\, \mathrm{Im}\, z \geq 0\} \cup \{\infty_\mathbb{C}\}$, where the relevant computations become slightly simpler. The biholomorphic mappings $\overline{\mathrm{H}} \to \overline{\mathrm{H}}$ are in fact Möbius transformations $\widetilde{\tau}$, given by $\widetilde{\tau}(z) := \frac{az+b}{cz+d}$ with *real* parameters $a, b, c, d \in \mathbb{R}$ such that $ad{-}bc > 0$. We next verify that for all $\chi_0, \chi_1, \chi_\infty \in \partial\overline{\mathrm{H}} = \mathbb{R} \cup \{\infty_\mathbb{C}\}$ which are ordered by either $\chi_0{<}\chi_1{<}\chi_\infty$ or $\chi_1{<}\chi_\infty{<}\chi_0$ or $\chi_\infty{<}\chi_0{<}\chi_1$ (with $\infty_\mathbb{C}$ counting as either $+\infty$ or $-\infty$) there exists a Möbius transformation $\widetilde{\tau}$ as above with $\widetilde{\tau}(0) = \chi_0$, $\widetilde{\tau}(1) = \chi_1$, and $\widetilde{\tau}(\infty) = \chi_\infty$. If $\chi_0, \chi_1, \chi_\infty$ are all finite, by fixing $c = 1$ this leads to the system of linear equations $b - \chi_0 d = 0$, $a + b - \chi_1 d = \chi_1$, $a = \chi_\infty$ for $a, b, d \in \mathbb{R}$. It is readily checked that this system is always uniquely solvable (since $\chi_0, \chi_1, \chi_\infty$ are distinct) and that the solution satisfies $ad{-}bc > 0$. In addition, if one of $\chi_0, \chi_1, \chi_\infty$ is infinite, a similar but slightly simpler reasoning works. Thus, we have solved the problem to map the three points $0, 1, \infty \in \partial\overline{\mathrm{H}} = \mathbb{R} \cup \{\infty_\mathbb{C}\}$ to three arbitrary 'correctly ordered' points $\chi_0, \chi_1, \chi_\infty \in \partial\overline{\mathrm{H}}$ by a Möbius transformation $\widetilde{\tau}$ as above. However, since these transformations form a group, it follows that we can also use them to map three arbitrary points in $\partial\overline{\mathrm{H}}$ to three other arbitrary points in $\partial\overline{\mathrm{H}}$ provided that the preimage and the image points are 'ordered' in the same way. Then, by using the biholomorphic transformation $\overline{\mathrm{H}} \to \overline{\mathrm{B}_1}$, $z \mapsto \frac{z-\mathbf{i}}{z+\mathbf{i}}$ and its inverse $\overline{\mathrm{B}_1} \to \overline{\mathrm{H}}$, $z \mapsto \mathbf{i}\frac{1+z}{1-z}$, we can solve the original problem (3.3) on the disk as well. With the mapping τ from (3.3) at hand, we can finally set $Y^* := Y\circ\tau$. It then follows that we have $Y^* \in \mathscr{C}(\Gamma)$ (in this connection observe that $\tau|_{\partial \mathrm{B}_1} \colon \partial\mathrm{B}_1 \to \partial\mathrm{B}_1$ is a restriction of a biholomorphic mapping; thus, it is homeomorphic and carries the orientation of the identity) and $Y^*(x_i) = Y(\tau(x_i)) = Y(\widetilde{x}_i) = y_i$ for $i = 1, 2, 3$. All in all, we have hence shown $Y^* \in \mathscr{C}^*(\Gamma)$. To conclude the reasoning we use the conformal invariance of the Dirichlet integral (Lemma 3.6), which yields

$$\mathrm{Dir}[Y^*] = \mathrm{Dir}[Y \circ \tau] = \mathrm{Dir}[Y]\,.$$

Thus, we have verified (3.2), and the proof of the lemma is complete. □

Lemma 3.8 (Courant-Lebesgue lemma). *Consider $Y \in \mathrm{C}^1(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^\ell\big)$. Then, for every $\delta\in\,]0,1[$ and $z_0\in\partial\mathrm{B}_1$ there exists a radius $\varrho_0\in\big]\delta,\sqrt{\delta}\big[$ with*

$$\int_{\mathrm{B}_1\cap\partial\mathrm{B}_{\varrho_0}(z_0)} |\mathrm{D}Y|\,\mathrm{d}\mathcal{H}^1 \le \bigg(\frac{4\pi}{\log\frac{1}{\delta}}\,\mathrm{Dir}[Y]\bigg)^{\frac12}\,.$$

In particular, for the two distinct points $z_0^-, z_0^+\in\partial\mathrm{B}_1$ with $|z_0^- - z_0| = |z_0^+ - z_0| = \varrho_0$, we have

$$|Y(z_0^+) - Y(z_0^-)| \le \bigg(\frac{4\pi}{\log\frac{1}{\delta}}\,\mathrm{Dir}[Y]\bigg)^{\frac12}\,.$$

Remark. *We remark that $\mathrm{W}^{1,2}$ functions of two variables do not necessarily have continuous traces. Correspondingly, even for smooth functions of two variables, there is no general way to control the modulus of continuity of the trace in terms of the $\mathrm{W}^{1,2}$ norm of the function. Also Lemma 3.8 does* not *yield such a control (since z_0^+ and z_0^- are* not *arbitrary points in a neighborhood of z_0), but it* almost *gives an explicit modulus of continuity for $Y|_{\partial\mathrm{B}_1}$.*

Proof of Lemma 3.8. We assume that $M := \mathrm{Dir}[Y]$ is finite and introduce the set of good radii

$$G := \bigg\{\varrho\in\,]\delta,\sqrt{\delta}[\;:\;\int_{\mathrm{B}_1\cap\partial\mathrm{B}_\varrho(z_0)}|\mathrm{D}Y|^2\,\mathrm{d}\mathcal{H}^1 \le \frac{4M}{\varrho\log\frac{1}{\delta}}\bigg\}\,.$$

Then we show $\mathcal{L}^1(G) > 0$. Indeed, suppose for contradiction that G is $\mathcal{L}^1$-negligible. Then we have

$$\int_{\mathrm{B}_1\cap\partial\mathrm{B}_\varrho(z_0)}|\mathrm{D}Y|^2\,\mathrm{d}\mathcal{H}^1 > \frac{4M}{\varrho\log\frac{1}{\delta}}\qquad\text{for }\mathcal{L}^1\text{-a.e. }\varrho\in\,]\delta,\sqrt{\delta}[\,,$$

and consequently radial integration shows

$$\mathrm{Dir}[Y] \ge \frac12\int_\delta^{\sqrt{\delta}}\int_{\mathrm{B}_1\cap\partial\mathrm{B}_\varrho(z_0)}|\mathrm{D}Y|^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}\varrho > \frac12\int_\delta^{\sqrt{\delta}}\frac{4M}{\varrho\log\frac{1}{\delta}}\,\mathrm{d}\varrho = \frac{2M}{\log\frac{1}{\delta}}\log\frac{\sqrt{\delta}}{\delta} = M\,.$$

This contradicts the choice of M, and thus we have verified the claim $\mathcal{L}^1(G) > 0$. In particular, G is non-empty and we can choose some $\varrho_0\in G$. Writing $\partial\mathrm{B}_1\cap\partial\mathrm{B}_{\varrho_0}(z_0) = \{z_0^-, z_0^+\}$, we have $|z_0^\pm - z_0| = \varrho_0 \in\,]\delta,\sqrt{\delta}[$, and moreover integration and Hölder's inequality yield

$$\begin{aligned}
|Y(z_0^+) - Y(z_0^-)| &\le \int_{\mathrm{B}_1\cap\partial\mathrm{B}_{\varrho_0}(z_0)}|\mathrm{D}Y|\,\mathrm{d}\mathcal{H}^1\\
&\le \bigg(\int_{\mathrm{B}_1\cap\partial\mathrm{B}_{\varrho_0}(z_0)}|\mathrm{D}Y|^2\,\mathrm{d}\mathcal{H}^1\bigg)^{\frac12}\Big(\mathcal{H}^1(\mathrm{B}_1\cap\partial\mathrm{B}_{\varrho_0}(z_0))\Big)^{\frac12}\\
&\le \bigg(\frac{4M}{\varrho_0\log\frac{1}{\delta}}\bigg)^{\frac12}(\pi\varrho_0)^{\frac12} = \bigg(\frac{4\pi}{\log\frac{1}{\delta}}\,\mathrm{Dir}[Y]\bigg)^{\frac12}.
\end{aligned}$$

This ends the proof of the lemma. □

Now we are ready to prove the existence result.

Proof of Theorem 3.4. We start by proving the claim (II) not only for minimizers in $\mathscr{C}(\Gamma)$ but even for minimizers with respect to a Dirichlet boundary condition. More precisely, understanding in the following all (in)equalities on $\partial\mathrm{B}_1$ in the sense of trace, we will show

$$\left.\begin{cases} X \in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N})\,,\ X|_{\partial\mathrm{B}_1} \in \mathrm{C}^0(\partial\mathrm{B}_1,\mathbb{R}^{2+N})\,, \\ \mathrm{Dir}[X] \le \mathrm{Dir}[Y] \text{ for all } Y \in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N}) \text{ with } Y = X \text{ on } \partial\mathrm{B}_1 \end{cases}\right\} \\ \implies X \in \mathrm{C}^\infty(\mathrm{B}_1,\mathbb{R}^{2+N}) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^{2+N}\big) \text{ and } \Delta X \equiv 0 \text{ on } \mathrm{B}_1\,. \tag{3.4}$$

Once (3.4) is obtained, the claim (II) is contained as a special case. In order to establish the conclusion of (3.4) for X as in its hypothesis, we first make use of the Euler equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Dir}[X{+}t\Phi] = \int_{\mathrm{B}_1} \mathrm{D}X\cdot\mathrm{D}\Phi\,\mathrm{d}x \qquad \text{for all } \Phi \in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell) \text{ with } \Phi \equiv 0 \text{ on } \partial\mathrm{B}_1\,.$$

In other words, X is weakly harmonic, and by Weil's lemma it is a C^∞ (component-wise) harmonic function on B_1. It remains to deduce from the continuity of the trace φ of X on $\partial\mathrm{B}_1$ that X is continuous up to $\partial\mathrm{B}_1$. To this end, one can use a simple reasoning with affine barriers. Indeed, fixing $i \in \{1,2,\ldots,2{+}N\}$, $x_0 \in \partial\mathrm{B}_1$, and $\varepsilon > 0$ and exploiting the continuity of φ, we can find an affine function $a\colon \mathbb{R}^2 \to \mathbb{R}$ with $a \ge \varphi_i$ on $\partial\mathrm{B}_1$ and $a(x_0) \le \varphi_i(x_0){+}\varepsilon$. By elementary properties of traces, we get $X_i{-}a \le 0$ and $(X_i{-}a)_+ = 0$ on $\partial\mathrm{B}_1$. Testing the Euler with $\Phi = (X_i{-}a)_+$, then leads in a straightforward way to $X_i{-}a \le 0$ on B_1 and $\limsup_{\mathrm{B}_1\ni x\to x_0} X_i(x) \le a(x_0) \le \varphi_i(x_0) + \varepsilon$. Relying also on an analogous estimate from below and sending $\varepsilon \searrow 0$, this shows $\lim_{\mathrm{B}_1\ni x\to x_0} X_i(x) = \varphi_i(x_0)$, and all in all we see that X continuously attains the boundary datum φ on $\partial\mathrm{B}_1$. This completes the proof of (3.4).

Next we turn to the proof of (I). By Lemma 3.7, we have $\mathscr{C}^*(\Gamma) \neq \emptyset$ and it suffices to prove the claim of (I) with $\mathscr{C}^*(\Gamma)$ in place $\mathscr{C}(\Gamma)$, where $\mathscr{C}^*(\Gamma)$ is defined with suitably fixed, distinct points $x_1,x_2,x_3 \in \partial\mathrm{B}_1$ and $y_1,y_2,y_3 \in \Gamma$. We now start with a minimizing sequence $(Y_k)_{k\in\mathbb{N}}$ for Dir in $\mathscr{C}^*(\Gamma)$ and first replace each Y_k with a minimizer X_k of Dir in the class $W_k := \{Z \in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N}) : Z = Y_k \text{ on } \partial\mathrm{B}_1\}$. Such minimizers X_k exist by a standard argument[2], and evidently $(X_k)_{k\in\mathbb{N}}$ is still a minimizing sequence for Dir in $\mathscr{C}^*(\Gamma)$. Moreover, by (3.4), we have $X_k \in \mathrm{C}^\infty(\mathrm{B}_1,\mathbb{R}^{2+N}) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^{2+N}\big)$ and $\Delta X_k \equiv 0$ on B_1. We next show (and this is really the crux of the proof) equi-continuity of the traces $X_k|_{\partial\mathrm{B}_1}$. To this end, we consider some $\varepsilon > 0$ with

$$\varepsilon < \min\{|y_2{-}y_1|,|y_3{-}y_1|,|y_3{-}y_2|\}\,.$$

Since Γ is homeomorphic to $\partial\mathrm{B}_1$, the following property then holds for a suitably small $\gamma > 0$:

> Whenever $p,q \in \Gamma$ satisfy $|q{-}p| \le \gamma$, then we have $\operatorname{diam}\Gamma(p,q) < \varepsilon$ for some closed arc $\Gamma(p,q)$ connecting p and q in Γ.

[2] Indeed, it follows from the Poincaré inequality with traces that every minimizing sequence for Dir in W_k is bounded in $\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N})$ and thus has a weakly convergent subsequence there. By continuity of the trace operator the weak limit is still in W_k, and by weak lower semicontinuity of the norm it minimizes Dir in W_k.

Furthermore, setting $M := \sup_{k\in\mathbb{N}} \mathrm{Dir}[X_k]$ we can clearly find some $\delta \in \,]0,1[$ with

$$\left(\frac{4\pi}{\log\frac{1}{\delta}}M\right)^{\frac{1}{2}} \le \gamma \qquad \text{and} \qquad 2\sqrt{\delta} \le \min\{|x_2{-}x_1|, |x_3{-}x_1|, |x_3{-}x_2|\}\,.$$

For arbitrary $z_0 \in \partial\mathrm{B}_1$ and fixed $k \in \mathbb{N}$, we now apply the Courant-Lebesgue lemma (Lemma 3.8) to X_k and consider the corresponding distinct points $z_0^\pm \in \partial\mathrm{B}_1$. By the lemma and the preceding choice of δ and M, we have $|X_k(z_0^+) - X_k(z_0^-)| \le \gamma$. By the choices of γ and ε, we deduce first $\operatorname{diam}\Gamma(X_k(z_0^-), X_k(z_0^+)) < \varepsilon$ and then that

$$\text{the arc } \Gamma(X_k(z_0^-), X_k(z_0^+)) \text{ contains at most one of the points } y_1, y_2, y_3\,.$$

However, since the radius $\varrho_0 = |z_0^\pm{-}z_0|$ from Lemma 3.8 satisfies $\varrho_0 < \sqrt{\delta}$, the choice of δ also guarantees that

$$\text{the arc } \overline{\mathrm{B}_\varrho(z_0)} \cap \partial\mathrm{B}_1 \text{ contains at most one of the points } x_1, x_2, x_3\,.$$

Now we observe that the arc $\overline{\mathrm{B}_\varrho(z_0)}\cap\partial\mathrm{B}_1$ connecting z_0^- and z_0^+ in $\partial\mathrm{B}_1$ is mapped by the weakly monotonous trace $X_k|_{\partial\mathrm{B}_1}$ onto some arc connecting $X_k(z_0^-)$ and $X_k(z_0^+)$ in Γ. The later arc $X_k(\overline{\mathrm{B}_\varrho(z_0)} \cap \partial\mathrm{B}_1)$ must in fact coincide with $\Gamma(X_k(z_0^-), X_k(z_0^+))$, since the only other candidate $\overline{\Gamma \setminus \Gamma(X_k(z_0^-), X_k(z_0^+))}$ contains at least two of the points $y_1 = X_k(x_1), y_2 = X_k(x_2), y_3 = X_k(x_3)$, while $X_k\big(\overline{\mathrm{B}_\varrho(z_0)} \cap \partial\mathrm{B}_1\big)$ contains at most one of them. In conclusion, we have shown

$$X_k(\mathrm{B}_\delta(z_0) \cap \partial\mathrm{B}_1) \subset X_k\big(\overline{\mathrm{B}_\varrho(z_0)} \cap \partial\mathrm{B}_1\big) = \Gamma(X_k(z_0^-), X_k(z_0^+)) \subset \mathrm{B}_\varepsilon(X_k(z_0))\,,$$

and all in all this proves equi-continuity of the traces $X_k|_{\partial\mathrm{B}_1}$. Since Γ is bounded, the traces are also equi-bounded, and by the Arzelà-Ascoli theorem a subsequence $(X_{k_\ell}|_{\partial\mathrm{B}_1})_{\ell\in\mathbb{N}}$ of the traces converges uniformly on $\partial\mathrm{B}_1$ to some $\varphi \in \mathrm{C}^0(\partial\mathrm{B}_1, \mathbb{R}^{2+N})$. Clearly, φ is a weakly monotonous parametrization of Γ with $\varphi(x_1) = y_1$, $\varphi(x_2) = y_2$, $\varphi(x_3) = y_3$, since uniform convergence preserves these properties. An application of the maximum principle to the harmonic component functions of $X_{k_{\ell_1}}{-}X_{k_{\ell_2}}$ now shows that $(X_{k_\ell})_{\ell\in\mathbb{N}}$ is a uniform Cauchy sequence even on all of $\overline{\mathrm{B}}_1$ and thus converges uniformly on $\overline{\mathrm{B}_1}$ to a limit $X \in \mathrm{C}^0(\overline{\mathrm{B}_1}, \mathbb{R}^{2+N})$ with $X|_{\partial\mathrm{B}_1} = \varphi$. Finally, by weak compactness, $(X_{k_\ell})_{\ell\in\mathbb{N}}$ converges to X also weakly in $\mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N})$. Hence, we have $X \in \mathscr{C}^*(\Gamma)$, and the lower semicontinuity of the norm gives

$$\mathrm{Dir}[X] \le \liminf_{\ell\to\infty} \mathrm{Dir}[X_{k_\ell}] = \inf_{\mathscr{C}^*(\Gamma)} \mathrm{Dir}\,.$$

In view of the reduction step at the beginning, this completes the proof of the claim (I). □

Remarks. *We stress that* ***the preceding proof cannot work with a two-point condition*** *in place of the three-point condition. This can be seen already in the simple case of the planar curve $\Gamma = \partial\mathrm{B}_1 \times \{0\} \subset \mathbb{R}^{2+N} = \mathbb{C} \times \mathbb{R}^N$, where a sequence of Dirichlet minimizers in $\mathscr{C}(\Gamma)$ is given by*

$$X_k(z) := \Big(\frac{z+1-k^{-1}}{1+(1-k^{-1})z}, 0\Big) \in \mathbb{C} \times \mathbb{R}^N \qquad \text{for } z \in \overline{\mathrm{B}_1} \subset \mathbb{C}\,.$$

Here the first complex-valued component is a biholomorphic mapping $\overline{\mathrm{B}_1} \to \overline{\mathrm{B}_1}$, which leaves the points ±1 fixed and converges for $k \to \infty$ to the constant function 1 locally uniformly on

$\overline{\mathrm{B}_1} \setminus \{-1\}$ and weakly in $\mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{C})$. Since constant functions are not in $\mathscr{C}(\Gamma)$, this means that the minimizing sequence X_k (and all its subsequences) do not have a limit in $\mathscr{C}(\Gamma)$.

The same degeneration phenomenon can occur for every given Jordan curve Γ. Indeed, for every $X \in \mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N})$ with bounded trace, a sequence of reparametrizations of X converges weakly in $\mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N})$ to a constant function.

Addendum on biholomorphic mappings

In order to complete the solution of Plateau's problem, we will employ the following classical result of complex analysis.

Theorem 3.9 (Riemann mapping theorem). *For every simply connected domain Ω in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$, there exists a biholomorphic mapping of Ω onto the unit disk B_1.*

Remarks.

(1) *Here, by a domain we mean a non-empty, open, and connected set.*

(2) *Theorem 3.9 is surprising insofar that arbitrary domains can have quite irregular boundaries, while biholomorphic mappings are subject to comparably strong constraints.*

(3) *The assumptions of Theorem 3.9 are, to a large extent, optimal: If Ω lacks one of the assumed topological properties (non-empty, open, connected, simply connected), then there is not even an homeomorphism of Ω onto B_1. Moreover, the theorem cannot be extended to the case $\Omega = \mathbb{C}$, since every holomorphic mapping $\mathbb{C} \to \mathrm{B}_1$ is constant by Liouville's theorem.*

(4) *It follows as a corollary that, for every pair of simply connected domains Ω and $\widetilde{\Omega}$ in $\mathbb{C}$ with $\Omega \neq \mathbb{C} \neq \widetilde{\Omega}$, there exists a biholomorphic mapping of Ω onto $\widetilde{\Omega}$.*

Different proofs of Theorem 3.9 can be found in many textbooks on complex analysis, but we do not discuss any of these here. However, we now establish another classical result, whose proof is somewhat harder to find in the literature and which concerns the boundary behavior of the Riemann mappings (on domains with a Jordan curve as boundary).

Theorem 3.10 (Carathéodory's extension theorem for conformal mappings). *Consider a bounded open set Ω in $\mathbb{C}$ such that $\partial\Omega$ is a Jordan curve. Then every biholomorphic mapping h of B_1 onto Ω extends to an homeomorphism of $\overline{\mathrm{B}_1}$ onto $\overline{\Omega}$ (which, in particular, maps $\partial\mathrm{B}_1$ homeomorphicly onto $\partial\Omega$).*

Remarks.

(1) *Every Jordan curve Γ in $\mathbb{C}$ decomposes $\mathbb{C}$ into an 'interior' and an 'exterior', namely a non-empty, bounded connected component and an unbounded connected component of $\mathbb{C} \setminus \Gamma$, both these components have Γ as their boundary, and the bounded 'interior' is additionally simply connected. These (extremely plausible) assertions are indeed guaranteed by the* ***Jordan curve theorem****, and in the situation of Theorem 3.10 they imply that Ω, being the 'interior' of $\partial\Omega$, is automatically connected and simply connected (as assumed in the Riemann mapping theorem).*

(2) *Also the assumptions of Theorem 3.9 are quite sharp. In particular, the requirement that $\partial\Omega$ is a Jordan curve is evidently necessary to obtain the conclusion of the theorem.*

(3) *A variant of Theorem 3.10 holds for a pair of suitable domains. This follows as a straightforward corollary from both Theorem 3.9 and Theorem 3.10.*

Proof of Theorem 3.10. Regarding h as an $\mathbb{R}^2$-valued function of two real variables and writing $\mathrm{D}h$ for the real derivative, we infer from the Cauchy-Riemann equations the equality $2\det\mathrm{D}h = |\mathrm{D}h|^2$. Changing variables and taking into account the boundedness of Ω, we thus deduce

$$\int_{\mathrm{B}_1} |\mathrm{D}h|^2\,\mathrm{d}x = 2\mathcal{L}^2(h(\mathrm{B}_1)) = 2\mathcal{L}^2(\Omega) < \infty\,.$$

Now we fix a point $z_0 \in \partial\mathrm{B}_1$. Then, the Courant-Lebesgue lemma (Lemma 3.8) provides us with a null sequence of radii $\varrho_k \le k^{-1}$ such that

$$\int_{\mathrm{B}_1\cap\partial\mathrm{B}_{\varrho_k}(z_0)} |\mathrm{D}h|\,\mathrm{d}\mathcal{H}^1 \le \left(\frac{4\pi}{\log k}\int_{\mathrm{B}_1} |\mathrm{D}h|^2\,\mathrm{d}x\right)^{\frac12}. \tag{3.5}$$

In particular, for every $k \in \mathbb{N}$, the Jordan arc

$$\Xi_k := h(\mathrm{B}_1 \cap \partial\mathrm{B}_{\varrho_k}(z_0)) \subset \Omega$$

has finite length and possesses two (at this stage not necessarily distinct) endpoints $p_k, q_k \in \overline{\Omega}$. Since $h\colon \mathrm{B}_1 \to \Omega$ is homeomorphic, endpoints in Ω are excluded and in fact we necessarily have $p_k, q_k \in \partial\Omega$. Moreover, since the right-hand side of (3.5) vanishes in the limit, we have

$$\lim_{k\to\infty} \operatorname{diam}\Xi_k = 0\,.$$

Next we exploit that $\partial\Omega$ is a Jordan curve. Denoting by Γ_k the (closed) arc of smaller diameter connecting p_k and q_k in $\partial\Omega$, we infer

$$\lim_{k\to\infty} \operatorname{diam}\Gamma_k = 0\,.$$

By the preceding remark, the 'interior' U_k of each Jordan curve $\Xi_k \cup \Gamma_k$ is open, bounded, and connected, and moreover we have[3] $U_k \subset \Omega$. It follows that

$$\lim_{k\to\infty} \operatorname{diam} U_k = \lim_{k\to\infty} \operatorname{diam}(\Xi_k \cup \Gamma_k) = 0$$

and also that

$$U_k \,\dot\cup\, \big(\Omega \setminus \overline{U_k}\big) = \Omega \setminus \Xi_k = h(\mathrm{B}_1 \cap \mathrm{B}_{\varrho_k}(z_0)) \,\dot\cup\, h\big(\mathrm{B}_1 \setminus \overline{\mathrm{B}_{\varrho_k}(z_0)}\big)\,.$$

[3] Here we use the very plausible fact that, for a Jordan curve J contained in the closure $\overline{\Omega}$ of the bounded domain Ω with Jordan boundary, the 'interior' U of J is contained in Ω. A formal justification of this fact goes as follows: Since $\mathbb{C} \setminus \overline{\Omega}$ is an open and connected subset of $\mathbb{C} \setminus J$, it is contained in one of the two connected components of $\mathbb{C}\setminus J$. Furthermore, since $\mathbb{C}\setminus\overline{\Omega}$ is unbounded, it can only be contained in the unbounded component $\mathbb{C} \setminus \overline{U}$ of $\mathbb{C} \setminus J$. Passing to the complements, we infer $\overline{U} \subset \overline{\Omega}$, and since U is open, U is then contained in the interior of $\overline{\Omega}$. By the Jordan curve theorem, we have $\partial\Omega = \partial(\mathbb{C} \setminus \overline{\Omega})$, the interior of $\overline{\Omega}$ is just Ω, and we arrive at $U \subset \Omega$.

As U_k, $h(\mathrm{B}_1 \cap \mathrm{B}_{\varrho_k}(z_0))$, $h\big(\mathrm{B}_1 \setminus \overline{\mathrm{B}_{\varrho_k}(z_0)}\big)$ are open and connected and $\Omega \setminus \overline{U_k}$ is at least open, by comparing the connected components on the two sides of the last equality, we infer that U_k equals either $h(\mathrm{B}_1 \cap \mathrm{B}_{\varrho_k}(z_0))$ or $h\big(\mathrm{B}_1 \setminus \overline{\mathrm{B}_{\varrho_k}(z_0)}\big)$. Taking into account that $\operatorname{diam} U_k$ vanishes in the limit, while $\operatorname{diam} h\big(\mathrm{B}_1 \setminus \overline{\mathrm{B}_{\varrho_k}(z_0)}\big)$ stays away from zero (since $h\big(\mathrm{B}_1 \setminus \overline{\mathrm{B}_{\varrho_k}(z_0)}\big)$ contains the open set $h(\mathrm{B}_{1/2})$ for $k \gg 1$), the second possibility drops out and we necessarily have

$$U_k = h(\mathrm{B}_1 \cap \mathrm{B}_{\varrho_k}(z_0)) \qquad \text{for } k \gg 1\,.$$

In conclusion, we have shown

$$\lim_{k\to\infty} h(\mathrm{B}_1 \cap \mathrm{B}_{\varrho_k}(z_0)) = 0\,,$$

and hence the intersection $\bigcap_{k=1}^{\infty} \overline{h(\mathrm{B}_1 \cap \mathrm{B}_{\varrho_k}(z_0))}$ of a decreasing sequence of non-empty closed sets consists of exactly one point $y_0 \in \partial\Omega$. Therefore, by setting $h(z_0) := y_0$ we can extend h continuously to $\mathrm{B}_1 \cup \{z_0\}$, and since $z_0 \in \partial \mathrm{B}_1$ is arbitrary, h extends to a continuous mapping on $\overline{\mathrm{B}_1}$ with $h(\partial \mathrm{B}_1) \subset \partial\Omega$. Moreover, since $h\big(\overline{\mathrm{B}_1}\big)$ is compact with $\Omega \subset h\big(\overline{\mathrm{B}_1}\big) \subset \overline{\Omega}$, we infer $h(\overline{\mathrm{B}_1}) = \overline{\Omega}$, and since h maps B_1 onto Ω, we get $h(\partial \mathrm{B}_1) = \partial\Omega$.

As the last step, it only remains to conclude that $h|_{\partial \mathrm{B}_1}$ is one-to-one and this is now proved by contradiction. Assume that we can find two distinct points $z_1, z_2 \in \partial \mathrm{B}_1$ with $h(z_1) = h(z_2)$. Then, for the union $L := \{tz_1 : t \in [0,1]\} \cup \{tz_2 : t \in [0,1]\}$ of two radial line segments, $h(L)$ is a Jordan curve which intersects $\partial\Omega$ in precisely one point. Writing U for the open and connected 'interior' of $h(L)$, we have[4] $U \subset \Omega$, and $h^{-1}(U)$ is a non-empty, open, and connected set in $\mathrm{B}_1 \setminus L$. Now $\mathrm{B}_1 \setminus L$ consists of two circular sectors, and by connectedness $h^{-1}(U)$ then equals one of these two sectors, which we call S. This implies $h(S) = U$ and $h(\partial S \cap \partial \mathrm{B}_1) \subset \overline{U} \cap \partial\Omega$. Since $\overline{U} \cap \partial\Omega$ consists of precisely one point, h is then constant on the circular arc $\partial S \cap \partial \mathrm{B}_1$ with non-empty interior Ξ in $\partial \mathrm{B}_1$. In this situation, the following Lemma 3.11 (applied to the two harmonic component functions of h) will ensure that h extends analytically to a neighborhood of $\mathrm{B}_1 \cup \Xi$ in $\mathbb{C}$. In particular, the complex derivative $h^{(\ell)}$ of h of arbitrary order $\ell \in \mathbb{N}_0$ satisfies the Cauchy-Riemann equations on $\mathrm{B}_1 \cup \Xi$, thus the real derivative $\mathrm{D}[h^{(\ell)}]$ of $h^{(\ell)}$ is a matrix of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, whose rank is either 0 or 2. Hence, the vanishing of the tangential derivative of h along Ξ implies the vanishing of $\mathrm{D}h$ and h' on Ξ, and it follows iteratively that all derivatives of h (of order ≥ 1) vanish on Ξ. By the identity theorem for analytic functions, h is then constant not only on Ξ but also on all of B_1. This contradicts the assumption that h is biholomorphic on B_1, and thus we have verified the claim that $h|_{\partial \mathrm{B}_1}$ is one-to-one. All in all, the extended h maps $\overline{\mathrm{B}_1}$ continuously and one-to-one onto $\overline{\Omega}$ and is thus[5] an homeomorphism. □

Here, towards the end of the preceding proof we have used the following lemma, which is also useful in the subsequent section.

Lemma 3.11. *Assume that $h \in \mathrm{C}^2(\mathrm{B}_1) \cap \mathrm{C}^0(\mathrm{B}_1 \cup \Xi)$ is harmonic on B_1 and constant on a non-empty, open circular arc Ξ in $\partial \mathrm{B}_1$. Then h has an harmonic (and thus analytic) extension to a neighborhood of $\mathrm{B}_1 \cup \Xi$ in $\mathbb{C}$.*

Idea of proof. We can assume that h vanishes on Ξ. Then, by setting $h(tx) := -h(x/t)$ for $x \in \Xi$ and $t > 1$, we extend h to $\mathrm{B}_1 \cup \{tx : x \in \Xi, t \geq 1\}$, and we claim that h remains harmonic

[4] This inclusion has already been justified in the preceding footnote.

[5] Continuity of h^{-1} follows elementarily. Indeed, every closed set in $\overline{\mathrm{B}}_1$ is compact and is mapped by h onto a compact and thus closed set in $\overline{\Omega}$; by passing to the complements it then follows that h is an open map, which means that its inverse h^{-1} is continuous.

across Ξ. Indeed, the last assertion is known as the **Schwarz reflection principle** and can be deduced from a more common (odd) reflection principle by biholomorphic transformation from the upper half-plane to the unit disk. □

3.3 Conformality relations and solution of Plateau's problem

Now we are ready to make the next step towards the solution of Plateau's problem.

Theorem 3.12 (Dirichlet minimizers satisfy the conformality relations). *Consider an oriented Jordan curve Γ in $\mathbb{R}^{2+N}$ and some $X \in \mathscr{C}(\Gamma)$ with $\mathrm{Dir}[X] \le \mathrm{Dir}[Y]$ for all $Y \in \mathscr{C}(\Gamma)$. Then X is conformal, that is*

$$|\partial_1 X| = |\partial_2 X| \qquad \text{and} \qquad \partial_1 X \cdot \partial_2 X \equiv 0 \qquad \text{on } \mathrm{B}_1\,.$$

Remark. *Since Theorem 3.4 guarantees $\Delta X \equiv 0$ on B_1, **the minimizer X** in Theorem 3.12 **is a conformally parametrized (possibly) branched minimal surface** in the sense of Definition 1.10.*

We now prove the theorem by a comparison technique based on **inner**[6] **variations.**

Proof of Theorem 3.12. We consider an arbitrary $\Phi \in \mathrm{C}^1\big(\overline{\mathrm{B}_1}, \mathbb{R}^2\big)$ and set $\Phi_t(x) := x + t\Phi(x)$ for $x \in \overline{\mathrm{B}_1}$ and a parameter $t \in \mathbb{R}$. Then, for $|t| \ll 1$, we obtain a C^1 diffeomorphism Φ_t from B_1 onto the simply connected domain $\Omega_t := \Phi_t(\mathrm{B}_1) \subset \mathbb{R}^2$ and from (neighborhoods of) $\overline{\mathrm{B}_1}$ onto (neighborhoods of) $\overline{\Omega_t}$. By the Riemann mapping and Carathéodory theorems (Theorems 3.9 and 3.10), we can find a biholomorphic mapping h_t of B_1 onto Ω_t which extends to an homeomorphism $h_t \colon \overline{\mathrm{B}_1} \to \overline{\Omega_t}$. Writing Σ_t for the inverse of Φ_t (then Σ_t is diffeomorphic from Ω_t to B_1 and from $\overline{\Omega_t}$ to $\overline{\mathrm{B}_1}$), we now aim at comparing X with its reparametrizations $X \circ \Sigma_t \circ h_t$. To this end, we exploit that $X \circ \Sigma_t \circ h_t$ has square-integrable first derivatives, which essentially comes out along the lines of the subsequent reasoning. Moreover, since h_t is biholomorphic and Σ_t remains close to the identity, $\Sigma_t \circ h_t$ is an orientation-preserving homeomorphism of $\partial\mathrm{B}_1$ onto itself, and the trace of $X \circ \Sigma_t \circ h_t$ yields weakly monotonous parametrization of Γ. Involving the trace version of Poincaré's inequality, we can thus conclude that $X \circ \Sigma_t \circ h_t \in \mathscr{C}(\Gamma)$ is an admissible comparison surface for the minimality property of X. By this minimality, the conformal invariance of the Dirichlet integral (Lemma 3.6), and change of variables, for $|t| \ll 1$, we find

$$\begin{aligned}
\mathrm{Dir}[X] \le \mathrm{Dir}[X \circ \Sigma_t \circ h_t] = \mathrm{Dir}_{\Omega_t}[X \circ \Sigma_t]& \\
= \frac{1}{2}\int_{\Omega_t} \big(|(\mathrm{D}X \circ \Sigma_t)\partial_1\Sigma_t|^2 + |(\mathrm{D}X \circ \Sigma_t)\partial_2\Sigma_t|^2\big)\,\mathrm{d}x& \\
= \frac{1}{2}\int_{\mathrm{B}_1} \big(|\mathrm{D}X(\partial_1\Sigma_t \circ \Phi_t)|^2 + |\mathrm{D}X(\partial_2\Sigma_t \circ \Phi_t)|^2\big)|\det \mathrm{D}\Phi_t|\,\mathrm{d}x\,.&
\end{aligned}$$

Since Φ_0 and Σ_0 equal the identity map, the expression on the right-hand side of the preceding formula simplifies for $t = 0$ to $\mathrm{Dir}[X]$. Thus, this expression as a function of t attains its minimum for $t = 0$, and by the first-order calculus criterion we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{\mathrm{B}_1} \big(|\mathrm{D}X(\partial_1\Sigma_t \circ \Phi_t)|^2 + |\mathrm{D}X(\partial_2\Sigma_t \circ \Phi_t)|^2\big)|\det \mathrm{D}\Phi_t|\,\mathrm{d}x$$

[6]Here the term 'inner variations' does *not at all* indicate that something happens 'only away from the boundary' but rather it refers to a family of comparison surfaces $X \circ \tau_t$, where the *inner function* τ_t *of the composition* varies in dependence of a real parameter t.

(where the existence of the $\frac{\mathrm{d}}{\mathrm{d}t}$-derivative comes out as a side benefit of the following calculations). Switching the order of differentiation and integration, using product rules and the fact that $\det \mathrm{D}\Phi_t$ is positive for $|t| \ll 1$, we next find

$$\begin{aligned} 0 &= \int_{\mathrm{B}_1} 2\mathrm{D}X(\partial_1\Sigma_0 \circ \Phi_0) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{D}X(\partial_1\Sigma_t \circ \Phi_t) \det \mathrm{D}\Phi_0 \,\mathrm{d}x \\ &\quad + \int_{\mathrm{B}_1} 2\mathrm{D}X(\partial_2\Sigma_0 \circ \Phi_0) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{D}X(\partial_2\Sigma_t \circ \Phi_t) \det \mathrm{D}\Phi_0 \,\mathrm{d}x \\ &\quad + \int_{\mathrm{B}_1} \big(|\mathrm{D}X(\partial_1\Sigma_0 \circ \Phi_0)|^2 + |\mathrm{D}X(\partial_2\Sigma_0 \circ \Phi_0)|^2\big) \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det \mathrm{D}\Phi_t \,\mathrm{d}x \end{aligned} \tag{3.6}$$

In order to simplify the expressions on the right-hand side of (3.6), we compute with the usual rules for $(2{\times}2)$-matrices first

$$\det \mathrm{D}\Phi_t = \det(\mathrm{Id}{+}t\mathrm{D}\Phi) = 1 + t(\partial_1\Phi_1{+}\partial_2\Phi_2) + t^2 \det \mathrm{D}\Phi\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det \mathrm{D}\Phi_t = \partial_1\Phi_1{+}\partial_2\Phi_2\,,$$

and then

$$\mathrm{D}\Sigma_t \circ \Phi_t = (\mathrm{D}\Phi_t)^{-1} = \frac{1}{\det \mathrm{D}\Phi_t} \begin{pmatrix} 1 + t\partial_2\Phi_2 & -t\partial_2\Phi_1 \\ -t\partial_1\Phi_2 & 1 + t\partial_1\Phi_1 \end{pmatrix},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{D}\Sigma_t \circ \Phi_t = \begin{pmatrix} \partial_2\Phi_2 & -\partial_2\Phi_1 \\ -\partial_1\Phi_2 & \partial_1\Phi_1 \end{pmatrix} - (\partial_1\Phi_1 + \partial_2\Phi)\mathrm{Id} = -\mathrm{D}\Phi\,.$$

Exploiting these identities and again the fact that Φ_0 and Σ_0 equal the identity map, (3.6) turns into

$$\begin{aligned} 0 &= 2\int_{\mathrm{B}_1} \mathrm{D}X\mathrm{e}_1 \cdot \mathrm{D}X(-\partial_1\Phi_1\mathrm{e}_1{-}\partial_1\Phi_2\mathrm{e}_2) \,\mathrm{d}x \\ &\quad + 2\int_{\mathrm{B}_1} \mathrm{D}X\mathrm{e}_2 \cdot \mathrm{D}X(-\partial_2\Phi_1\mathrm{e}_1{-}\partial_2\Phi_2\mathrm{e}_2) \,\mathrm{d}x \\ &\quad + \int_{\mathrm{B}_1} \big(|\mathrm{D}X\mathrm{e}_1|^2{+}|\mathrm{D}X\mathrm{e}_2|^2\big)(\partial_1\Phi_1{+}\partial_2\Phi_2) \,\mathrm{d}x\,, \end{aligned}$$

where $\mathrm{e}_1, \mathrm{e}_2$ denotes the standard basis of $\mathbb{R}^2$. Rearranging terms, we next arrive at

$$0 = \int_{\mathrm{B}_1} \big(|\partial_1X|^2{-}|\partial_2X|^2\big)(\partial_2\Phi_2{-}\partial_1\Phi_1) \,\mathrm{d}x - 2\int_{\mathrm{B}_1} (\partial_1X \cdot \partial_2X)(\partial_1\Phi_2{+}\partial_2\Phi_1) \,\mathrm{d}x\,,$$

where $\Phi \in \mathrm{C}^1(\overline{\mathrm{B}_1}, \mathbb{R}^2)$ is still arbitrary. Next we consider arbitrary functions $f_1, f_2 \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1)$ and solutions $u_1, u_2 \in \mathrm{C}^\infty(\mathbb{R}^2)$ (which can be obtained, for instance, as Newton potentials of f_1, f_2) of the Poisson equations $\Delta u_1 = f_1$ and $\Delta u_2 = f_2$ on $\mathbb{R}^2$. We then apply the above to

$$\Phi := \begin{pmatrix} -\partial_1u_1 + \partial_2u_2 \\ \partial_1u_2 + \partial_2u_1 \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} \partial_2\Phi_2 - \partial_1\Phi_1 &= \Delta u_1 = f_1 \\ \partial_1\Phi_2 + \partial_2\Phi &= \Delta u_2 = f_2 \end{aligned}$$

and come out with

$$0 = \int_{\mathrm{B}_1} \big(|\partial_1X|^2{-}|\partial_2X|^2\big) f_1 \,\mathrm{d}x - 2\int_{\mathrm{B}_1} (\partial_1X \cdot \partial_2X) f_2 \,\mathrm{d}x\,.$$

As $f_1, f_2 \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1)$ are arbitrary, the fundamental lemma of the calculus of variations, then yields

$$|\partial_1 X|^2 - |\partial_2 X|^2 \equiv 0 \qquad \text{and} \qquad \partial_1 X \cdot \partial_2 X \equiv 0 \qquad \text{on } \mathrm{B}_1$$

as claimed. □

Remark. ***The proof of Theorem 3.12 crucially exploits*** *that X minimizes* Dir *in the class $\mathscr{C}(\Gamma)$ encoding* ***the Plateau boundary condition*** *and not only in a smaller class with a fixed Dirichlet boundary condition. Indeed, observe in this connection that by adding a well-posed boundary condition for the solutions u_1, u_2 above one cannot ensure zero boundary values of the resulting Φ.*

By combining Theorem 3.4 and Theorem 3.12, we can solve the Plateau problem in the class $\mathscr{C}(\Gamma)$. However, a slightly nicer existence statement is obtained by incorporating the following improvement at the boundary.

Theorem 3.13 (**the trace of X is an homeomorphism**)**.** *Consider an oriented Jordan curve Γ in $\mathbb{R}^{2+N}$ and a conformally parametrized (possibly) branched minimal surface $X \in \mathscr{C}(\Gamma)$. Then we have $X \in \mathrm{C}^0\big(\overline{\mathrm{B}_1}\big)$, and the trace $X|_{\partial \mathrm{B}_1}$ is an orienting homeomorphism for Γ.*

Proof. The claim $X \in \mathrm{C}^0\big(\overline{\mathrm{B}_1}\big)$ follows by comparing the harmonic (component functions of) X with affine barriers. This reasoning has already been described in the proof of Theorem 3.4, and we do not repeat the details here.

Next we prove that $X|_{\partial \mathrm{B}_1}$ is an homeomorphism of $\partial \mathrm{B}_1$ onto Γ. If this were false, then, in view of the weak monotonicity of $X|_{\partial \mathrm{B}_1}$, X would necessarily be constant on a non-empty, open circular arc Ξ in $\partial \mathrm{B}_1$. By Lemma 3.11 we could then extend X as an harmonic function to a neighborhood of $\mathrm{B}_1 \cup \Xi$ in $\mathbb{C}$, and the conformality relations $|\partial_1 X| = |\partial_2 X|$ and $\partial_1 X \cdot \partial_2 X \equiv 0$ would hold even on Ξ. Consequently, we would get $\operatorname{rank}(\mathrm{D}X) \in \{0, 2\}$ on Ξ, and the vanishing of the tangential derivative of X along Ξ would imply the vanishing of the whole derivative $\mathrm{D}X$ on Ξ. To get an analogous conclusion for the higher derivatives, one could, for instance, write $X = \operatorname{Re} H$ as the real part of $\mathbb{C}^{2+N}$-valued holomorphic function H on a neighborhood of $\mathrm{B}_1 \cup \Xi$ (compare Theorem 1.11). By the Cauchy-Riemann equations, $\mathrm{D}X \equiv 0$ on Ξ implies $H' \equiv 0$ on Ξ, and then, as in the proof of Theorem 3.10, all derivatives of H (of order ≥ 1) would vanish on Ξ. By the identity theorem, this would require that H and X are constant on $\overline{\mathrm{B}_1}$, which contradicts the weak monotonicity of $X|_{\partial \mathrm{B}_1}$. This contradiction shows that $X|_{\partial \mathrm{B}_1}$ is an homeomorphism of $\partial \mathrm{B}_1$ onto Γ, and the proof of the theorem is complete. □

Summarizing the outcome of Theorem 3.4, Proposition 3.5, Theorem 3.12, and Theorem 3.13, we finally arrive at the following statement, which is formulated without reference to the class $\mathscr{C}(\Gamma)$.

Main Theorem 3.14 (**solvability of the two-dimensional Plateau problem**)**.** *Consider the unit disk B_1 in $\mathbb{R}^2$ and a rectifiable oriented Jordan curve Γ in $\mathbb{R}^{2+N}$. Then there exists an $X \in \mathrm{C}^\infty(\mathrm{B}_1, \mathbb{R}^{2+N}) \cap \mathrm{C}^0\big(\overline{\mathrm{B}_1}, \mathbb{R}^{2+N}\big)$ with the following two properties:*

- *X is a conformally parametrized (possibly) branched minimal surface, that is*

$$\Delta X \equiv 0\,, \qquad |\partial_1 X| = |\partial_2 X|\,, \qquad \textit{and} \qquad \partial_1 X \cdot \partial_2 X \equiv 0 \qquad \textit{on } \mathrm{B}_1\,,$$

- *$X|_{\partial \mathrm{B}_1}$ is an orienting homeomorphism for Γ, in particular $X(\partial \mathrm{B}_1) = \Gamma$.* □

Remarks.

(1) *The conclusion of Main Theorem 3.14 remains true even for arbitrary (not necessarily rectifiable) Jordan curves Γ — in spite of the possibility that $\mathscr{C}(\Gamma)$ may then be empty. This refined solvability result essentially goes back to the classical work of J. Douglas [22] and is detailed in* [18, Section 4.12].

(2) *The usage of the Riemann mapping and Carathéodory theorems in the proof of Theorem 3.12 can be bypassed by a more elaborate choice of variations. Indeed, one can even reverse the line of argument and rely on the solvability of Plateau's problem in order to establish the theorems of Riemann and Carathéodory (on bounded domains with Jordan boundary); see* [18, Sections 4.5, 4.11].

(3) *By Main Theorem 3.14, Plateau's problem can generally be solved by a minimal surface (which is additionally a Dirichlet minimizer), but it is not yet clear whether even a solution by a surface of minimal area is possible. In the next section we turn to the solvability problem in this stronger sense, that is to the least area Plateau problem.*

3.4 Solution of the least area Plateau problem

Before coming two the main results of this section we record two basic lemmas.

The first lemma formally affirms the geometrically very plausible idea that the value $\mathrm{A}_\Omega[Y]$ of the area functional remains invariant under *arbitrary* reparametrizations of the surface $Y(\Omega)$. We emphasize that this invariance property is stronger than the conformal invariance of the Dirichlet integral, which allows only for *biholomorphic* reparametrizations.

Lemma 3.15 (parameter invariance of A). *For every $Y\in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^N)$, every open subset Ω of $\mathbb{R}^2$, and every C^1 diffeomorphism τ of Ω onto B_1, there holds*

$$\mathrm{A}_\Omega[Y\circ\tau]=\mathrm{A}[Y]\,.$$

Proof. A calculation with the chain rule and change of variables shows

$$\begin{aligned}
\mathrm{A}_\Omega[Y\circ\tau]&=\int_\Omega\sqrt{\det(\mathrm{D}(Y\circ\tau)^*\mathrm{D}(Y\circ\tau))}\,\mathrm{d}x\\
&=\int_\Omega\sqrt{\det(\mathrm{D}\tau^*(\mathrm{D}Y\circ\tau)^*(\mathrm{D}Y\circ\tau)\mathrm{D}\tau)}\,\mathrm{d}x\\
&=\int_\Omega\sqrt{\det((\mathrm{D}Y\circ\tau)^*(\mathrm{D}Y\circ\tau))}\,|\det\mathrm{D}\tau|\,\mathrm{d}x\\
&=\int_{\mathrm{B}_1}\sqrt{\det(\mathrm{D}Y^*\mathrm{D}Y)}\,\mathrm{d}x=\mathrm{A}[Y]\,.
\end{aligned}$$

□

The second lemma concerns the connection between A and Dir.

Lemma 3.16. *For every open subset Ω of $\mathbb{R}^2$ and $Y\in\mathrm{W}^{1,2}(\Omega,\mathbb{R}^{2+N})$, we have*

$$\mathrm{A}_\Omega[Y]\le\mathrm{Dir}_\Omega[Y]\,,$$

with equality if and only if Y is conformal.

Proof. By definition of A and Dir and by Young's inequality, we have

$$\begin{aligned}\mathrm{A}_\Omega[Y] &= \int_\Omega \sqrt{|\partial_1 Y|^2|\partial_2 Y|^2-(\partial_1 Y\cdot\partial_2 Y)^2}\,\mathrm{d}x\\ &\le \int_\Omega |\partial_1 Y||\partial_2 Y|\,\mathrm{d}x \le \int_\Omega \tfrac12\big(|\partial_1 Y|^2+|\partial_2 Y|^2\big)\,\mathrm{d}x = \mathrm{Dir}_\Omega[Y]\,.\end{aligned}$$

Here, the first inequality is an equality if and only if $\partial_1 Y\cdot\partial_2 Y\equiv 0$ holds on Ω, and the second inequality is an equality if and only if $|\partial_1 Y|$ equals $|\partial_2 Y|$ on Ω. All in all, we thus have $\mathrm{A}_\Omega[Y]=\mathrm{Dir}_\Omega[Y]$ if and only if Y is conformal. □

In particular, whenever $\mathscr{C}(\Gamma)$ is non-empty, Lemma 3.16 yields the inequality

$$\inf_{\mathscr{C}(\Gamma)} \mathrm{A} \le \inf_{\mathscr{C}(\Gamma)} \mathrm{Dir}\,, \tag{3.7}$$

and from Theorem 3.4 we know that the right-hand infimum is, in fact, a minimum. In the sequel, we additionally rule out the existence of non-conformal mappings $Y\in\mathscr{C}(\Gamma)$ with $\mathrm{A}[Y]<\inf_{\mathscr{C}(\Gamma)}\mathrm{Dir}$. Once this is achieved, we infer that equality holds in (3.7) and that also the left-hand infimum is attained. A classical approach to these assertions is based on the following result of C.B. Morrey [47].

Theorem 3.17 (**Morrey's lemma on ε-conformal mappings**). *Consider $Y\in\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^\ell\big)$. Then, for every $\varepsilon>0$, there exists an homeomorphism τ_ε of $\overline{\mathrm{B}_1}$ onto itself such that there hold $\tau_\varepsilon\in\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^2)$, $Y\circ\tau_\varepsilon\in\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^\ell\big)$, and*

$$\mathrm{Dir}[Y\circ\tau_\varepsilon]\le\mathrm{A}[Y]+\varepsilon\,.$$

Remarks.

(1) *If we could find a change of coordinates τ_0 with $\mathrm{Dir}[Y\circ\tau_0]\le\mathrm{A}[Y]$, then Lemmas 3.15 and 3.16 would imply that $Y\circ\tau_0$ were conformal. This fact serves as a motivation to call the reparametrized surfaces $Y\circ\tau_\varepsilon$ in Theorem 3.17 almost-conformal or ε-conformal mappings (even though the above-mentioned τ_0 need not exist under the present assumptions on Y).*

(2) *In particular, the mappings τ_ε of Theorem 3.17 map $\partial\mathrm{B}_1$ homeomorphicly onto itself. Possibly replacing τ_ε with $x\mapsto\tau_\varepsilon(x_1,-x_2)$, one can additionally achieve that these homeomorphisms $\tau_\varepsilon|_{\partial\mathrm{B}_1}$ are orientation-preserving.*

With Morrey's lemma at hand, one can quickly solve the least area Plateau problem.

Corollary 3.18 (**solvability of the least area Plateau problem**). *Consider an oriented Jordan curve Γ in $\mathbb{R}^{2+N}$ with $\mathscr{C}(\Gamma)\ne\emptyset$. Then, setting $\overline{\mathscr{C}}(\Gamma):=\mathscr{C}(\Gamma)\cap\mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^{2+N}\big)$, one has*

$$\inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{A}=\inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{Dir}=\inf_{\mathscr{C}(\Gamma)}\mathrm{Dir}\,,$$

in fact all these infima are minima, and in particular A *has a minimum in $\overline{\mathscr{C}}(\Gamma)$.*

Proof of Corollary 3.18. The equality $\inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{A}=\inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{Dir}$ follows from Lemma 3.16 and Theorem 3.17. Moreover, the equality $\inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{Dir}=\inf_{\mathscr{C}(\Gamma)}\mathrm{Dir}$ and the fact that both Dir-infima are minima result from Theorem 3.4, which yields a minimizer $X \in \overline{\mathscr{C}}(\Gamma)$ of Dir in $\mathscr{C}(\Gamma)$. Finally, Theorem 3.12 shows that X is conformal, and via Lemma 3.16 we then arrive at

$$\mathrm{A}[X] = \mathrm{Dir}[X] = \inf_{\mathscr{C}(\Gamma)} \mathrm{Dir} = \inf_{\overline{\mathscr{C}}(\Gamma)} \mathrm{A}\,,$$

so that X minimizes A in $\overline{\mathscr{C}}(\Gamma)$. □

In order to complete the solution of the least area problem, we still miss the proof of Theorem 3.17, and thus we next provide a short and very rough sketch of proof for this result.

Sketch of proof for Theorem 3.17. One writes

$$\mathrm{D}Y^*\mathrm{D}Y = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where $E, G \in \mathrm{L}^1(\mathrm{B}_1)$ are non-negative and $F \in \mathrm{L}^1(\mathrm{B}_1)$ satisfies $F^2 \le EG$. For arbitrarily fixed $\delta > 0$, one further sets

$$\Lambda := \begin{pmatrix} G+\delta E & F \\ F & E+\delta G \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ b & c \end{pmatrix} := \begin{cases} \frac{\Lambda}{\sqrt{\det\Lambda}} & \text{on } \mathrm{B}_1 \cap \{\Lambda \ne 0\} \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{on } (\mathbb{R}^2 \setminus \mathrm{B}_1) \cup \{\Lambda = 0\} \end{cases}.$$

Then it is not difficult to see that $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ has determinant 1 and is uniformly bounded and uniformly elliptic on $\mathbb{R}^2$ (with constants which depend only on δ).

Now the proof is based on the investigation of the first-order PDE system with L^∞ coefficients

$$\nabla T_2 = \begin{pmatrix} b & -c \\ a & -b \end{pmatrix} \nabla T_1 \qquad \text{for } T\colon \mathbb{R}^2 \to \mathbb{R}^2\,. \tag{3.8}$$

Indeed, one needs to show that (3.8) has a solution T which is $\mathrm{W}^{1,2}$ and homeomorphic on a neighborhood of B_1, with $\mathrm{W}^{1,2}$ inverse σ which solves the closely related system

$$\partial_1\sigma = \left[\begin{pmatrix} -b & -a \\ c & b \end{pmatrix} \circ T^{-1}\right] \partial_2\sigma\,.$$

The construction of T and σ can be achieved either by specializing various advanced PDE results to the present situation or more 'by hands' as in Morrey's original work [47]. So, the existence of $\mathrm{W}^{1,2}$ solutions T, for instance, follows from L^2 theory for the second-order PDE

$$\partial_1(a\partial_1 u - b\partial_2 u) - \partial_2(b\partial_1 u - c\partial_2 u) \equiv 0 \qquad \text{for } u\colon \mathbb{R}^2 \to \mathbb{R} \tag{3.9}$$

with L^∞ coefficients, since, for every solution u of (3.9), there exists a T with $T_1 = u$ and $\nabla T_2 = (b\partial_1 u - c\partial_2 u, a\partial_1 u - b\partial_2 u)$, and this T solves (3.8). Continuity of u and T then follows, for instance, from a well-known regularity theorem of E. De Giorgi and J. Nash. The assertion that T is homeomorphic, finally, can be deduced by passing to complex variables and the Beltrami equation

$$\frac{\partial T}{\partial \overline{z}} = \mu \frac{\partial T}{\partial z} \qquad \text{for } T\colon \mathbb{C} \to \mathbb{C}\,. \tag{3.10}$$

This equation, with the complex coefficient $\mu = \frac{1+b^2-c^2+2\mathrm{i}bc}{b^2+(c+1)^2}$, $|\mu| < 1$ is equivalent with (3.8), but in connection with (3.10) the existence of homeomorphic solutions with suitably prescribed boundary values is quite well-known. Concerning the finding of T, we now omit all further details which go beyond the brief and rough explanations above, and we turn to the conclusion of the proof.

Indeed, once T and thus σ are at hand, one uses the Riemann and Carathéodory theorems to find a biholomorphic $h\colon \mathrm{B}_1 \to \Omega := T(\mathrm{B}_1)$ which is homeomorphic from $\overline{\mathrm{B}_1}$ to $\overline{\Omega}$. Via conformal invariance and a suitable transformation rule, one the obtains

$$\mathrm{Dir}[Y\circ\sigma\circ h] = \frac{1}{2}\int_\Omega |\mathrm{D}(Y\circ\sigma)|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega |(\mathrm{D}Y\circ\sigma)\mathrm{D}\sigma|^2\,\mathrm{d}x = \frac{1}{2}\int_{\mathrm{B}_1} \frac{|\mathrm{D}Y(\mathrm{D}\sigma\circ T)|^2}{|\det(\mathrm{D}\sigma\circ T)|}\,\mathrm{d}x\,.$$

The integrand of the last integral is a function of $(\partial_1\sigma\circ T)$, $(\partial_2\sigma\circ T)$, $E=|\partial_1 Y|^2$, $F=\partial_1 Y\cdot\partial_2 Y$, and $G=|\partial_2 Y|^2$. When one uses the system for σ and the choice of $\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right)$, the dependence on $(\partial_1\sigma\circ T)$ can be eliminated. After a lengthy computation, also $\partial_2\sigma\circ T$ drops out, and it turns out that the integrand is in fact bounded by $2\sqrt{EG{-}F^2}+o(\delta)$. Therefore, one can finally conclude

$$\mathrm{Dir}[Y\circ\sigma\circ h] \le \int_{\mathrm{B}_1}\sqrt{EG{-}F^2}\,\mathrm{d}x + o(\delta) = \int_{\mathrm{B}_1}\sqrt{\det(\mathrm{D}Y^*\mathrm{D}Y)}\,\mathrm{d}x = \mathrm{A}[Y]+o(\delta)\,,$$

and thus, for every $\varepsilon>0$, one can choose τ_ε as the mapping $\sigma\circ h$ which corresponds to a suitably small $\delta>0$. □

Instead of elaborating on details of the preceding argument, we **now present another alternative approach to the least area Plateau problem** and a quite complete proof for its solvability which stems from a more recent work of Hildebrandt–von der Mosel [38]; compare also [37]. Indeed, this second approach remains much more elementary and also yields a slightly refined conclusion inasmuch as it allows to minimize A even in $\mathscr{C}(\Gamma)$ instead of $\overline{\mathscr{C}}(\Gamma)$. The precise statement follows.

Theorem 3.19 (**solvability of the least area Plateau problem; once more**). *Consider an oriented Jordan curve Γ in $\mathbb{R}^{2+N}$ with $\mathscr{C}(\Gamma)\neq\emptyset$. Then, setting $\overline{\mathscr{C}}(\Gamma) := \mathscr{C}(\Gamma)\cap \mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^{2+N}\big)$, one has*

$$\inf_{\mathscr{C}(\Gamma)}\mathrm{A} = \inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{A} = \inf_{\overline{\mathscr{C}}(\Gamma)}\mathrm{Dir} = \inf_{\mathscr{C}(\Gamma)}\mathrm{Dir}\,,$$

in fact all these infima are minima, and in particular A *has a conformal minimum in $\mathscr{C}(\Gamma)$.*

Before coming to the proof of Theorem 3.19, we record a corollary on the relation between minimizers of A and Dir.

Corollary 3.20. *Consider an oriented Jordan curve Γ in $\mathbb{R}^{2+N}$ with $\mathscr{C}(\Gamma)\neq\emptyset$. Then, for $X\in\mathscr{C}(\Gamma)$, there holds*

$$X \text{ is conformal and minimizes } \mathrm{A} \text{ in } \mathscr{C}(\Gamma) \iff X \text{ minimizes } \mathrm{Dir} \text{ in } \mathscr{C}(\Gamma)\,.$$

Proof of Corollary 3.20. If X is a conformal minimizer of A in $\mathscr{C}(\Gamma)$, then via Lemma 3.16 and Theorem 3.19 we get

$$\mathrm{Dir}[X] = \mathrm{A}[X] = \inf_{\mathscr{C}(\Gamma)}\mathrm{A} = \inf_{\mathscr{C}(\Gamma)}\mathrm{Dir}\,,$$

so that X also minimizes Dir in $\mathscr{C}(\Gamma)$.

If X minimizes Dir in $\mathscr{C}(\Gamma)$, then via Lemma 3.16 and Theorem 3.19 we get

$$\mathrm{A}[X] \le \mathrm{Dir}[X] = \inf_{\mathscr{C}(\Gamma)}\mathrm{Dir} = \inf_{\mathscr{C}(\Gamma)}\mathrm{A}\,,$$

so that X also minimizes A in $\mathscr{C}(\Gamma)$. Moreover, we infer $\mathrm{A}[X] = \mathrm{Dir}[X]$, and Lemma 3.16 implies that X is conformal. $\square$

The main idea in the proof of Theorem 3.19 is to work with **convex combinations of A and Dir**, given by

$$\mathrm{A}^s[Y] := (1-s)\,\mathrm{A}[Y] + s\,\mathrm{Dir}[Y] \qquad \text{for } Y \in \mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N}) \text{ and } s \in [0,1]\,.$$

The starting point for the proof of the theorem are then the following semicontinuity properties.

Lemma 3.21 (Weak lower semicontinuity of A and A^s). *If Y^k converges to Y weakly in $\mathrm{W}^{1,2}(\mathrm{B}_1, \mathbb{R}^{2+N})$, then we have*

$$\mathrm{A}[Y] \le \liminf_{k\to\infty} \mathrm{A}[Y^k] \qquad \textit{and} \qquad \mathrm{A}^s[Y] \le \liminf_{k\to\infty} \mathrm{A}^s[Y^k] \textit{ for all } s \in [0,1]\,.$$

Proof of Lemma 3.21. We first recall that the analogous semicontinuity property holds for Dir (since the norm is weakly lower semicontinuous in every normed space). Thus, once the semicontinuity property is available for A, it immediately follows for the convex combinations A^s, and it suffices to deal with A itself in the sequel.

We now define a function with values in the skew-symmetric $((2+N)\times(2+N))$-matrices by setting

$$\Lambda^2(\mathrm{D}Y) := (\partial_1 Y_i\, \partial_2 Y_j - \partial_1 Y_j\, \partial_2 Y_i)_{i,j=1,2\ldots,2+N}\,.$$

In other words, the entries of $\Lambda^2(\mathrm{D}Y)$ are the (2×2)-minors of $\mathrm{D}Y$ (and specifically for $N{=}1$ the non-trivial entries correspond to the components of the vector product $\partial_1 Y \times \partial_2 Y$). We compute

$$\begin{aligned}
|\Lambda^2(\mathrm{D}Y)|^2 &= \sum_{i,j=1}^{2+N} (\partial_1 Y_i\, \partial_2 Y_j - \partial_1 Y_j\, \partial_2 Y_i)^2 \\
&= \sum_{i=1}^{2+N} (\partial_1 Y_i)^2 \sum_{j=1}^{2+N} (\partial_2 Y_j)^2 - 2 \sum_{i=1}^{2+N} (\partial_1 Y_i\, \partial_2 Y_i) \sum_{j=1}^{2+N} (\partial_2 Y_j\, \partial_1 Y_j) + \sum_{j=1}^{2+N} (\partial_1 Y_j)^2 \sum_{i=1}^{2+N} (\partial_2 Y_i)^2 \\
&= 2\big[|\partial_1 Y|^2 |\partial_2 Y|^2 - (\partial_1 Y \cdot \partial_2 Y)^2\big]\,,
\end{aligned}$$

and thus we see that $|\Lambda^2(\mathrm{D}Y)|$ is connected to the area functional by

$$\int_{\mathrm{B}_1} |\Lambda^2(\mathrm{D}Y)|\,\mathrm{d}x = \sqrt{2}\,\mathrm{A}[Y]\,. \tag{3.11}$$

Next we aim at verifying a weak continuity property of the operation $\mathrm{D}Y \mapsto \Lambda^2(\mathrm{D}Y)$. To this end, we first assume $Y \in \mathrm{C}^2(\mathrm{B}_1, \mathbb{R}^{2+N})$ and consider an arbitrary $\Phi \in \mathrm{C}^1_{\mathrm{cpt}}(\mathrm{B}_1, \mathbb{R}^{(2+N)\times(2+N)})$. Integration by parts in the first variable shows

$$\begin{aligned}
\int_{\mathrm{B}_1} \Phi \cdot \Lambda^2(\mathrm{D}Y)\,\mathrm{d}x &= \sum_{i,j=1}^{2+N} \int_{\mathrm{B}_1} \Phi_{ij}(-\partial_1 Y_j\, \partial_2 Y_i + \partial_1 Y_i\, \partial_2 Y_j)\,\mathrm{d}x \\
&= \sum_{i,j=1}^{2+N} \int_{\mathrm{B}_1} \big[\partial_1 \Phi_{ij}(Y_j\, \partial_2 Y_i - Y_i\, \partial_2 Y_j) + \Phi_{ij}(Y_j\, \partial_1\partial_2 Y_i - Y_i\, \partial_1\partial_2 Y_j)\big]\,\mathrm{d}x\,,
\end{aligned}$$

and similarly integration by parts in the second variable yields

$$\int_{\mathrm{B}_1} \Phi \cdot \Lambda^2(\mathrm{D}Y)\,\mathrm{d}x = \sum_{i,j=1}^{2+N} \int_{\mathrm{B}_1} \left[\partial_2\Phi_{ij}(-Y_j\,\partial_1 Y_i + Y_i\,\partial_1 Y_j) + \Phi_{ij}(Y_i\,\partial_1\partial_2 Y_j - Y_j\,\partial_1\partial_2 Y_i)\right] \mathrm{d}x\,.$$

Adding up the last two formulas and dividing by 2, we arrive at

$$\int_{\mathrm{B}_1} \Phi\cdot\Lambda^2(\mathrm{D}Y)\,\mathrm{d}x = \frac{1}{2}\sum_{i,j=1}^{2+N}\int_{\mathrm{B}_1} \left[\partial_1\Phi_{ij}(Y_j\,\partial_2 Y_i - Y_i\,\partial_2 Y_j) + \partial_2\Phi_{ij}(-Y_j\,\partial_1 Y_i + Y_i\,\partial_1 Y_j)\right] \mathrm{d}x\,, \tag{3.12}$$

and by the Meyers-Serrin approximation theorem, the resulting formula (3.12) remains valid for every $Y \in \mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N})$. Now we come to the weakly convergent sequence of the lemma. Clearly, $\mathrm{D}Y^k$ converges to $\mathrm{D}Y$ weakly in $\mathrm{L}^2(\mathrm{B}_1,\mathbb{R}^{(2+N)\times 2})$, and by Rellich's theorem Y^k converges to Y strongly in $\mathrm{L}^2(\mathrm{B}_1,\mathbb{R}^{2+N})$. Using this and (3.12) for both Y^k and Y, one straightforwardly verifies

$$\lim_{k\to\infty}\int_{\mathrm{B}_1} \Phi\cdot\Lambda^2(\mathrm{D}Y^k)\,\mathrm{d}x = \int_{\mathrm{B}_1}\Phi\cdot\Lambda^2(\mathrm{D}Y)\,\mathrm{d}x \qquad \text{for all } \Phi\in \mathrm{C}^1_{\mathrm{cpt}}(\mathrm{B}_1,\mathbb{R}^{(2+N)\times(2+N)})\,. \tag{3.13}$$

In view of the bound $|\Lambda^2(\mathrm{D}Y^k)| \le \mathrm{const}(N)|\mathrm{D}Y^k|^2$, the L^2-boundedness of the weakly convergent sequence $(\mathrm{D}Y^k)_{k\in\mathbb{N}}$ implies that the sequence $(\Lambda^2(\mathrm{D}Y^k)\mathcal{L}^2)_{k\in\mathbb{N}}$ of weighted Lebesgue measures is bounded in the space $\mathrm{RM}(\mathrm{B}_1,\mathbb{R}^{(2+N)\times(2+N)})$ of $\mathbb{R}^{(2+N)\times(2+N)}$-valued finite Radon measures. It follows that every subsequence of $(\Lambda^2(\mathrm{D}Y^k)\mathcal{L}^2)_{k\in\mathbb{N}}$ contains yet another subsequence which weakly-$*$ converges in $\mathrm{RM}(\mathrm{B}_1,\mathbb{R}^{(2+N)\times(2+N)})$, and then (3.13) allows to identify the limits as $\Lambda^2(\mathrm{D}Y)\mathcal{L}^2$, so that actually the whole sequence $(\Lambda^2(\mathrm{D}Y^k)\mathcal{L}^2)_{k\in\mathbb{N}}$ converges to $\Lambda^2(\mathrm{D}Y)\mathcal{L}^2$ weakly-$*$ in $\mathrm{RM}(\mathrm{B}_1,\mathbb{R}^{(2+N)\times(2+N)})$. Thanks to (3.11), the latter convergence, and the weak-$*$ lower semicontinuity of the total variation norm $\nu\mapsto|\nu|(\mathrm{B}_1)$, we then end up with

$$\begin{aligned}\mathrm{A}[Y] &= \frac{1}{\sqrt{2}}\int_{\mathrm{B}_1}|\Lambda^2(\mathrm{D}Y)|\,\mathrm{d}x = \frac{1}{\sqrt{2}}\big|\Lambda^2(\mathrm{D}Y)\mathcal{L}^2\big|(\mathrm{B}_1)\\ &\le \frac{1}{\sqrt{2}}\liminf_{k\to\infty}\big|\Lambda^2(\mathrm{D}Y^k)\mathcal{L}^2\big|(\mathrm{B}_1) = \liminf_{k\to\infty}\mathrm{A}[Y^k]\,.\end{aligned}$$

This completes the proof of the lemma. □

With Lemma 3.21 at hand, we next demonstrate that the functionals A^s with $s>0$ can essentially take over the role of the Dirichlet integral in most arguments of Sections 3.2 and 3.3. This is detailed in the following proof.

Proof of Theorem 3.19. We proceed in four steps.

Step 1: For every $s\in\,]0,1]$, there exists a minimizer X^s of A^s in $\mathscr{C}(\Gamma)$.

By Lemmas 3.6 and 3.15, both Dir and A are conformally invariant, hence the same is true for A^s, and by the arguments already given in the proof of Lemma 3.7 we can reduce to the class $\mathscr{C}^*(\Gamma)$ (which is defined by imposing the three-point condition for suitably fixed $x_1,x_2,x_3\in\partial\mathrm{B}_1$ and $y_1,y_2,y_3\in\Gamma$). Thus, we now consider a minimizing sequence $(X_k)_{k\in\mathbb{N}}$ for A^s in $\mathscr{C}^*(\Gamma)$. Then we have

$$\sup_{k\in\mathbb{N}}\mathrm{Dir}[X_k] \le \frac{1}{s}\sup_{k\in\mathbb{N}}\mathrm{A}^s[X_k] < \infty\,,$$

and via the Poincaré inequality with traces we deduce that $(X_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N})$. Consequently, there is a subsequence such that X_{k_ℓ} converges to X weakly in $\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^{2+N})$. Weak convergence is preserved by the continuous linear trace operator, and thus $X_{k_\ell}|_{\partial\mathrm{B}_1}$ converges to $X|_{\partial\mathrm{B}_1}$ weakly in $\mathrm{L}^2(\partial\mathrm{B}_1,\mathbb{R}^{2+N};\mathcal{H}^1)$. Relying on a version[7] of the Courant-Lebesgue lemma, we can closely follow the reasoning in the proof of Theorem 3.4 in order to show equi-continuity of the traces $X_k|_{\partial\mathrm{B}_1}$. By the Arzelà-Ascoli theorem, we then infer that $X_{k_\ell}|_{\partial\mathrm{B}_1}$ converges to $X|_{\partial\mathrm{B}_1}$ also uniformly on $\partial\mathrm{B}_1$, and this suffices to conclude $X\in\mathscr{C}^*(\Gamma)$. By Lemma 3.21, we moreover have

$$\mathrm{A}^s[X]\le\liminf_{\ell\to\infty}\mathrm{A}^s[X_{k_\ell}]=\inf_{\mathscr{C}^*(\Gamma)}\mathrm{A}^s\,,$$

and thus X minimizes A^s in $\mathscr{C}^*(\Gamma)$. In view of the reduction step at the beginning, this establishes the claim of Step 1.

Step 2: For every $s\in\,]0,1]$, the minimizer X^s is conformal.

We proceed as in the proof of Theorem 3.12. For arbitrary $\Phi\in\mathrm{C}^1\big(\overline{\mathrm{B}_1,\mathbb{R}^2}\big)$, we set $\Phi_t(x):=x+t\Phi(x)$ and $\Omega_t:=\Phi_t(\mathrm{B}_1)$. Then, for $|t|\ll1$, the mapping Φ_t is diffeomorphic with inverse $\Sigma_t\colon\Omega_t\to\mathrm{B}_1$, and Ω_t is a simply connected domain in $\mathbb{C}$. By Riemann's theorem we can thus find a biholomorphic mapping h_t of B_1 onto Ω_t, and we get $X^s\circ\Sigma_t\circ h_t\in\mathscr{C}(\Gamma)$ for $|t|\ll1$. Similarly to the proof of Theorem 3.12, the first-order criterion for the minimality of X^s now yields

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{A}^s[X^s\circ\Sigma_t\circ h_t]=(1{-}s)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{A}[X^s\circ\Sigma_t\circ h_t]+s\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Dir}[X^s\circ\Sigma_t\circ h_t]\,.$$

However, the A-term on the right-hand side does always vanish, since $\mathrm{A}[X^s\circ\Sigma_t\circ h_t]=\mathrm{A}[X^s]$ holds by Lemma 3.15. Therefore, we are left with

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Dir}[X^s\circ\Sigma_t\circ h_t]\qquad\text{for all }\Phi\in\mathrm{C}^1\big(\overline{\mathrm{B}_1},\mathbb{R}^2\big)\,,$$

and it has already been demonstrated in the proof of Theorem 3.12 that this information implies the conformality of X^s.

Step 3: X^1 minimizes A (and Dir $=\mathrm{A}^1$) in $\mathscr{C}(\Gamma)$, and we have

$$\inf_{\mathscr{C}(\Gamma)}\mathbf{A}=\inf_{\mathscr{C}(\Gamma)}\mathbf{Dir}\,.$$

For $s\in\,]0,1]$, we first use Lemma 3.16, then the minimality of X^1, then the equality $\mathrm{Dir}[X^s]=\mathrm{A}[X^s]$, which results from the conformality of X^s and Lemma 3.16, and finally the minimality of X^s. In this way, we deduce

$$\mathrm{A}[X^1]\le\mathrm{Dir}[X^1]=\mathrm{A}^1[X^1]\le\mathrm{A}^1[X^s]=\mathrm{A}^s[X^s]\le\mathrm{A}^s[Y]\qquad\text{for all }Y\in\mathscr{C}(\Gamma)\,.$$

[7] In fact, at this point we need a version of the Courant-Lebesgue lemma for functions $Y\in\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell)$ with continuous trace on $\partial\mathrm{B}_1$. To extend Lemma 3.8 to this generality, one can construct $\mathrm{W}^{1,2}$-convergent smooth approximations of Y with fixed trace as follows. For $Y\in\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell)$ with continuous trace φ, there exists an $X\in\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^\infty(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^\ell\big)$ with the same trace φ; for instance, X can be found by Dirichlet minimization as discussed in the proof of Theorem 3.4. As pointed out in the addendum on traces, we have $Y{-}X\in\mathrm{W}^{1,2}_0(\mathrm{B}_1,\mathbb{R}^\ell)$, thus $Y{-}X$ can be approximated by smooth functions with compact support, and consequently Y can be suitably approximated by functions in $\mathrm{W}^{1,2}(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^\infty(\mathrm{B}_1,\mathbb{R}^\ell)\cap\mathrm{C}^0\big(\overline{\mathrm{B}_1},\mathbb{R}^\ell\big)$ with fixed trace φ.

Sending $s \searrow 0$, we infer

$$\mathrm{A}[X^1] \le \mathrm{A}[Y] \qquad \text{for all } Y \in \mathscr{C}(\Gamma)\,,$$

and thus X^1 minimizes A in $\mathscr{C}(\Gamma)$. Using this together with the conformality of X^1 and Lemma 3.16, we get

$$\inf_{\mathscr{C}(\Gamma)} \mathrm{Dir} \le \mathrm{Dir}[X^1] = \mathrm{A}[X^1] = \inf_{\mathscr{C}(\Gamma)} \mathrm{A}\,,$$

and in view of (3.7), all claims of Step 3 are verified.

Step 4: Conclusion of the proof.
Since X^1 minimizes both A and Dir and since Theorem 3.4 gives $X^1 \in \overline{\mathscr{C}}(\Gamma)$, it follows that all four infima in Theorem 3.19 coincide and are in fact minima. This ends the proof. □

3.5 A brief view towards uniqueness and (boundary) regularity

In general, one may *not* hope that the Plateau problem for a given boundary curve has only one solution. Indeed, classical examples show that non-uniqueness may occur already in codimension $N = 1$ (and thus in the ambient space $\mathbb{R}^3$), and we refer to the beginning [18, Section 4] where such examples are discussed and depicted. Nevertheless, for *specific* boundary curves Γ one may still obtain uniqueness results, and this actually happens in the next statement.

Theorem 3.22 (**Radó's uniqueness theorem for the codimension-one case**)**.** *Consider an oriented*[8] *Jordan curve* Γ *of the form*

$$\Gamma = \mathrm{Graph}\,\varphi$$

in $\mathbb{R}^3$*, where* $\varphi \in \mathrm{C}^0(\partial\Omega)$ *is defined on the boundary of a* convex*, open, and bounded set* Ω *in* $\mathbb{R}^2$*. Then the solution of Plateau's problem for the curve* Γ *in the sense of Theorem 3.14 is unique up to biholomorphic change of coordinates. Indeed, each solution* X *takes the form* $X = \mathrm{G}_u \circ \tau$ *where* $\tau \colon \mathrm{B}_1 \to \Omega$ *is biholomorphic and where* $\mathrm{G}_u(x) := (x, u(x))$ *is the graph mapping of the unique solution* $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0\big(\overline{\Omega}\big)$ *to the Dirichlet problem for the minimal surface equation*

$$\begin{aligned} \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} &\equiv 0 \qquad \text{on } \Omega\,, \\ u &= \varphi \qquad \text{on } \partial\Omega\,. \end{aligned}$$

A proof of Theorem 3.22 can be found in [18, Section 4.9], for instance.

Remarks.

(1) *Clearly, uniqueness also holds, if* Γ *takes the assumed form only after translation and rotation in* $\mathbb{R}^3$*.*

(2) ***The connection between parametric and non-parametric solutions extends to the higher-codimension case.*** *Indeed, for an oriented Jordan curve* $\Gamma = \mathrm{Graph}\,\varphi$ *in* $\mathbb{R}^{2+N}$ *with* $N \ge 2$*,* $\varphi \in \mathrm{C}^0(\partial\Omega)$*, and a convex, open, bounded* Ω *in* $\mathbb{R}^2$*, parametric solutions* X

[8] We understand that Γ is oriented by the composition of the graph mapping of φ and an orientation-preserving homeomorphism $\partial\mathrm{B}_1 \to \partial\Omega$.

still take the form $X = \mathrm{G}_u \circ \tau$ with a non-parametric solution u. But now u ***solves the Dirichlet problem for the minimal surface system****, and solutions of this system need not be unique. Thus, there is* ***no way to conclude uniqueness of parametric solutions*** *X. For more details we refer to the work of Lawson–Osserman [41].*

Turning to the regularity of the solutions X, we already know that the interior regularity $X \in \mathrm{C}^\infty(\mathrm{B}_1, \mathbb{R}^{2+N})$ (and even analyticity of X) is always at hand. A more difficult question is the one for boundary regularity, that is the question whether X smoothly extends to (a neighborhood of) $\overline{\mathrm{B}_1}$. A quite complete answer is provided by the following result.

Theorem 3.23 (**Hildebrandt's boundary regularity theorem**). *Suppose that $c\colon \partial\mathrm{B}_1 \to \mathbb{R}^{2+N}$ is a regular*[9] *$\mathrm{C}^{m,\alpha}$ curve in $\mathbb{R}^{2+N}$ with $m \in \mathbb{N}$ and $\alpha \in \,]0,1[$. Then every solution X of Plateau's problem for the curve $\Gamma := c(\partial\mathrm{B}_1)$ in the sense of Theorem 3.14 satisfies*

$$X \in \mathrm{C}^{m,\alpha}(\overline{\mathrm{B}_1}, \mathbb{R}^{2+N})\,.$$

For a proof of Theorem 3.23, we refer to [19, Section 2.3].

Remark. *The theorem was originally proved by S. Hildebrandt [36] only for $m \geq 4$. The extension to arbitrary values $m \geq 1$ has been achieved in subsequent work of Heinz-Tomi [35], J.C.C. Nitsche [48], and D. Kinderlehrer [40].*

Finally, we briefly discuss the question for the **geometric regularity** of solutions X to Plateau's problem. This is the question whether one can rule out branch points of X and thus ensure that X is an immersion at least. In particular, one is interested in excluding what is known in the literature as true branch points: points where $\mathrm{D}X$ vanishes not only due to the choice of an unfavorable parametrization X but where rather two geometrically different sheets of the surface $X(\mathrm{B}_1)$ are glued together. As mentioned in Section 2, the program of proving geometric regularity can indeed be carried on for solutions of the least area problem in codimension $N = 1$, and in this case all interior branch points (no matter whether true or false ones) are excluded by the results in [50, 32, 6, 7, 33]. However, these results extend neither to non-minimizing solutions in codimension $N = 1$ nor to minimizers in codimension $N \geq 2$: This is shown by the example of the Henneberg surface (already discussed in Section 1.4) and by the parametric surface $X\colon \mathrm{B}_1 \to \mathbb{C}^2 = \mathbb{R}^4$ with $X(z) = (z^2, z^3)$ (which is a minimal surface by Theorem 1.11 and which is even area-minimizing by a theorem of H. Federer [23]).

[9] Saying that c is regular indicates that $\frac{\mathrm{d}}{\mathrm{d}t}c(\mathrm{e}^{\mathrm{i}t}) \neq 0$ for all $t \in \mathbb{R}$.

Bibliography

Chapters 1 and 3 of these notes mostly follow [18, Chapters 1–4], while Chapter 2 is partially inspired by the presentations [46, 15]. Finally, Chapter 4.3 draws from [8, Chapter 3.4, 3.5] and [55, Chapter 1.7, 1.8], while Chapter 4.4 is loosely based on [17], [42, Chapters 17.4, 21–26], and [56, Chapters 4, 5].

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