Lectures Notes

Calculus of Variations

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Chapter 1

Introduction

The **Calculus of Variations** is the mathematical discipline which **studies extrema and critical points** of functions

$$\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$$

on an ∞-dimensional subset $\mathcal{A}$ of a (normed) **function space** $\mathcal{X}$. One usually calls $\mathcal{A}$ the **admissible (function) class** or the class of competitors, and $\mathcal{F}$ a **functional** (which is short for 'function of functions'). Since the discipline with classical origins in the 17th century has developed into a huge and very diverse field, for the purposes of this lecture we necessarily have to single out specific topics. Hence, as a matter of fact we mostly dispense with classical (field) theory, geometric variational problems, and various applications, but rather we focus on the fundamental case of a **first-order integral functional** or **first-order variational integral**

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,, w, \mathrm{D}w)\,\mathrm{d}x := \int_\Omega F(x, w(x), \mathrm{D}w(x))\,\mathrm{d}x$$
$$\text{for (weakly) differentiable } w\colon \Omega \to \mathbb{R}^N\,,$$

where the dimensions $n, N \in \mathbb{N}$ of domain and target, an open set $\Omega \subset \mathbb{R}^n$, and a suitable integrand $F\colon \Omega{\times}\mathbb{R}^N{\times}\mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$ are given data. We will mostly consider $\mathcal{F}$ on first-order Sobolev spaces and develop theory which covers general dimensions n.

Basic conventions. In these notes we use the conventions $\mathbb{N} := \{1,2,3,\ldots\}$, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, and $\overline{\mathbb{R}} := [-\infty,\infty]$.

Matrices. Given $n, N \in \mathbb{N}$ we write $\mathbb{R}^{N\times n}$ for the space of real matrices with N rows and n columns and $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^N)$ for the space of $\mathbb{R}$-linear maps $\mathbb{R}^n \to \mathbb{R}^N$, but we often identify (matrices and linear maps in) these spaces. By default we use on $\mathbb{R}^n$ the Euclidean inner product $\cdot$ and the Euclidean norm $|\cdot|$, and we carry this over to $\mathbb{R}^{N\times n} = \mathcal{L}(\mathbb{R}^n,\mathbb{R}^N)$ by identification with $\mathbb{R}^{Nn}$. The latter convention results in fact in the usage of the Hilbert-Schmidt inner product $z \cdot \xi = \mathrm{trace}(z^{\mathrm{t}}\xi) = \sum_{\ell=1}^N \sum_{i=1}^n z_{\ell i}\xi_{\ell i}$ for $z, \xi \in \mathbb{R}^{N\times n}$ and the Frobenius norm $|z| = \sqrt{\mathrm{trace}(z^{\mathrm{t}}z)} = \sqrt{\sum_{\ell=1}^N \sum_{i=1}^n z_{\ell i}^2}$ for $z \in \mathbb{R}^{N\times n}$.

Derivatives. For (weakly) differentiable $w\colon \Omega \to \mathbb{R}^N$ on open $\Omega \subset \mathbb{R}^n$, we generally write $\partial_i w$ with index $i \in \{1,2,\ldots,n\}$, $\partial^\alpha w$ with multi-index $\alpha \in \mathbb{N}_0^n$ of order $|\alpha| := \sum_{i=1}^n \alpha_i$, and $\partial_v w$ with direction vector $v \in \mathbb{R}^n$ for the partial and directional derivatives of w (which are all mappings $\Omega \to \mathbb{R}^N$). Moreover, we denote by $\mathrm{D}w\colon \Omega \to \mathbb{R}^{N\times n}$ the Jacobi matrix or total derivative of w, while the gradient $\nabla w\colon \Omega \to \mathbb{R}^n$ is used for scalar functions $w\colon \Omega \to \mathbb{R}$ only.

Integration with no specific measure indicated is understood as integration with respect to the Lebesgue measure.

Weak derivatives. For open $\Omega \subset \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, one calls $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ the weak ∂^α partial derivative of $w \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ on Ω if $\int_\Omega w \cdot \partial^\alpha\varphi\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega v \cdot \varphi\,\mathrm{d}x$ holds for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ (where the subscript $_{\mathrm{cpt}}$ stands for 'compact support' and requires the existence of a compact $K \subset \Omega$ with $\varphi \equiv 0$ on $\Omega \setminus K$). If such v happens to exist, it is uniquely determined as an $\mathrm{L}^1_{\mathrm{loc}}$ function. One then writes $\partial^\alpha w$ for v and says that $\partial^\alpha w$ exists weakly on Ω. As for classical derivatives, one understands $\partial_i = \partial^{\mathrm{e}_i}$ also for weak ones.

As we will later verify in large generality, the investigation of the integral $\mathcal{F}$ is closely connected, in case $N = 1$, to a scalar second-order PDE or, in case $N \ge 2$, to a **system of N second-order PDEs**. For the sake of illustration, we first make this connection precise in the prominent exemplary case of the **Dirichlet integral** or **Dirichlet energy** $\mathcal{E}_2$, which is given by

$$\mathcal{E}_2[w] := \tfrac{1}{2}\int_\Omega |\mathrm{D}w|^2 \,\mathrm{d}x \qquad \text{for } w \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$$

and corresponds to the choice of the integrand $F(x,y,z) := \frac{1}{2}|z|^2$ above. Clearly, $\mathcal{E}_2$ reaches its minimum on all of $\mathcal{X} = \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ exactly on the constant functions, but a more interesting principle is at hand if the admissible class $\mathcal{A}$ is taken as a **Dirichlet class**

$$\mathrm{W}^{1,2}_{u_0}(\Omega,\mathbb{R}^N) := u_0 + \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N) = \{u_0 + w : w \in \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)\}$$

in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ with boundary values prescribed by a fixed $u_0 \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$. The principle applies if a (candidate for a) minimizing u in a Dirichlet class — which can then clearly be written as $\mathrm{W}^{1,2}_u(\Omega,\mathbb{R}^N)$ — is already at hand and reads as follows:

Theorem (**Dirichlet principle**). *For open $\Omega \subset \mathbb{R}^N$ and $u \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$, we have:*

$$\begin{aligned} &u \textit{ minimizes } \mathcal{E}_2 \textit{ in } \mathrm{W}^{1,2}_u(\Omega,\mathbb{R}^N)\,, \textit{ that is, } \mathcal{E}_2[u] \le \mathcal{E}_2[w] \textit{ for all } w \in \mathrm{W}^{1,2}_u(\Omega,\mathbb{R}^N) \\ &\iff u \textit{ is weakly harmonic on } \Omega\,, \textit{ that is, } \int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi \,\mathrm{d}x = 0 \textit{ for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)\,. \end{aligned}$$

The weak harmonicity of u in this principle means in fact that u is a weak solution to the Laplace equation $\Delta u \equiv 0$ on Ω (which involves the Laplace operator $\Delta = \operatorname{div}\nabla = \sum_{i=1}^n \partial_i^2$ and decouples to N scalar Laplace equations for the components of u). It follows from the well-known Weyl lemma on harmonic functions that even weak solutions of this model elliptic equation are automatically analytic on Ω and thus are classical solutions. Therefore, we have:

Corollary. *If $u \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ minimizes $\mathcal{E}_2$ in $\mathrm{W}^{1,2}_u(\Omega,\mathbb{R}^N)$, then u is analytic.*

Next we turn to the proof of the principle, which will later be widely generalized.

Proof of the Dirichlet principle. From the definitions of $\mathcal{E}_2$ and $\mathrm{W}^{1,2}_u(\Omega,\mathbb{R}^N)$ together with the elementary equality $\frac{1}{2}|\mathrm{D}(u{+}t\varphi)|^2 = \frac{1}{2}|\mathrm{D}u|^2 + t\,\mathrm{D}u\cdot\mathrm{D}\varphi + t^2\frac{1}{2}|\mathrm{D}\varphi|^2$ we infer

$$\begin{aligned} &\mathcal{E}_2[u] \le \mathcal{E}_2[w] \quad \text{for all } w \in \mathrm{W}^{1,2}_u(\Omega,\mathbb{R}^N) \\ &\iff \mathcal{E}_2[u] \le \mathcal{E}_2[u{+}t\varphi] \quad \text{for all } t \in \mathbb{R} \text{ and all } \varphi \in \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N) \\ &\iff 0 \le t\int_\Omega \mathrm{D}u\cdot\mathrm{D}\varphi\,\mathrm{d}x + t^2\mathcal{E}_2[\varphi] \quad \text{for all } t \in \mathbb{R} \text{ and all } \varphi \in \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)\,. \end{aligned}$$

Sobolev spaces. For $m \in \mathbb{N}_0$, $p \in [1,\infty]$, one introduces the (localized) Sobolev space $\mathrm{W}^{m,p}_{(\mathrm{loc})}(\Omega,\mathbb{R}^N)$ as

$$\begin{aligned} \mathrm{W}^{m,p}(\Omega,\mathbb{R}^N) &:= \{w \in \mathrm{L}^p(\Omega,\mathbb{R}^N) : \partial^\alpha w \in \mathrm{L}^p(\Omega,\mathbb{R}^N) \text{ exists weakly for all } \alpha \in \mathbb{N}_0^n, |\alpha| \le m\}\,, \\ \mathrm{W}^{m,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) &:= \{w \in \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N) : \partial^\alpha w \in \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \text{ exists weakly for all } \alpha \in \mathbb{N}_0^n, |\alpha| \le m\}\,, \end{aligned}$$

where the former is a Banach space with norm $\|w\|_{m,p;\Omega} := \sum_{|\alpha|\le m}\|\partial^\alpha w\|_{p;\Omega}$, for instance. The subspace $\mathrm{W}^{m,p}_0(\Omega,\mathbb{R}^N)$ of functions with zero boundary values is defined, for $p < \infty$, as the closure of $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ in $\mathrm{W}^{m,p}(\Omega,\mathbb{R}^N)$ with its norm.

Here, the last line means that the quadratic polynomials in $t \in \mathbb{R}$ given there have a minimum at $t=0$. Since we have $\mathcal{E}_2[\varphi] \geq 0$, this happens precisely if the linear term vanishes (To see this, either take $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$ or take into account that the linear term dominates for $|t| \ll 1$.), and we can continue the preceding chain of equivalences by

$$\ldots \iff \int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)\,.$$

Since $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ is dense in $\mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$ (by definition) and $\varphi \mapsto \int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi \,\mathrm{d}x$ is continuous on $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ (by the Cauchy-Schwarz inequality $|\int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi \,\mathrm{d}x| \leq \|\mathrm{D}u\|_{2;\Omega} \|\mathrm{D}\varphi\|_{2;\Omega}$), we conclude

$$\ldots \iff \int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)\,,$$

and the proof is complete. □

Before entering more seriously into the theory of variational problems, we first provide three series of examples. The first two series provide model examples of variational integrals $\mathcal{F}$ (where sometimes we also *mention* the associated PDE without yet *proving* any relation however) and of admissible classes $\mathcal{A}$, respectively. In the third series we discuss some classical one-dimensional variational problems and analytical frameworks for their investigation.

Examples (of **variational integrals**). Consider $n, N \in \mathbb{N}$ and open $\Omega \subset \mathbb{R}^n$. Then the following are model examples of variational integrals.

(1) A first generalization of the Dirichlet integral $\mathcal{E}_2$ are **quadratic integrals**

$$\tfrac{1}{2} \int_\Omega A(\mathrm{D}w, \mathrm{D}w) \,\mathrm{d}x = \tfrac{1}{2} \sum_{\ell,m=1}^N \sum_{i,j=1}^n A^{\ell m}_{ij} \int_\Omega \partial_i w_\ell \, \partial_j w_m \,\mathrm{d}x \qquad \text{for } w \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$$

(integrand $F(x,y,z) = \frac{1}{2} A(z,z)$), where A is a bilinear form on $(N{\times}n)$-matrices and the numbers $A^{\ell m}_{ij} \in \mathbb{R}$ are the components of A. If A is symmetric or, in other words, $A^{\ell m}_{ij} = A^{m\ell}_{ji}$, this integral is connected with the linear PDE system

$$\sum_{\ell=1}^N \sum_{i,j=1}^n A^{\ell m}_{ij} \partial_j \partial_i u_\ell \equiv 0 \qquad \text{on } \Omega\,, \qquad\qquad \text{for } m = 1, 2, \ldots, N\,.$$

(1') A second generalization of $\mathcal{E}_2$, for simplicity discussed only in the scalar case $N=1$, are integrals of the type

$$\int_\Omega \bigg(\frac{1}{2} \sum_{i,j=1}^n a_{ij} \partial_i w \, \partial_j w - \tfrac{1}{2} c w^2 + f w \bigg) \,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,2}(\Omega)$$

(integrand $F(x,y,z) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) z_i z_j - \frac{1}{2} c(x) y^2 + f(x) y$), where $a_{ij}, c \in \mathrm{L}^\infty(\Omega)$ and $f \in \mathrm{L}^2(\Omega)$ are (coefficient) functions on Ω. In the symmetric case $a_{ij} = a_{ji}$ this integral is connected with the linear PDE of comparably general form

$$\sum_{i,j=1}^n \partial_j (a_{ij} \partial_i u) + cu = f \qquad \text{on } \Omega\,.$$

(2) Yet another generalization of $\mathcal{E}_2$ is the ***p*-Dirichlet integral** or ***p*-energy**

$$\mathcal{E}_p[w] := \tfrac{1}{p}\int_\Omega |\mathrm{D}w|^p \,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$$

(integrand $F(x,y,z) = \frac{1}{p}|z|^p$), typically considered for $p \in [1,\infty)$, where specifically the **1-energy** $\mathcal{E}_1$ is **also called** the **total variation** or the $\mathcal{TV}$ functional. The p-energy is connected with the PDE system (linear for $p=2$, non-linear in all other cases)

$$\operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) \equiv 0 \qquad \text{on } \Omega\,,$$

known as the ***p*-Laplace system** and, in case $N=1$, simply as the p-Laplace equation.

(2') Variants of the p-energy, usually considered for $p \in (1,\infty)$, are the **non-degenerate *p*-energy**

$$\tfrac{1}{p}\int_\Omega \big(1+|\mathrm{D}w|^2\big)^{\frac{p}{2}}\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$$

(integrand $F(x,y,z) = \frac{1}{p}(1+|z|^2)^{\frac{p}{2}}$) with its PDE system $\operatorname{div}((1+|\mathrm{D}u|^2)^{\frac{p-2}{2}}\mathrm{D}u) \equiv 0$ and the **Riemannian *p*-energy** (motivated by geometric situations with Riemannian metric on domain and target)

$$\tfrac{1}{p}\int_\Omega \bigg(\sum_{\ell,m=1}^N \sum_{i,j=1}^n a_{ij} g^{\ell m}(w)\partial_i w_\ell\,\partial_j w_m\bigg)^{\frac{p}{2}}\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$$

(integrand $F(x,y,z) = \frac{1}{p}\big[\sum_{\ell,m=1}^N \sum_{i,j=1}^n a_{ij}(x)g^{\ell m}(y)z_{\ell i}z_{mj}\big]^{\frac{p}{2}}$), where the bounded coefficients $(a_{ij})_{i,j=1,2,\ldots,n}\colon \Omega \to \mathbb{R}^{n\times n}$ and $(g^{\ell m})_{\ell,m=1,2,\ldots,N}\colon \mathbb{R}^N \to \mathbb{R}^{N\times N}$ take values in non-negative symmetric matrices.

(3) One more important model integral is the **non-parametric area integral** or **area integral for graphs**

$$\mathcal{S}[w] := \int_\Omega \sqrt{1+|\mathcal{M}(\mathrm{D}w)|^2}\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,\min\{N,n\}}(\Omega,\mathbb{R}^N)$$

(integrand $F(x,y,z) = \sqrt{1+|\mathcal{M}(z)|^2}$), where $\mathcal{M}(z) \in \mathbb{R}^\tau$, $\tau := \sum_{k=1}^{\min\{N,n\}}\binom{N}{k}\binom{n}{k}$, denotes the vector of all minors of the matrix $z \in \mathbb{R}^{N\times n}$, that is, $\mathcal{M}(z)$ contains the Nn entries of z as $(1{\times}1)$-minors, the $\binom{N}{2}\binom{n}{2}$ numbers obtained as $(2{\times}2)$-minors, the $\binom{N}{3}\binom{n}{3}$ numbers obtained as $(3{\times}3)$-minors, and then all minors of higher orders up to the maximal order $\min\{N,n\}$. The interest in this integral stems from the fact that $\mathcal{S}[w]$ equals, at least for $w \in \mathrm{C}^1(\Omega,\mathbb{R}^N)$, the n-dimensional surface area $\mathcal{H}^n(\operatorname{Graph} w)$ of $\operatorname{Graph} w := \{(x,w(x)) : x \in \Omega\} \subset \mathbb{R}^{n+N}$.

Hausdorff measure. For $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, the n-dimensional (spherical) Hausdorff measure of a set $A \subset \mathbb{R}^M$ is defined as

$$\mathcal{H}^n(A) := \lim_{\delta\searrow 0}\bigg(\inf\bigg\{\sum_{i=1}^\infty \omega_n r_i^n : A \subset \bigcup_{i=1}^\infty \mathrm{B}_{r_i}(x_i) \text{ with } r_i \in [0,\delta)\bigg\}\bigg)\,,$$

where we used $\mathrm{B}_r(x) = \{y \in \mathbb{R}^M : |y-x| < r\}$ for balls in $\mathbb{R}^M$ and ω_n for the n-dimensional Lebesgue measure of the unit ball $\mathrm{B}_1(0)$ in $\mathbb{R}^n$. It can be shown that $\mathcal{H}^n$ is σ-additive on the Borel σ-algebra of $\mathbb{R}^M$ and naturally measures the n-dimensional surface area on (mildly regular) n-dimensional surfaces in $\mathbb{R}^M$.

In the **scalar case $N = 1$**, where the graph of w is a hypersurface in $\mathbb{R}^n$, the non-parametric area integral takes the simpler form

$$\mathcal{S}[w] = \int_\Omega \sqrt{1+|\nabla w|^2}\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,1}(\Omega)$$

and is connected with the non-linear PDE known as the **minimal surface equation**, that is

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \equiv 0 \qquad \text{on } \Omega\,.$$

(4) Finally, our last model integral is the quantity

$$\mathcal{V}[w] := \int_\Omega |\mathcal{M}_{\min\{N,n\}}(\mathrm{D}w)|\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,\min\{N,n\}}(\Omega,\mathbb{R}^N)\,,$$

(integrand $F(x,y,z) = |\mathcal{M}_{\min\{N,n\}}(z)|$), where $\mathcal{M}_{\min\{N,n\}}(z) \in \mathbb{R}^\eta$, $\eta := \binom{N}{\min\{N,n\}}\binom{n}{\min\{N,n\}}$, denotes the vector of all minors of $z \in \mathbb{R}^{N\times n}$ of the maximal order $\min\{N,n\}$. The integral can be rewritten in a different form by checking (with some multilinear algebra)

$$|\mathcal{M}_{\min\{N,n\}}(z)| = \begin{cases} \sqrt{\det(z^{\mathrm{t}}z)} & \text{if } N \ge n \\ |\det z| & \text{if } N = n \\ \sqrt{\det(zz^{\mathrm{t}})} & \text{if } N \le n \end{cases} \qquad\qquad \text{for } z \in \mathbb{R}^{N\times n}\,.$$

In **case $N \ge n$**, the quantity $\mathcal{V}[w]$ equals, at least for injective $w \in \mathrm{C}^1(\Omega,\mathbb{R}^N)$, the surface area $\mathcal{H}^n(\operatorname{Image} w)$ of the n-dimensional surface parametrized by w and is known as the **parametric area integral**. In the non-scalar **two-dimensional case $N \ge 2 = n$** the integral reads $\mathcal{V}[w] = \int_\Omega \sqrt{|\partial_1 w|^2|\partial_2 w|^2 - (\partial_1 w\cdot\partial_2 w)^2}\,\mathrm{d}x$ for $w \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$, and in the classical case $N = 3$, $n = 2$ it can also be expressed as $\mathcal{TV}[w] = \int_\Omega |\partial_1 w\times\partial_2 w|\,\mathrm{d}x$ with the vector product $\times$ on $\mathbb{R}^3$.

In **case $N \le n$**, we have $\mathcal{V}[w] = \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(\{w = y\})\,\mathrm{d}y$, at least for $w \in \mathrm{C}^1(\Omega,\mathbb{R}^N)$.

Finally, in **case $\min\{N,n\} = 1$**, that is, in both the scalar and the one-dimensional situation, the integral $\mathcal{V}$ **coincides with the total variation $\mathcal{E}_1$**.

Examples (of **admissible classes** and **side conditions/constraints** for competitors)**.** Here we comment on typical choices of the admissible class $\mathcal{A}$ or, in other words on additional side conditions which can be (reasonably) imposed on the competitors $w \in \mathcal{A}$.

(i) The **most common side conditions** are certainly **boundary conditions**, and the most basic one is the **Dirichlet boundary condition** $w = \varphi$ on $\partial\Omega$. In the context of Sobolev functions on Ω, which a priori do not have well-defined pointwise values, one can give the Dirichlet boundary condition a precise meaning by relying on Sobolev trace theory. However, one can often get around such issues if one avoids the mentioning of the datum φ as a function on $\partial\Omega$ and only expresses that the competitors share the 'boundary values' of a given Sobolev function u_0 on Ω. In fact, this means that one chooses — precisely as done in connection with the Dirichlet principle — the admissible class $\mathcal{A}$ as a **Dirichlet class**

$$\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N) := u_0 + \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N) = \{u_0 + w : w \in \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)\}$$

in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ with given fixed $u_0 \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$.

In principle, one may also work with competitors satisfying Neumann boundary conditions, Plateau boundary conditions, or more general free boundary conditions, but the treatment of such cases goes beyond the scope of this lecture.

(ii) Other typical conditions are **holonomic side conditions**, that is, **equality constraints**

$$g(x, w(x)) = 0 \qquad \text{for } x \in \Omega$$

with a given function $g\colon \Omega\times\mathbb{R}^N \to \mathbb{R}^k$, $k \in \mathbb{N}$. These side conditions include, as a regular case, **manifold constraints**

$$w(x) \in M \qquad \text{for } x \in \Omega$$

with a given $(n-k)$-dimensional submanifold M in $\mathbb{R}^n$. In the context of Sobolev functions such conditions are usually implemented by choosing $\mathcal{A}$ as a subclass of $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ which satisfies the conditions **almost-everywhere** with respect to the Lebesgue measure.

(iii) One also considers **non-holonomic side conditions** such as ***in*equality constraints**

$$g(x, w(x)) \geq 0 \qquad \text{for } x \in \Omega$$

with given $g\colon \Omega\times\mathbb{R}^N \to \mathbb{R}$. Typical cases are **obstacle conditions**

$$w(x) \notin O \qquad \text{for } x \in \Omega$$

and in the scalar case $N = 1$ also

$$w(x) \geq \psi(x) \qquad \text{for } x \in \Omega\,,$$

where the 'obstacle' is an open set $O \subset \mathbb{R}^N$ and a scalar function $\psi\colon \Omega \to \mathbb{R}$, respectively. In the Sobolev context also such conditions are typically imposed **almost-everywhere**.

(iv) A last common type of side conditions are **integral constraints**

$$\int_\Omega g(x, w(x))\,\mathrm{d}x = 0\,,$$

with given $g\colon \Omega\times\mathbb{R}^N \to \mathbb{R}^k$, $k \in \mathbb{N}$, and specifically **volume constraints**. For instance, in the scalar case $N = 1$ one may require

$$\int_\Omega |w(x)|\,\mathrm{d}x = V\,,$$

which means that the $(n{+}1)$-dimensional volume enclosed by $\operatorname{Graph} w$ and $\Omega\times\{0\}$ coincides with a given $V \in [0, \infty)$. Since they have a decisive role in the classical isoperimetric problem (see Example (II) below), integral and volume constraints are often called **isoperimetric side conditions**.

(v) Clearly one **can combine the above constraints**, and indeed one often imposes conditions of either type (ii), type (iii), or type (iv) together with a Dirichlet boundary condition.

(vi) In principle, holonomic, non-holonomic, or integral side conditions may involve also the derivative $\mathrm{D}w$ and take the form of (ii), (iii), or (iv) with $h(x, w(x), \mathrm{D}w(x))$ in place of $g(x, w(x))$. Such constraints are, however, much more difficult to handle and can usually be treated only in very particular cases. Thus, we mostly dispense with $\mathrm{D}w$-dependent versions of (ii), (iii), (iv).

Examples (of **basic variational problems**). Now we introduce and partially discuss some classical variational problems. Though the focus of the lecture is on a general theory in arbitrary dimension n, at this stage we take the opportunity to mention for once some more basic problems with geometric flavor in the one-dimensional case $n = 1$.

In order to give a proper description in the Sobolev framework, we recall that, in the **single-variable case**, every function in $\mathrm{W}^{1,1}((\alpha,\beta),\mathbb{R}^N)$ with $-\infty < \alpha < \beta < \infty$ has a unique representative in $\mathrm{C}^0([\alpha,\beta],\mathbb{R}^N)$ (which in fact is even *absolutely* continuous). In particular, the same is true for every $w \in \mathrm{W}^{1,p}((\alpha,\beta),\mathbb{R}^N)$ with $p \in [1,\infty]$. In the sequel, we **widely identify $\mathbf{W^{1,p}}$ functions with the continuous representative** and always use this representative to **make sense of values in points** from $[\alpha,\beta]$ **or pointwise concepts**. With this understanding we introduce, for $p \in [1,\infty]$, $y_1, y_2 \in \mathbb{R}^N$, the notations

$$\begin{aligned}
\mathrm{W}^{1,p}_{\mathrm{per}}((\alpha,\beta),\mathbb{R}^N) &:= \{w \in \mathrm{W}^{1,p}((\alpha,\beta),\mathbb{R}^N) \,:\, w(\beta) = w(\alpha)\}\,,\\
\mathrm{W}^{1,p}_{y_1,y_2}((\alpha,\beta),\mathbb{R}^N) &:= \{w \in \mathrm{W}^{1,p}((\alpha,\beta),\mathbb{R}^N) \,:\, w(\alpha) = y_1\,,\ w(\beta) = y_2\}\,.
\end{aligned}$$

(I) The **planar geodesic problem** is the one of finding the **shortest curve which connects two given points in the plane $\mathbb{R}^2$**. Evidently the **solution is the straight line segment** from the one point to the other, and one may view this problem as a more or less trivial one. Nonetheless we proceed to specify two different variational settings for this problem, and we remark that similar frameworks can serve as a basis also for more interesting variants: For instance, one can add an obstacle ('curves may not enter a certain region $O \subset \mathbb{R}^2$') or pose the problem in a non-Euclidean space instead of $\mathbb{R}^2$. We will partially cover such variants later on but for the moment restrict the discussion to the plain version initially mentioned:

(a) In the first framework, **curves are**, as usual, **images** of functions $w\colon [\alpha,\beta] \to \mathbb{R}^2$ on a fixed interval $[\alpha,\beta]$ with $-\infty < \alpha < \beta < \infty$, and their length is given by the **parametric length integral** (case $N = 2$, $n = 1$ of the parametric area)

$$\mathcal{V}[w] = \int_\alpha^\beta |w'|\,\mathrm{d}t \qquad\qquad \text{for } w \in \mathrm{W}^{1,1}((\alpha,\beta),\mathbb{R}^2)\,.$$

The problem of the shortest curve from $y_1 \in \mathbb{R}^2$ to $y_2 \in \mathbb{R}^2$ then results in the minimization problem for $\mathcal{V}[w]$ among $w \in \mathrm{W}^{1,1}_{y_1,y_2}((\alpha,\beta),\mathbb{R}^2)$ (i.e. with boundary conditions $w(\alpha) = y_1$, $w(\beta) = y_2$). In fact it can be shown (see the exercises):

Mini Theorem (**planar geodesic problem with parametric curves**). *Fix α, β, y_1, y_2 as above. Then, for $u \in \mathrm{W}^{1,1}_{y_1,y_2}((\alpha,\beta),\mathbb{R}^2)$, it holds:*

$$\begin{aligned}
&u \textit{ minimizes } \mathcal{V} \textit{ in } \mathrm{W}^{1,1}_{y_1,y_2}((\alpha,\beta),\mathbb{R}^2)\,,\ \textit{ i.e. } \mathcal{V}[u] \le \mathcal{V}[w] \textit{ for all } w \in \mathrm{W}^{1,1}_{y_1,y_2}((\alpha,\beta),\mathbb{R}^2)\\
&\iff \begin{cases} u \textit{ is a monotone parametrization of the } \textbf{\textit{line segment}} \textit{ from } y_1 \textit{ to } y_2\,, \textit{ that is,}\\ u(t) = (1{-}\tau(t))y_1 + \tau(t)y_2 \textit{ for } t \in [\alpha,\beta] \textit{ with non-decreasing } \tau \in \mathrm{W}^{1,1}_{0,1}((\alpha,\beta))\,. \end{cases}
\end{aligned}$$

Since the same curve can evidently be parametrized[1] in different ways, this is the expected solution.

[1]The fact that only the geometric solution but not its parametrization is unique is closely connected with the (re)parametrization invariance of the parametric length integral. This type of invariance is, in fact, a basic difficulty in the theory of parametric variational problems.

(b) In the second framework, **curves are graphs** of functions $w\colon [x_1,x_2] \to \mathbb{R}$ on a fixed interval $[x_1,x_2]$ with $-\infty < x_1 < x_2 < \infty$, and their length is given by the **non-parametric length integral** (case $N=n=1$ of the non-parametric area)

$$\mathcal{L}[w] := \int_{x_1}^{x_2} \sqrt{1+(w')^2}\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,1}((x_1,x_2))\,.$$

The minimization of $\mathcal{L}[w]$ among functions $w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ with fixed $y_1,y_2 \in \mathbb{R}$ (i.e. with boundary conditions $w(x_1)=y_1$, $w(x_2)=y_2$) then models the problem of the shortest curve from (x_1,y_1) to (x_2,y_2), and it can be shown (see again the exercises):

Mini Theorem (planar geodesic problem with non-parametric curves). *Fix x_1, x_2, y_1, y_2 as above. Then, for $u \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$, it holds:*

$$\begin{aligned} &u \text{ minimizes } \mathcal{L} \text{ in } \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))\,, \text{ that is, } \mathcal{L}[u] \le \mathcal{L}[w] \text{ for all } w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2)) \\ \iff\ & u \text{ is } \textbf{affine}, \text{ that is, } u(x) = \frac{y_2-y_1}{x_2-x_1}x + \frac{y_1x_2-y_2x_1}{x_2-x_1} \text{ for } x \in [x_1,x_2]\,. \end{aligned}$$

As affine functions have line segments as graphs, this is again the expected solution.

(II) The **planar isoperimetric problem** is the problem to establish (any of) the following three principles for **simple closed curves** in $\mathbb{R}^2$ with the **length L** of such a curve and the **area A** of the region enclosed by it:

(Iso1) For fixed length L, the largest possible area A is $A = \frac{1}{4\pi}L^2$, with equality if and only if the curve is a circle.

(Iso2) For fixed area A, the shortest possible length L is $L = \sqrt{4\pi A}$, with equality if and only if the curve is a circle.

(Iso3) There holds the **isoperimetric inequality** $A \le \frac{1}{4\pi}L^2$, with equality if and only if the curve is a circle.

In fact, a bit of elementary reasoning shows that the three statements are equivalent. While (Iso1), which fixes the perimeter of the enclosed region, is responsible for the name of the problem, we here **prefer to discuss** (Iso2), which is a bit easier to access. Again we provide two frameworks in which the principle (or slight variants thereof) can be made precise:

(a) In the **parametric framework**, we work once more with curves $w\colon [\alpha,\beta] \to \mathbb{R}^2$ with length given by the parametric length integral $\mathcal{V}[w]$. The closedness of the curves is taken into account by working in $\mathrm{W}^{1,1}_{\mathrm{per}}((\alpha,\beta),\mathbb{R}^2)$, and for the enclosed *oriented* area (which is, roughly speaking, positive if the curve runs counter-clockwise and negative if it runs clockwise) one deduces, by either geometric considerations or the divergence theorem, the formulas $\int_\alpha^\beta w_1w_2'\,\mathrm{d}t = -\int_\alpha^\beta w_2w_1'\,\mathrm{d}t = \frac{1}{2}\int_\alpha^\beta (w_1w_2'-w_2w_1')\,\mathrm{d}t$. All in all, the isoperimetric principle of type (Iso2) thus asks to minimize $\mathcal{V}[w]$ in the admissible class

$$\mathcal{A}_A := \left\{ w \in \mathrm{W}^{1,1}_{\mathrm{per}}((\alpha,\beta),\mathbb{R}^2) \,:\, \int_\alpha^\beta w_1w_2'\,\mathrm{d}t = A \right\},$$

and one can then prove (compare with the exercises) the claimed optimality of the circle:

Theorem (planar isoperimetric problem with parametric curves). *Fix* $-\infty < \alpha < \beta < \infty$ *and* $A \in \mathbb{R}$*. Then, for* $u \in \mathcal{A}_A$*, it holds*:

$$u \text{ minimizes } \mathcal{V} \text{ in } \mathcal{A}_A\,, \text{ that is, } \mathcal{V}[u] \le \mathcal{V}[w] \text{ for all } w \in \mathcal{A}_A$$

$$\iff \begin{cases} u \text{ is a simple parametrization of a } \textbf{circle } \mathrm{S}_r(y_0)\,,\ r \ge 0\,,\ y_0 \in \mathbb{R}^2\,, \text{ that is,} \\ u(t) = y_0 + r\binom{\cos\tau(t)}{\sin\tau(t)} \text{ with monotone } \tau \in \mathrm{W}^{1,1}((\alpha,\beta))\,,\ |\tau(\beta) - \tau(\alpha)| = 2\pi\,. \end{cases}$$

We remark that in the situation of the theorem, the constraint necessarily implies $|A| = \pi r^2$, in case $A > 0$ with non-decreasing τ (i.e. counter-clockwise parametrization), and in case $A < 0$ with non-increasing τ (i.e. clockwise parametrization).

A well-known variant of the isoperimetric problem is the **Dido problem**[2] in which the curves w are not required to be closed but to have both endpoints on the first-coordinate axis, that is, to satisfy $w_2(\beta) = w_2(\alpha) = 0$. In this situation, the above integrals describe the area enclosed by the curve *and the first-coordinate axis*, and — in a sense analogous to the theorem — the solutions are simply parametrized semi-circles.

(b) If we try to phrase the planar isoperimetric problem in the **non-parametric framework**, we encounter into the obvious problem that the graph of a function $w\colon [x_1, x_2] \to \mathbb{R}$ cannot be a *closed* curve. Thus, the original isoperimetric problem cannot be treated in this framework. However, one can consider variants which seek to minimize the non-parametric length integral $\mathcal{L}[w]$ under the constraint $\int_{x_1}^{x_2} w\,\mathrm{d}x = A$ for the oriented area enclosed between the graph of w *and the* x*-axis*. If we add boundary conditions $w(x_1) = y_1$, $w(x_2) = y_2$ to this problem (Otherwise the only solution is the constant $\frac{A}{x_2 - x_1}$, and it is less interesting.), we arrive at the minimization problem for $\mathcal{L}$ in the admissible class

$$\widehat{\mathcal{A}}_A := \left\{ w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2)) \,:\, \textstyle\int_{x_1}^{x_2} w\,\mathrm{d}x = A \right\}.$$

For this version of the problem, one gets again the (more or less) expected answer (with a proof treated later in the exercises):

Theorem (planar isoperimetric problem with non-parametric curves). *Fix* $-\infty < x_1 < x_2 < \infty$*, and* $y_1, y_2, A \in \mathbb{R}$*. Then, for* $u \in \widehat{\mathcal{A}}_A$*, it holds*:

$$u \text{ minimizes } \mathcal{L} \text{ in } \widehat{\mathcal{A}}_A\,, \text{ that is, } \mathcal{L}[u] \le \mathcal{L}[w] \text{ for all } w \in \widehat{\mathcal{A}}_A$$

$$\iff \text{Graph}\, u \text{ is the straight line or a } \textbf{circular arc} \text{ from } (x_1,y_1) \text{ to } (x_2,y_2) \text{ in } \mathbb{R}^2\,.$$

In connection with this statement it is interesting to observe that for A too large or too small there exists no[3] line segment or circular arc from (x_1, y_1) to (x_2, y_2) which

[2]The Dido variant of the isoperimetric problem is connected with the saga of the founding of ancient Carthage by Queen Dido. In short summary, the saga says that the queen was granted as much territory for the founding of the city as she could enclose with an oxhide (the skin of an ox). She cut the oxhide into thin stripes and made a long cord out of these. Then in order to acquire as much as possible territory adjacent to the sea — with the coast idealized as the straight axis in the mathematical problem — she stretched out the cord in roughly the shape of a semi-circle with endpoints on the coast, thus solving the corresponding variant of the isoperimetric problem in our first formulation.

[3]In order to better understand the non-existence cases, one can think of the constraint as a constraint for the area between a circular arc and the straight line from (x_1, y_1) to (x_2, y_2). Then, a given area $\neq 0$ is always realized by a circular arc from (x_1, y_1) to (x_2, y_2) (around a center on the perpendicular bisector of the straight line), but in case of too large or too small area the arc does not stay in $[x_1, x_2] \times \mathbb{R}$ and is not representable as a graph. Reasoning in this way one can identify the limit cases (reached if the arc gets vertical at either (x_1, y_1) or (x_2, y_2)) and compute explicit bounds for the admissible A in terms of x_1, x_2, y_1, y_2.

encloses the correct area and is a graph over the x-axis. Thus, the theorem also shows that in these cases there exists no solution of the minimization problem.

Finally, we remark that the case $y_2 = y_1 = 0$ of the theorem is, of course, related to the Dido problem but does not describe completely the same geometric situation. The essential difference is that in the Dido problem there are no fixed quantities comparable to x_1, x_2 but rather such quantities are also optimized in such a way that the semi-circle with radius $\frac{1}{2}(x_2{-}x_1)$ encloses the area A. This is the reason why the solutions in the Dido problem are precisely the semi-circles, while in the last theorem we also get more general circular arcs.

(III) The **brachistochrone problem**[4] is the problem to determine the shape of a curve with given endpoints $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that an idealized point mass which starts at (x_1, y_1) with zero speed and then slides on the curve (under the influence of gravity and without any friction) reaches (x_2, y_2) in the shortest time.

In order to set up a mathematical model for this problem, we can reasonably assume $y_2 \le y_1$ (since otherwise the conservation-of-energy principle implies that there is no solution) and $x_1 < x_2$ (since $x_1 = x_2$ is covered by free vertical fall and $x_1 > x_2$ can be reduced to $x_1 < x_2$ by reflection). It is then usual to pass directly to a **non-parametric framework**, in which the relevant curves are graphs of functions $w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1, x_2))$, and **derive a model with the help of energy conservation**. Indeed, if one views the first coordinate of the position vector of the point mass as a C^1 function $q \colon [\alpha, \beta] \to [x_1, x_2]$ of time with $q(\alpha) = x_1$, $q(\beta) = x_2$ (where $\alpha < \beta$ in $\mathbb{R}$), then the full position vector and the velocity vector of a point mass sliding on $\operatorname{Graph} w$ are $\binom{q}{w(q)}$ and $\binom{q'}{w'(q)q'}$, respectively, and the scalar velocity is $\sqrt{1{+}w'(q)^2}\, q'$. Thus, the conservation of the sum of kinetic and potential energy means that $\frac{1}{2}(1{+}w'(q)^2)(q')^2 + gw(q)$ is constant on (α, β) and equals gy_1 (where $g > 0$ is the gravitational acceleration and the mass has already canceled out). If we add the physically obvious assumption that q' is positive on (α, β), then q has a continuous inverse $\tau \colon [x_1, x_2] \to [\alpha, \beta]$, which is C^1 with $\tau' > 0$ on (x_1, x_2) and represents the time as a function of the first coordinate of position vector. The conservation of energy now means $\frac{1+(w')^2}{2(\tau')^2} + gw = gy_1$ (and in particular $w < y_1$) on (x_1, x_2). Solving this equation for τ, we arrive at $\tau' = \sqrt{\frac{1+(w')^2}{2g(y_1-w)}}$ and can finally compute the total time to slide from (x_1, y_1) to (x_2, y_2) as $\tau(x_2){-}\tau(x_1) = \int_{x_1}^{x_2} \tau'(x)\,\mathrm{d}x = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1+w'(x)^2}{y_1-w(x)}}\,\mathrm{d}x$. All in all, the brachistochrone problem thus reduces to the minimization of the total time functional

$$\mathcal{T}[w] := \int_{x_1}^{x_2} \sqrt{\frac{1{+}(w')^2}{y_1{-}w}}\,\mathrm{d}x \qquad \text{among all } w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1, x_2))$$

(where we understand $\mathcal{T}[w] = \infty$ if $w \ge y_1$ holds on a set of positive Lebesgue measure).

[4] Johann Bernoulli set out the brachistochrone problem as public problem for the mathematical community as early as in 1696. While no solutions were received within the original time limit of six month, the mathematicians Newton, Jakob Bernoulli (Johann's brother), Leibniz, von Tschirnhaus, and de L'Hospital provided solutions in the following year. It is said that Newton solved the problem, once it came to his knowledge, during a single night and sent in the solution anonymously, but Bernoulli recognized “the lion from his claw mark”.

The brachistochrone problem can then be solved in the following sense of the following theorem (whose proof is omitted for the moment but will be discussed later in the exercises):

Theorem (on the **brachistochrone problem**). *Fix $x_1 < x_2$ and $y_2 \le y_1$ in $\mathbb{R}$. Then, for $u \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$, it holds:*

$$u \text{ minimizes } \mathcal{T} \text{ in } \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))\,,\text{ that is, } \mathcal{T}[u] \le \mathcal{T}[w] \text{ for all } w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$$
$$\iff \operatorname{Graph} u \text{ is a } \textbf{cycloidal arc} \text{ with upward-pointing cusp at } \left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right) \text{ but no interior cusp.}$$

Here a standard cycloid in $\mathbb{R}^2$ is the trajectory described by a point on the outer rim of circular wheel which rolls without slipping along a straight line. The cycloids with upward-pointing cusps at height y_1, as relevant in the theorem, can actually be thought to originate from a wheel rolling on a horizontal ceiling at height y_1 (i.e. touching $\mathbb{R}\times\{y_1\}$ from below). An arc of such a cycloid with cusp at the left endpoint $(x_1, y_1) \in \mathbb{R}^2$ and no cusp in the interior can be parametrized as

$$\left\{ \left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right) + R\left(\begin{smallmatrix} \varphi - \sin\varphi \\ -1 + \cos\varphi \end{smallmatrix}\right) \,:\, \varphi \in [0, \varphi_2] \right\}$$

with radius $R \in (0,\infty)$ and right-endpoint parameter $\varphi_2 \in (0, 2\pi]$. In the situation of the theorem, R and φ_2 are then uniquely determined by the requirements $x_2 = x_1 + R(\varphi_2 - \sin\varphi_2)$, $y_2 = y_1 + R(-1 + \cos\varphi_2)$ for the right endpoint (x_2, y_2). We remark that (x_2, y_2) can be reached with slope ≤ 0 (in which case it is the lowest point of the curve) but can also be reached with slope > 0 (in which case the lowest point was before) and can even be reached with slope $+\infty$ (in which case it is a second cusp). The last-mentioned alternative occurs if and only if y_2 equals y_1 (and, as one should expect, the arc is then symmetric with respect to the line $\{\frac{x_1+x_2}{2}\}\times\mathbb{R}$).

Chapter 2

Existence of minimizers (via the direct method)

In the remainder of these notes, whenever nothing else is said, we consider two arbitrary numbers $n, N \in \mathbb{N}$ as given and fixed.

2.1 The direct method

An important issue in the calculus of variations — just as in finite-dimensional calculus — is the proof that minima or maxima exist at all. In the early days of the discipline the necessity of such proofs has not always been recognized, and indeed it became generally accepted only in the 19th century when striking examples for the non-existence of minima were found and the need for a more solid foundation of the theory became apparent. In this lecture we postpone the discussion of such examples to a later point but rather concentrate — for the moment — on the most common method for proving existence of extrema. This method, known as the **direct method in the calculus of variations**, is based on the same basic arguments which are commonly used to prove the extreme value theorem in finite-dimensional calculus, and it **requires certain abstract compactness and semicontinuity properties as its two main ingredients**.

To fix ideas, we return to a (very) general functional $\mathcal{F}$ and focus on the existence problem for its minimizers (always keeping in mind that maximizers can be treated analogously). In this setting the following theorem summarizes the core principle of the direct method:

Theorem (abstract existence theorem for minimizers). *Consider a Hausdorff topological space $\mathcal{A} \neq \emptyset$ and a function $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$.*

(I) ***Topological statement:*** *If*

(a) *the **sublevel sets** $\{w \in \mathcal{A} : \mathcal{F}[w] \leq s\}$ with $s \in \mathbb{R}$ are **relatively compact** in $\mathcal{A}$*

and

(b) *the **functional** $\mathcal{F}$ is **lower semicontinuous** on $\mathcal{A}$,*

then there exists some $u \in \mathcal{A}$ such that $\mathcal{F}[u] \leq \mathcal{F}[w]$ for all $w \in \mathcal{A}$.

(II) ***Sequential statement:*** *If*

(a) ***every sequence*** $(w_k)_{k\in\mathbb{N}}$ *in* $\mathcal{A}$ ***with*** $\sup_{k\in\mathbb{N}} \mathcal{F}[w_k] < \infty$ ***has a subsequence converging to a limit*** *in* $\mathcal{A}$ (*what can be taken as a definition of* $\{w \in \mathcal{A} : \mathcal{F}[w] \le s\}$ *with* $s \in \mathbb{R}$ *being relatively* sequentially *compact in* $\mathcal{A}$)

and

(b) *the* ***functional*** $\mathcal{F}$ *is* **sequentially** ***lower semicontinuous*** *on* $\mathcal{A}$,

then there exists some $u \in \mathcal{A}$ *such that* $\mathcal{F}[u] \le \mathcal{F}[w]$ *for all* $w \in \mathcal{A}$.

Remarks (on the abstract existence theorem and its assumptions)**.**

(1) The **verification of the abstract assumptions** (Ia), (Ib) or (IIa), (IIb) for more concrete functionals $\mathcal{F}$ **requires theory in its own right** and will be addressed at length in the subsequent sections.

(2) One often requires in the definition of a minimizer $u \in \mathcal{A}$ that it satisfies, in addition to $\mathcal{F}[u] \le \mathcal{F}[w]$ for all $w \in \mathcal{A}$, also $\mathcal{F}[u] < \infty$. Clearly, the last condition is at hand in the existence result as soon as some $v \in \mathcal{A}$ with $\mathcal{F}[v] < \infty$ exists at all and thereby the trivial case $\mathcal{F} \equiv \infty$ on $\mathcal{A}$ is excluded. This is the reason why $\boldsymbol{\mathcal{F} \not\equiv \infty}$ is **sometimes added as a third assumption** in statements of the above type.

(3) The **compactness requirements** (Ia), (IIa)[1] are trivially satisfied for (sequentially) compact $\mathcal{A}$ (what however is usually not at hand) and otherwise compensate for the lack of compactness of $\mathcal{A}$ by requiring that $\mathcal{F}$ tends to $+\infty$ 'away from all compacts'[2]. For integral functionals the compactness requirements will **eventually** be **obtained from compactness results in Sobolev spaces together with growth assumptions on the integrand** $\boldsymbol{F}$.

(4) The decisive **semicontinuity requirements** (Ib), (IIb) mean by very definition that the sublevel sets $\mathcal{F}^{-1}([-\infty, s]) = \{w \in \mathcal{A} : \mathcal{F}[w] \le s\}$ with $s \in \overline{\mathbb{R}}$ are (sequentially) closed in $\mathcal{A}$. An equivalent characterization is that the supergraph $\{(w,s) \in \mathcal{A}\times\mathbb{R} : \mathcal{F}[w] \le s\}$ of $\mathcal{F}$ is (sequentially) closed in $\mathcal{A}\times\mathbb{R}$ with the product topology, and the sequential notion is also characterized by the inequality $\mathcal{F}[w] \le \liminf_{k\to\infty} \mathcal{F}[w_k]$ for every convergent sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ with limit $w \in \mathcal{A}$. For integral functionals $\mathcal{F}$ (with suitable assumptions on the integrand F), the **verification** of the semicontinuity requirements **is the main concern of the present chapter** and will be discussed at length.

(5) For metric spaces or at least metrizable topologies, the topological and sequential notions fully coincide. For general topologies, closedness still implies sequential closedness and

[1] We briefly recall the background definitions: Compactness of a set means that every open cover of the set contains a finite subcover of the set, and relative compactness of a set in $\mathcal{A}$ means that its closure in $\mathcal{A}$ is compact. Sequential compactness of a set means that every sequence in the set contains a subsequence convergent to a limit in the set, while in general topologies it may be debatable what is the right definition of relative sequential compactness.

[2] The relevant behavior of $\mathcal{F}$ can also be expressed in the one-point compactification $\mathcal{A} \,\dot\cup\, \{\omega_{\mathcal{A}}\}$ of $\mathcal{A}$, where (Ia) means $\lim_{\mathcal{A}\ni w\to\omega_{\mathcal{A}}} \mathcal{F}[w] = +\infty$ and (IIa) means $\lim_{k\to\infty} \mathcal{F}[w_k] = +\infty$ for all sequences $(w_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ with $\lim_{k\to\infty} w_k = \omega_{\mathcal{A}}$.

semicontinuity implies sequential semicontinuity, but not vice versa[3]. Moreover, there is no universal relationship between compactness and sequential compactness. All in all, one **often prefers to use the sequential notions**, since the setting with sequences is slightly less abstract and sequential semicontinuity is sometimes easier to obtain.

We now implement the core reasoning of the direct method and prove the statements in the above existence theorem

Proof of (I) *by* ***topological implementation of the direct method.*** In case $\mathcal{F} \equiv \infty$, every competitor is a minimizer. Thus, we can assume $M := \inf_{\mathcal{A}} \mathcal{F} < \infty$. Then, by definition of the infimum, all sublevel sets $\{w \in \mathcal{A} : \mathcal{F}[w] \le s\}$ with $s \in (M, \infty)$ are non-empty. Moreover, since $s \le \widetilde{s}$ implies $\{w \in \mathcal{A} : \mathcal{F}[w] \le s\} \subset \{w \in \mathcal{A} : \mathcal{F}[w] \le \widetilde{s}\}$, also finite intersections of these sets are non-empty. Most importantly, by (Ia) and (Ib), the sublevel sets are closed and relatively compact in $\mathcal{A}$, thus indeed compact in $\mathcal{A}$. All in all, with **Cantor's intersection theorem**[4] we conclude that $\{w \in \mathcal{A} : \mathcal{F}[w] \le M\} = \bigcap_{s \in (M,\infty)} \{w \in \mathcal{A} : \mathcal{F}[w] \le s\}$ is still non-empty and contains some $u \in \mathcal{A}$. This, however, means $\mathcal{F}[u] \le \inf_{\mathcal{A}} \mathcal{F} < \infty$. So, u is indeed a minimizer. □

Proof of (II) *by* ***implementation of the direct method with sequences.*** We assume once more $M := \inf_{\mathcal{A}} \mathcal{F} < \infty$. Then the definition of the infimum yields a **'minimizing sequence'** $(u_k)_{k \in \mathbb{N}}$ in $\mathcal{A}$ with $\lim_{k \to \infty} \mathcal{F}[u_k] = M$. In particular, this means $\sup_{k \ge k_0} \mathcal{F}[u_k] < \infty$ for sufficiently large k_0, and (IIa) implies the existence of a subsequence $(u_{k_\ell})_{\ell \in \mathbb{N}}$ which converges to some $u \in \mathcal{A}$. By (IIb) we then get

$$\mathcal{F}[u] \le \liminf_{\ell \to \infty} \mathcal{F}[u_{k_\ell}] = \lim_{k \to \infty} \mathcal{F}[u_k] = M\,.$$

This means $\mathcal{F}[u] \le \inf_{\mathcal{A}} \mathcal{F} < \infty$, and thus u is a minimizer. □

Remark (on **suitable topologies for the direct method**)**.** In **practical applications of the abstract existence theorem**, one has the freedom to **choose the topology** on the admissible class $\mathcal{A}$, and choosing a suitable one is **absolutely crucial**. Indeed, one has to cope with opposing requirements, since for (Ia), (IIa) to be satisfied the topology should be rather coarse/weak (few open and closed sets, many convergent sequences), while for (Ib), (IIb) it should be fine/strong (many open and closed sets, few convergent sequences). It turns out that the **norm topology** on subsets $\mathcal{A}$ **of an ∞-dimensional normed space is too strong**, since, in this topology, compact sets need to be totally bounded[5] and stay in ε-neighborhoods of

[3]Indeed, the following pathological example shows that sequential closedness does not imply closedness in general. Consider the space $\mathcal{A} = (\mathbb{N} \times \mathbb{N}) \,\dot\cup\, \{\omega\}$ equipped with the Hausdorff topology in which all point of $\mathbb{N} \times \mathbb{N}$ are isolated and the open neighborhoods of ω are all sets of the form $\mathcal{A} \setminus \bigcup_{i \in \mathbb{N}} \big(\{i\} \times S_i\big)$ with $S_i \subset \mathbb{N}$ and all but finitely many S_i finite. Then $\mathbb{N} \times \mathbb{N}$ is sequentially closed in $\mathcal{A}$ (since every sequence in $\mathbb{N} \times \mathbb{N}$ has a subsequence in some $\bigcup_{i=1}^{\infty} \big(\{i\} \times S_i\big)$ with S_i as before) but obviously not closed in $\mathcal{A}$. As a consequence, the characteristic function $\mathbb{1}_{\{\omega\}}$ is sequentially lower semicontinuous but not lower semicontinuous on $\mathcal{A}$.

[4]Cantor's intersection theorem is the following statement (which is dual to the covering definition of compactness and easy to deduce from this definition): Consider a family $(K_i)_{i \in I}$ of compact sets in a Hausdorff topological space with arbitrary index set I. If $\bigcap_{i \in J} K_i \neq \emptyset$ holds for every finite subset J of I, then one also has $\bigcap_{i \in I} K_i \neq \emptyset$.

[5]A set in a metric space is said to be totally bounded if, for every $\varepsilon > 0$, the set can be covered by *finitely many* balls with radii smaller than ε. The significance of this notion lies in the fact (which generalizes the Heine-Borel theorem) that a set in a metric space is compact if and only if it is complete and totally bounded.

finite-dimensional subspaces, while balls and all sets with non-empty interior cannot be compact. Rather the **right topology to choose is usually the weak topology** (and sometimes the weak$*$ topology) of a normed space, since this topology **comes with good compactness properties** (without being too weak to be meaningful).

(By the way, the necessity to work with the weak topology also motivates the formulation of the above existence theorem in Hausdorff topological spaces rather than merely metric ones. Indeed, since the weak topology of an ∞-dimensional normed space is Hausdorff and on norm-bounded sets also metrizable, but non-metrizable on the whole space, a metric-space version would not be general enough — at least if one aims at applying the theorem without any further ado.)

In the sequel we come back to integral functionals $\mathcal{F}$ and aim at verifying the hypotheses of the abstract existence theorem for them. We start with the semicontinuity requirement, which we deal with in the next section.

2.2 Weak lower semicontinuity

We first recall that **in every normed space $\mathcal{X}$**, the Hahn-Banach theorem implies that closed balls $\{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \le s\}$ are also weakly closed and thus **the norm $\|\cdot\|_{\mathcal{X}} \colon \mathcal{X} \to [0,\infty)$ is lower semicontinuous with respect to the weak topology** on $\mathcal{X}$. Since semicontinuity generally implies the corresponding sequential semicontinuity, the norm is then also sequentially weakly lower semicontinuous, that is, weak convergence $x_k \underset{k\to\infty}{\rightharpoonup} x$ in $\mathcal{X}$ implies $\|x\|_{\mathcal{X}} \le \liminf_{k\to\infty} \|x_k\|_{\mathcal{X}}$.

From this general fact, one can directly read off weak lower semicontinuity in simple cases. For instance, for open $\Omega \subset \mathbb{R}^n$ and $p \in [1,\infty)$, one gets the sequential weak lower semicontinuity properties

$$w_k \underset{k\to\infty}{\rightharpoonup} w \text{ weakly in } \mathrm{L}^p(\Omega,\mathbb{R}^N) \qquad \Longrightarrow \qquad \int_\Omega |w|^p \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega |w_k|^p \,\mathrm{d}x\,,$$

$$w_k \underset{k\to\infty}{\rightharpoonup} w \text{ weakly in } \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N) \qquad \Longrightarrow \qquad \mathcal{E}_p[w] \le \liminf_{k\to\infty} \mathcal{E}_p[w_k]$$

with the p-energy $\mathcal{E}_p$ introduced earlier.

However, here we aim at a theory which includes more general functionals. We start with:

Terminology (measures, measurability, standard σ-algebras)**.** *By a measure, a measurable set or function, and the a.e.-'quantor' with no measure specified we usually mean the Lebesgue*

Weak topology. The weak topology of a normed space $\mathcal{X}$ is the coarsest/weakest topology on $\mathcal{X}$ in which all elements of $\mathcal{X}^*$ are continuous. In other words, sets of the form $\{y \in \mathcal{X} : |\langle x_k^*\,; y{-}x\rangle| < \varepsilon \text{ for } k = 1,2,\ldots,\ell\}$ with $\ell \in \mathbb{N}$, $x_1^*, x_2^*, \ldots, x_\ell^* \in \mathcal{X}^*$, $\varepsilon > 0$ are weakly open basis neighborhoods of a point $x \in \mathcal{X}$, and every weakly open set in $\mathcal{X}$ is a union of such sets.

Weak convergence. The convergence of a sequence $(x_k)_{k\in\mathbb{N}}$ in a normed space $\mathcal{X}$ to a limit $x \in \mathcal{X}$ in the weak topology of $\mathcal{X}$ means $\lim_{k\to\infty}\langle x^*; x_k\rangle = \langle x^*; x\rangle$ for all $x^* \in \mathcal{X}^*$. It is expressed by writing $x_k \rightharpoonup x$ weakly in $\mathcal{X}$.

Weak convergence in $\mathbf{L}^p$ and $\mathbf{W}^{1,p}$. Consider a measurable $\Omega \subset \mathbb{R}^n$ and $p \in [1,\infty)$. Weak convergence $w_k \rightharpoonup w$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ is characterized by $\lim_{k\to\infty}\int_\Omega w_k \cdot v \,\mathrm{d}x = \int_\Omega w \cdot v \,\mathrm{d}x$ for all $v \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^N)$, where $p' := \frac{p}{p-1} \in (1,\infty]$ is the conjugate exponent to p. Weak convergence $w_k \rightharpoonup w$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ is equivalent to the weak convergences $w_k \rightharpoonup w$ and $\partial_i w_k \rightharpoonup \partial_i w$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ for all $i \in \{1,2,\ldots,n\}$.

measure, a Lebesgue measurable set or function, and the Lebesgue-a.e.-'quantor', respectively. We sometimes write $|\Omega|$ for the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$. We denote by $\mathcal{M}^n$ the σ-algebra of measurable subsets of $\mathbb{R}^n$ and by $\mathcal{B}(\mathcal{A})$ the Borel-σ-algebra of a topological space $\mathcal{A}$. Finally, we write $\mathcal{S}_1 \otimes \mathcal{S}_2$ for the product-σ-algebra of σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ (which is the σ-algebra generated by Cartesian products $S_1 \times S_2$ with $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$).

Proposition (*strong* lower semicontinuity). *Consider an open subset Ω of $\mathbb{R}^n$ and an $\mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N)$-measurable function $G \colon \Omega \times \mathbb{R}^N \to [0,\infty]$ such that $G(x,\,\cdot\,) \colon \mathbb{R}^N \to [0,\infty]$ is lower semicontinuous on $\mathbb{R}^N$ for a.e. $x \in \Omega$. Then the functional $\mathcal{G} \colon \mathrm{L}^1(\Omega,\mathbb{R}^N) \to [0,\infty]$, defined by*

$$\mathcal{G}[w] := \int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x \qquad \text{for } w \in \mathrm{L}^1(\Omega,\mathbb{R}^N)\,,$$

is lower semicontinuous on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ with its norm topology.

Remarks.

(1) The proposition **does *not* provide *weak* lower semicontinuity**, which — as explained before — is the property we really aim at, **but only strong lower semicontinuity**. Still it will turn out to be a useful preliminary result.

(2) The functional $\mathcal{G}$ can also be defined, by the same formula, on $\mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and is then lower semicontinuous with respect to $\mathrm{L}^1_{\mathrm{loc}}$-convergence. This can be shown easily by exhausting Ω with open $\widetilde{\Omega} \Subset \Omega$ and applying the proposition on the $\widetilde{\Omega}$.

In view of Hölder's inequality, $\mathcal{G}$ can also be regarded, for every $p \in [1,\infty]$, as a functional on the subspace $\mathrm{L}^p_{(\mathrm{loc})}(\Omega,\mathbb{R}^N)$ of $\mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and is there lower semicontinuous with respect to $\mathrm{L}^p_{(\mathrm{loc})}$-convergence.

Analogous extensions can be formulated for many subsequent results but from now on will not be explicitly mentioned in most cases.

(3) The $\mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N)$-measurability of G is required in order to guarantee that the composition $G(\,\cdot\,,w) \colon \Omega \to [0,\infty]$ remains measurable for measurable $w \colon \Omega \to \mathbb{R}^N$. (Indeed, in order to understand this assumption, recall that a measurable w maps, by definition, measurably from $(\Omega,\mathcal{M}^n)$ to $(\mathbb{R}^N,\mathcal{B}(\mathbb{R}^N))$. Consequently, $(\,\cdot\,,w)$ maps measurably from $(\Omega,\mathcal{M}^n)$ to $(\Omega \times \mathbb{R}^N, \mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N))$ and fits together with the $\mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N)$-measurable G.)

Proof of the proposition. Consider a strongly convergent sequence $w_k \underset{k\to\infty}{\longrightarrow} w$ in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$. For a suitable subsequence, we have $\lim_{\ell\to\infty} \mathcal{G}[w_{k_\ell}] = \liminf_{k\to\infty} \mathcal{G}[w_k] \in [0,\infty]$, and by a standard result on L^p-convergent sequences, for a further subsequence, we get $w_{k_{\ell_m}} \underset{m\to\infty}{\longrightarrow} w$ a.e. on Ω. With the lower semicontinuity assumption on the integrand G, we infer

$$G(\,\cdot\,,w) \le \liminf_{m\to\infty} G(\,\cdot\,,w_{k_{\ell_m}}) \qquad \text{a.e. on } \Omega\,.$$

By the previous choices and Fatou's lemma (which relies on the non-negativity of G), we conclude

$$\mathcal{G}[w] = \int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x \le \liminf_{m\to\infty} \int_\Omega G\big(\,\cdot\,,w_{k_{\ell_m}}\big)\,\mathrm{d}x = \lim_{m\to\infty} \mathcal{G}\big[w_{k_{\ell_m}}\big] = \liminf_{k\to\infty} \mathcal{G}[w_k]\,.$$

This proves (sequential) lower semicontinuity of $\mathcal{G}$ on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$, where 'sequential' can also be omitted, since we are dealing as yet with the norm topology. $\square$

Before coming to more relevant semicontinuity results, we introduce and discuss a technically convenient standard class of integrands:

Definition (Carathéodory functions). *A function $G\colon \Omega\times\mathbb{R}^N \to \mathcal{Z}$ with measurable $\Omega \subset \mathbb{R}^n$ and a metric space $\mathcal{Z}$ is called a **Carathéodory function** or a **Carathéodory integrand** if, for a.e. $x \in \Omega$, the function $G(x,\,\cdot\,)\colon \mathbb{R}^N \to \mathcal{Z}$ is continuous on $\mathbb{R}^N$ and, for every $y \in \mathbb{R}^N$, the function $G(\,\cdot\,,y)\colon \Omega \to \mathcal{Z}$ is measurable.*

Roughly speaking, the definition means that a Carathéodory function of $(x,y) \in \Omega\times\mathbb{R}^N$ is **measurable in $\boldsymbol{x \in \Omega}$ and continuous in $\boldsymbol{y \in \mathbb{R}^N}$**. In order to contextualize and prove a convenient property of Carathéodory functions we next recall two results from general measure theory, which we here state for the Lebesgue measure only:

- **Lusin's theorem:** Consider an open $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$, a metric space $\mathcal{Z}$, and a measurable function $f\colon \Omega \to \mathcal{Z}$. Then, for every $\varepsilon > 0$, there exists a compact $A \subset \Omega$ with $|\Omega \setminus A| < \varepsilon$ such that $f|_A$ is continuous.
- **Egoroff's theorem:** Consider a measurable $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$, a metric space $\mathcal{Z}$, and measurable functions $f_k, f\colon \Omega \to \mathcal{Z}$ such that $f_k \underset{k\to\infty}{\longrightarrow} f$ a.e. on Ω. Then, for every $\varepsilon > 0$, there exists a compact $B \subset \Omega$ with $|\Omega \setminus B| < \varepsilon$ such that $f_k \underset{k\to\infty}{\longrightarrow} f$ uniformly on B.

In fact, a Lusin-type statement applies also for Carathéodory functions:

Lemma (Scorza-Dragoni lemma). *Consider an open $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$, a metric space $\mathcal{Z}$, and a Carathéodory function $G\colon \Omega\times\mathbb{R}^N \to \mathcal{Z}$. Then, for every $\varepsilon > 0$, there exists a compact $A \subset \Omega$ with $|\Omega \setminus A| < \varepsilon$ such that $G|_{A\times\mathbb{R}^N}$ is continuous.*

Remarks.

(1) Lusin's theorem and the Scorza-Dragoni lemma extend to the case $|\Omega| = \infty$, in which $A \subset \Omega$ can, however, be taken just closed in $\mathbb{R}^n$ not compact. This can be shown by applying the preceding statements on $\Omega \cap (\mathrm{B}_k(0) \setminus \overline{\mathrm{B}_{k-1}(0)})$, $k \in \mathbb{N}$, and taking the union of the resulting compacts.

For Egoroff's theorem, in contrast, the requirement $|\Omega| < \infty$ cannot be dropped.

(2) The Scorza-Dragoni lemma implies that **Carathéodory integrands $G\colon \Omega\times\mathbb{R}^N \to \mathcal{Z}$ are $\boldsymbol{\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)}$-measurable**, and this then ensures that $G(\,\cdot\,,w)\colon \Omega \to \mathcal{Z}$ stays measurable for all measurable $w\colon \Omega \to \mathbb{R}^N$.

Proof of the $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)$-measurability. By applying the lemma for $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, we find compact (or, in case $|\Omega| = \infty$, at least closed) $A_k \subset \Omega$ with $\lim_{k\to\infty} |\Omega \setminus A_k| = 0$ such that $G|_{A_k\times\mathbb{R}^N}$ is continuous for all $k \in \mathbb{N}$. We write G_k for a function which coincides with G on $A_k\times\mathbb{R}^N$ and is elsewhere constant with an arbitrary value, and we observe that G_k is even $\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(\mathbb{R}^N)$-measurable. Possibly replacing A_k with $\bigcup_{j=1}^k A_j$, we now additionally assume $A_1 \subset A_2 \subset A_3 \subset \ldots$. Then we have $G = \lim_{k\to\infty} G_k$ on $\big(\bigcup_{j=1}^\infty A_j\big)\times\mathbb{R}^N$, and in view of $|\Omega \setminus \bigcup_{j=1}^\infty A_j| = 0$ the $\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(\mathbb{R}^N)$-measurability of G_k implies $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)$-measurability of G. $\square$

(3) The Scorza-Dragoni lemma and Lusin's and Egoroff's theorems all **apply in the case of the target space $\boldsymbol{\mathcal{Z}} = \overline{\mathbb{R}}$**, since the topology of $\overline{\mathbb{R}}$ can be metrized (for instance by setting $\mathrm{d}_{\overline{\mathbb{R}}}(x,y) := |\arctan y - \arctan x|$ with the understanding $\arctan(\pm\infty) := \pm\frac{\pi}{2}$).

Proof of the Scorza-Dragoni lemma. For the moment, we fix $\varepsilon > 0$ and $M \in \mathbb{N}$. For every $k \in \mathbb{N}$, by setting

$$\delta_k(x) := \sup\left\{\mathrm{d}(G(x,y),G(x,\widetilde{y}))\ :\ y,\widetilde{y} \in \mathrm{B}_M(0)\,,\ |\widetilde{y}{-}y| < \frac{1}{k}\right\} \qquad \text{for } x \in \Omega$$

(where $\mathrm{B}_M(0)$ stands for the ball with radius M and center 0 in $\mathbb{R}^N$ and $\mathrm{d} = \mathrm{d}_{\mathcal{Z}}$ denotes the metric of $\mathcal{Z}$) we obtain a measurable[6] function $\delta_k\colon \Omega \to [0,\infty]$. Since the Carathéodory property gives uniform continuity of $G(x,\,\cdot\,)|_{\overline{\mathrm{B}_M(0)}}$ for a.e. $x \in \Omega$, we infer $\lim_{k\to\infty} \delta_k = 0$ a.e. on Ω. Egoroff's theorem then yields a compact $B \subset \Omega$ with $|\Omega \setminus B| < \frac{\varepsilon}{2}$ such that the convergence $\lim_{k\to\infty} \delta_k = 0$ is uniform on B. Next we choose a countable dense subset $D = \{\widetilde{y}_i : i \in \mathbb{N}\}$ of $\mathrm{B}_M(0)$, and from Lusin's theorem we obtain, for every $i \in \mathbb{N}$, a compact $A_i \subset \Omega$ with $|\Omega \setminus A_i| < 2^{-i-1}\varepsilon$ such that $G(\,\cdot\,,\widetilde{y}_i)|_{A_i}$ is continuous. Consequently, for $\widetilde{A} := B \cap \bigcap_{i=1}^\infty A_i$, we get $|\Omega \setminus \widetilde{A}| < \varepsilon$, and we now aim at showing that

$$G|_{\widetilde{A}\times\mathrm{B}_M(0)} \text{ is continuous.} \tag{$*$}$$

In order to verify $(*)$ we consider a convergent sequence $(x_k,y_k) \underset{k\to\infty}{\longrightarrow} (x,y)$ in $\widetilde{A}{\times}\mathrm{B}_M(0)$ and an arbitrary $\gamma > 0$. In view of the uniform convergence of δ_k on $B \supset \widetilde{A}$, we can fix $k_0 \in \mathbb{N}$ with $\delta_{k_0} \le \gamma$ on $\widetilde{A}$ and in the next step then $\widetilde{y} \in D$ with $|\widetilde{y}{-}y| < \frac{1}{k_0}$. For all sufficiently large k we have $|y_k{-}y| < \frac{1}{k_0}$ and, thanks to continuity of $G(\,\cdot\,,\widetilde{y})|_{\widetilde{A}}$, also $\mathrm{d}(G(x_k,\widetilde{y}),G(x,\widetilde{y})) < \gamma$. Putting together all these choices, we end up with the estimate

$$\begin{aligned}
&\mathrm{d}(G(x_k,y_k),G(x,y))\\
&\le \mathrm{d}(G(x_k,y_k),G(x_k,y)) + \mathrm{d}(G(x_k,y),G(x_k,\widetilde{y})) + \mathrm{d}(G(x_k,\widetilde{y}),G(x,\widetilde{y})) + \mathrm{d}(G(x,\widetilde{y}),G(x,y))\\
&< 2\delta_{k_0}(x_k) + \gamma + \delta_{k_0}(x) \le 4\gamma\,.
\end{aligned}$$

Since $\gamma > 0$ is arbitrary, this means $\lim_{k\to\infty} G(x_k,y_k) = G(x,y)$ and establishes the continuity claim $(*)$.

To obtain the full claim of the lemma, we consider only $\varepsilon > 0$ (but no longer M) as fixed. The previous reasoning can then be applied to obtain, for every $M \in \mathbb{N}$, a compact set $\widetilde{A}(M) \subset \Omega$ with $\big|\Omega{\setminus}\widetilde{A}(M)\big| < 2^{-M}\varepsilon$ such that $G|_{\widetilde{A}(M)\times\mathrm{B}_M(0)}$ is continuous. It follows that $A := \bigcap_{M=1}^\infty \widetilde{A}(M)$ is still compact with $|\Omega \setminus A| < \varepsilon$ and that $G|_{A\times\mathbb{R}^N}$ is continuous. This finishes the proof. $\square$

At this stage we return to the main semicontinuity question and provide the first truly useful semicontinuity result:

[6] In order to establish measurability of δ_k, observe that the continuity of $G(x,\,\cdot\,)$ for a.e. $x \in \Omega$ together with the continuity of the metric d yields $\delta_k(x) = \sup\big\{\mathrm{d}(G(x,y),G(x,\widetilde{y}))\,:\,y,\widetilde{y} \in \mathbb{Q}^N \cap \mathrm{B}_M(0)\,,\,|\widetilde{y}{-}y| < \frac{1}{k}\big\}$ for a.e. $x \in \Omega$. Thus, δ_k coincides a.e. with a *countable* supremum of measurable functions and inherits measurability.

Theorem (***weak*** **lower semicontinuity of convex zero-order functionals**)**.** *Consider an open $\Omega \subset \mathbb{R}^n$ and a Carathéodory function $G\colon \Omega\times\mathbb{R}^N \to [0,\infty]$ such that $G(x,\,\cdot\,)\colon \mathbb{R}^N \to [0,\infty]$ is convex on $\mathbb{R}^N$ for a.e. $x \in \Omega$. Then the functional $\mathcal{G}\colon \mathrm{L}^1(\Omega,\mathbb{R}^N) \to [0,\infty]$, defined by*

$$\mathcal{G}[w] := \int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x \qquad \text{for } w \in \mathrm{L}^1(\Omega,\mathbb{R}^N)\,,$$

is lower semicontinuous on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ with the weak *topology.*

The proof of this theorem rests on the following functional analysis principle:

Lemma (**Mazur lemma**, topological version)**.** *Every* ***closed convex set*** *A in a normed space $\mathcal{X}$ is* ***even weakly closed*** *in $\mathcal{X}$.*

Specifically, the lemma shows that every **closed subspace** in a normed space $\mathcal{X}$ is even **weakly closed** in $\mathcal{X}$.

Proof of the lemma. For $x_0 \in \mathcal{X}\setminus A$, the Hahn-Banach theorem in the separation version yields an $x^* \in \mathcal{X}^*$ and an $s \in \mathbb{R}$ such that $\langle x^*\,;x_0\rangle < s \le \langle x^*\,;a\rangle$ for all $a \in A$. This means $x_0 \in \{x \in \mathcal{X} : \langle x^*\,;x\rangle < s\} \subset \mathcal{X}\setminus A$, and thus the weakly open set $\{x \in \mathcal{X} : \langle x^*\,;x\rangle < s\}$ is in fact a weakly open neighborhood of x_0 in $\mathcal{X}\setminus A$. Consequently, $\mathcal{X}\setminus A$ is weakly open in $\mathcal{X}$, and A is weakly closed in $\mathcal{X}$. □

Proof of the theorem. By the earlier proposition on strong lower semicontinuity, we know that $\mathcal{G}$ is lower semicontinuous on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$, and by Remark (4) in Section 2.1 this means that the supergraph $\mathrm{S}_\mathcal{G} := \{(w,s) \in \mathrm{L}^1(\Omega,\mathbb{R}^N)\times\mathbb{R} : \mathcal{G}[w] \le s\}$ of $\mathcal{G}$ is closed in $\mathrm{L}^1(\Omega,\mathbb{R}^N)\times\mathbb{R}$. In addition, we find[7] that the functional $\mathcal{G}$ is convex on $\mathbb{R}^N$, and this means that the supergraph $\mathrm{S}_\mathcal{G}$ is also convex. By a decisive application of the Mazur lemma, the closed convex set $\mathrm{S}_\mathcal{G}$ in $\mathrm{L}^1(\Omega,\mathbb{R})\times\mathbb{R}$ is then even weakly closed. Relying on Remark (4) in Section 2.1 once more, we obtain that $\mathcal{G}$ is weakly lower semicontinuous. □

We also mention a sequential version of the lemma, which leads to similar conclusions:

Lemma (**Mazur lemma**, sequential version)**.** *If a sequence $(x_\ell)_{\ell\in\mathbb{N}}$* ***converges*** **weakly** *in a normed space $\mathcal{X}$ to a limit x, then for every $k \in \mathbb{N}$ there exist an upper bound $m(k) \in \mathbb{N}_{\ge k}$ and coefficients $\lambda_{k,k},\lambda_{k,k+1},\ldots,\lambda_{k,m(k)-1},\lambda_{k,m(k)} \in [0,1]$ with $\sum_{\ell=k}^{m(k)}\lambda_{k,\ell} = 1$ such that the sequence $(y_k)_{k\in\mathbb{N}}$ of the* ***convex combinations*** *$y_k := \sum_{\ell=k}^{m(k)}\lambda_{k,\ell}x_\ell$* ***converges*** **strongly** *in $\mathcal{X}$ to x.*

Proof. For every $k \in \mathbb{N}$, the set

$$A_k := \left\{\sum_{\ell=k}^{m}\lambda_\ell x_\ell \,:\, m \in \mathbb{N}_{\ge k}\,,\,\lambda_k,\lambda_{k+1},\ldots,\lambda_m \in [0,1]\,,\,\sum_{\ell=k}^{m}\lambda_\ell = 1\right\},$$

(that is, the convex hull of the end-piece $\{x_\ell : \ell \in \mathbb{N}_{\ge k}\}$ of $(x_\ell)_{\ell\in\mathbb{N}}$) is easily seen to be convex. Therefore, the closure $\overline{A_k}$ (that is, the closed convex hull) is closed and convex in $\mathcal{X}$ and by the first version of Mazur's lemma also weakly closed in $\mathcal{X}$. In view of $\{x_\ell : \ell \in \mathbb{N}_{\ge k}\} \subset A_k \subset \overline{A_k}$, it follows that also the weak limit x is in $\overline{A_k}$. By definition of the closure, this implies the existence of some $y_k \in A_k$ with $\|y_k-x\|_\mathcal{X} < \frac{1}{k}$, and the sequence $(y_k)_{k\in\mathbb{N}}$ satisfies the claim. □

[7]For $w,\widetilde{w} \in \mathrm{L}^1(\Omega,\mathbb{R}^N)$ and $\lambda \in [0,1]$, the convexity assumption on the integrand G gives $G(\,\cdot\,,\lambda w+(1-\lambda)\widetilde{w}) \le \lambda G(\,\cdot\,,w)+(1-\lambda)G(\,\cdot\,,\widetilde{w})$ a.e. on Ω, which then implies $\mathcal{G}[\lambda w+(1-\lambda)\widetilde{w}] \le \lambda\mathcal{G}[w]+(1-\lambda)\mathcal{G}[\widetilde{w}]$ and thus convexity of the functional $\mathcal{G}$. For $(w,s),(\widetilde{w},\widetilde{s}) \in \mathrm{S}_\mathcal{G}$ and $\lambda \in [0,1]$, we infer $\mathcal{G}[\lambda w+(1-\lambda)\widetilde{w}] \le \lambda\mathcal{G}[w]+(1-\lambda)\mathcal{G}[\widetilde{w}] \le \lambda s+(1-\lambda)\widetilde{s}$ and thus $\lambda(w,s)+(1-\lambda)(\widetilde{w},\widetilde{s}) = (\lambda w+(1-\lambda)\widetilde{w},\lambda s+(1-\lambda)\widetilde{s}) \in \mathrm{S}_\mathcal{G}$, which proves the claimed convexity of $\mathrm{S}_\mathcal{G}$.

Remark. The **sequential version of the Mazur lemma** can be equivalently recast[8] by saying that every **closed convex set** A in a normed space $\mathcal{X}$ is also ***sequentially* weakly closed** in $\mathcal{X}$ and thus turns out to be a slightly weaker variant of the earlier topological statement.

The sequential version of the Mazur lemma can be applied to prove a version of the last theorem with exactly the same assumptions but merely ***sequential* weak lower semicontinuity** on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ as the slightly weaker outcome. We emphasize that this slightly weaker version is **fully sufficient in connection with the direct method** and leads to the same existence theory for minimizers. We now give the alternative proof.

Alternative proof of the theorem with merely sequential *weak lower semicontinuity as outcome.* We consider a weakly convergent sequence

$$w_j \underset{j\to\infty}{\longrightarrow} w \qquad \text{weakly in } \mathrm{L}^1(\Omega,\mathbb{R}^N)\,.$$

By definition of $\liminf$, we have $\lim_{\ell\to\infty}\mathcal{G}[w_{j_\ell}]=\liminf_{j\to\infty}\mathcal{G}[w_j]\in[0,\infty]$ for a suitable subsequence. By the sequential Mazur lemma, we find $m(k)\in\mathbb{N}_{\ge k}$ and $\lambda_{k,\ell}\in[0,1]$ with $\sum_{\ell=k}^{m(k)}\lambda_{k,\ell}=1$ such that

$$\sum_{\ell=k}^{m(k)}\lambda_{k,\ell}w_{j_\ell} \underset{k\to\infty}{\longrightarrow} w \qquad \text{strongly in } \mathrm{L}^1(\Omega,\mathbb{R}^N)\,.$$

With the proposition on strong lower semicontinuity and the convexity of the functional $\mathcal{G}$, we then estimate

$$\begin{aligned}\mathcal{G}[w]\le\liminf_{k\to\infty}\mathcal{G}\left[\sum_{\ell=k}^{m(k)}\lambda_{k,\ell}w_{j_\ell}\right]&\le\liminf_{k\to\infty}\sum_{\ell=k}^{m(k)}\lambda_{k,\ell}\mathcal{G}\big[w_{j_\ell}\big]\\&\le\liminf_{k\to\infty}\sup_{\ell\in\mathbb{N}_{\ge k}}\mathcal{G}\big[w_{j_\ell}\big]=\lim_{\ell\to\infty}\mathcal{G}\big[w_{j_\ell}\big]=\liminf_{j\to\infty}\mathcal{G}[w_j]\,.\end{aligned}$$

This proves sequential weak lower semicontinuity of $\mathcal{G}$ on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$. □

Our true aim in this lecture in the treatment of *first-order* functionals. For these, we directly infer from the previous *zero-order* theorem:

Corollary (*weak* lower semicontinuity of convex 1$^{\text{st}}$-order functionals). *Consider an open subset Ω of $\mathbb{R}^n$ and a Carathéodory*[9] *function $F\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to[0,\infty]$ such that $F(x,\,\cdot\,,\,\cdot\,)\colon\mathbb{R}^N\times\mathbb{R}^{N\times n}\to[0,\infty]$ is convex on $\mathbb{R}^N\times\mathbb{R}^{N\times n}$ for a.e. $x\in\Omega$. Then the functional $\mathcal{F}\colon\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)\to[0,\infty]$, defined by*

$$\mathcal{F}[w]:=\int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x \qquad\qquad \textit{for } w\in\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)\,,$$

is lower semicontinuous on $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ with the weak *topology.*

[8] Indeed, the sequential weak closedness of closed convex sets implies the sequential Mazur lemma by the reasoning in the proof of the lemma, where sequential weak closedness of A_k suffices. Conversely, the sequential Mazur lemma implies the sequential weak closedness of closed convex sets as follows: Consider a closed convex A and a sequence $(x_\ell)_{\ell\in\mathbb{N}}$ in A which converges weakly in $\mathcal{X}$ to x. Then, the sequence $(y_k)_{k\in\mathbb{N}}$ of convex combinations from the lemma remains in the convex A, and, by closedness of A, the strong limit x of $(y_k)_{k\in\mathbb{N}}$ is still in A. Thus, A is sequentially weakly closed.

[9] Here we understand the Carathéodory property the way that, for a.e. $x\in\Omega$, the function $F(x,\,\cdot\,,\,\cdot\,)$ is continuous on $\mathbb{R}^N\times\mathbb{R}^{N\times n}$ and, for all $(y,z)\in\mathbb{R}^N\times\mathbb{R}^{N\times n}$, the function $F(\,\cdot\,,y,z)$ is measurable. In the sequel a similar understanding goes without saying.

Proof. We abbreviate $\mathrm{L}^1 := \mathrm{L}^1(\Omega,\mathbb{R}^N)\times\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, $\mathrm{W}^{1,1} := \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ and write $\mathcal{F}=\mathcal{G}\circ J$ as composition of the zero-order functional $\mathcal{G}\colon \mathrm{L}^1\to[0,\infty]$, given by $\mathcal{G}[w,\overline{w}] := \int_\Omega F(\,\cdot\,,w,\overline{w})\,\mathrm{d}x$, and the continuous linear mapping $J\colon \mathrm{W}^{1,1}\to\mathrm{L}^1$, given by $Jw := (w,\mathrm{D}w)$. Since L^1 can be identified with $\mathrm{L}^1(\Omega,\mathbb{R}^{N+Nn})$, the last theorem gives weak lower semicontinuity of $\mathcal{G}$, that is, weak closedness of $\mathcal{G}^{-1}([-\infty,s])$ in L^1 for all $s\in\overline{\mathbb{R}}$. Moreover, the continuous linear J is also weak-weak continuous[10], that is, continuous with respect to the weak topology on both $\mathrm{W}^{1,1}$ and L^1. Thus, we conclude that $\mathcal{F}^{-1}([-\infty,s]) = J^{-1}(\mathcal{G}^{-1}([-\infty,s]))$ is weakly closed in $\mathrm{W}^{1,1}$ for all $s\in\overline{\mathbb{R}}$, which means that $\mathcal{F}$ is weakly lower semicontinuous. □

Remark. The preceding weak lower semicontinuity results for integral functionals $\mathcal{G}$ and $\mathcal{F}$ of order zero and one, respectively, involve convexity assumptions for the integrands G and F. In fact, it has been **crucial** that these assumptions are strong enough to guarantee **convexity of $\mathcal{G}$ and $\mathcal{F}$ as functionals**. Next we will show — as it seems reasonable in view of the initial strong lower semicontinuity statement — that **convexity in lower-order variables (and more generally in variables of strong convergence) can actually be dropped**. Since in such cases the functionals as a whole are no longer convex, this cannot be proved with the Mazur lemma alone, but rather it will require additional estimates.

Theorem (*weak-strong* lower semicontinuity). *Consider an open subset Ω of $\mathbb{R}^n$ and a Carathéodory function $G\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{\widetilde{N}}\to[0,\infty]$ such that $G(x,y,\,\cdot\,)\colon \mathbb{R}^{\widetilde{N}}\to[0,\infty]$ is convex on $\mathbb{R}^{\widetilde{N}}$ for all $(x,y)\in(\Omega\setminus E)\times\mathbb{R}^N$ with a null set $E\subset\Omega$. Then the functional $\mathcal{G}\colon \mathrm{L}^1(\Omega,\mathbb{R}^N)\times\mathrm{L}^1(\Omega,\mathbb{R}^{\widetilde{N}})\to[0,\infty]$, defined by*

$$\mathcal{G}[w,\widetilde{w}] := \int_\Omega G(\,\cdot\,,w,\widetilde{w})\,\mathrm{d}x \qquad \text{for } w\in\mathrm{L}^1(\Omega,\mathbb{R}^N)\,,\ \widetilde{w}\in\mathrm{L}^1(\Omega,\mathbb{R}^{\widetilde{N}})\,,$$

is sequentially lower semicontinuous on $\mathrm{L}^1(\Omega,\mathbb{R}^N)\times\mathrm{L}^1(\Omega,\mathbb{R}^{\widetilde{N}})$ with the **strong *topology on the first factor*** *$\mathrm{L}^1(\Omega,\mathbb{R}^N)$ and the* **weak *topology on the second factor*** *$\mathrm{L}^1(\Omega,\mathbb{R}^{\widetilde{N}})$.*

Proof. Exhausting Ω, if necessary, with open $\widetilde{\Omega}\Subset\Omega$, we can assume $|\Omega|<\infty$. We consider convergent sequences

$$w_k \underset{k\to\infty}{\longrightarrow} w \text{ strongly in } \mathrm{L}^1(\Omega,\mathbb{R}^N)\,, \qquad \widetilde{w}_k \underset{k\to\infty}{\longrightarrow} \widetilde{w} \text{ weakly in } \mathrm{L}^1(\Omega,\mathbb{R}^{\widetilde{N}})$$

with $\liminf_{k\to\infty}\mathcal{G}[w_k,\widetilde{w}_k]<\infty$. We pass to a subsequence such that $\lim_{\ell\to\infty}\mathcal{G}[w_{k_\ell},\widetilde{w}_{k_\ell}] = \liminf_{k\to\infty}\mathcal{G}[w_k,\widetilde{w}_k]$ and $\sup_{\ell\in\mathbb{N}}\mathcal{G}[w_{k_\ell},\widetilde{w}_{k_\ell}]<\infty$. We now claim:

$$\begin{gathered}\text{For every } \varepsilon>0 \text{ there is an } \ell\in\mathbb{N} \text{ and a measurable } A\subset\Omega \text{ with } |\Omega\setminus A|<\varepsilon \text{ such that}\\ |G(\,\cdot\,,w_{k_\ell},\widetilde{w}_{k_\ell})-G(\,\cdot\,,w,\widetilde{w}_{k_\ell})|<\varepsilon \qquad \text{on } A\,.\end{gathered} \tag{$**$}$$

In order to verify ($**$), we fix $\varepsilon>0$ and construct the set A in four steps. In a preliminary step (which is needed only if G takes the value ∞) we fix $L\in\mathbb{N}$ large enough that

$$\widetilde{A}_{0,\ell} := \{x\in\Omega : G(x,w_{k_\ell}(x),\widetilde{w}_{k_\ell}(x))<L\}$$

[10] To see that J is weak-weak continuous, one first observes that $J^{-1}(\{x^*<s\}) = \{x^*\circ J<s\}$ is weakly open in $\mathrm{W}^{1,1}$ for all $x^*\in(\mathrm{L}^1)^*$ and $s\in\mathbb{R}$ (since $x^*\circ J\in(\mathrm{W}^{1,1})^*$). Since weakly open sets in $(\mathrm{L}^1)^*$ can be written as unions of finite intersections of sets of form $\{x^*<s\}$ and preimage commutes with both union and intersection, one can conclude that $J^{-1}(O)$ is weakly open for all weakly open O. This is the claimed continuity

satisfies $|\Omega \setminus \widetilde{A}_{0,\ell}| < \frac{\varepsilon}{4}$ for all $\ell \in \mathbb{N}$. Indeed, choosing a single L, which works for all ℓ, is possible here, since $|\{x \in \Omega : G(x, w_{k_\ell}(x), \widetilde{w}_{k_\ell}(x)) \geq L\}| \leq \frac{1}{L}\mathcal{G}[w_{k_\ell}, \widetilde{w}_{k_\ell}]$ converges for $L \to \infty$ uniformly in $\ell \in \mathbb{N}$ to 0 by the initial choice of a subsequence. In the next step, we infer from the Scorza-Dragoni lemma the existence of a compact set $\widetilde{A}_1 \subset \Omega$ with $\big|\Omega \setminus \widetilde{A}_1\big| < \frac{\varepsilon}{4}$ such that $G|_{\widetilde{A}_1 \times \mathbb{R}^N \times \mathbb{R}^{\widetilde{N}}}$ is continuous. In a further step we fix $M \in \mathbb{N}$ large enough that

$$\widetilde{A}_{2,\ell} := \{x \in \Omega \,:\, |\widetilde{w}_{k_\ell}(x)| < M \,,\, |w_{k_\ell}(x)| < M \,,\, |w(x)| < M\}$$

satisfies $\big|\Omega \setminus \widetilde{A}_{2,\ell}\big| < \frac{\varepsilon}{4}$ for all $\ell \in \mathbb{N}$. Again, choosing a single M, which works for all ℓ, is possible here, but now this results from the observation that $|\{x \in \Omega : |\widetilde{w}_{k_\ell}(x)| \geq M\}| \leq \frac{1}{M}\|\widetilde{w}_{k_\ell}\|_{1;\Omega}$ converges for $M \to \infty$ uniformly in $\ell \in \mathbb{N}$ to 0 thanks to the boundedness of the weakly convergent sequence $(\widetilde{w}_k)_{k\in\mathbb{N}}$ and from an analogous treatment of $|\{x \in \Omega : |w_{k_\ell}(x)| \geq M\}|$ and $|\{x \in \Omega : |w(x)| \geq M\}|$. As a preparation for the last step we exploit uniform continuity of G on the compact set $\{(x, y, \widetilde{y}) \in A_1 \times \mathbb{R}^N \times \mathbb{R}^{\widetilde{N}} : |\widetilde{y}| \leq M \,,\, |y| \leq M\}$ to fix $\delta > 0$ such that we have the implication

$$\left.\begin{array}{c} x \in \widetilde{A}_1 \,,\, |\widetilde{y}| < M \,,\, |y| < M \,,\, |y'| < M \,, \\ G(x, y, \widetilde{y}) < L \,,\, |y - y'| < \delta \end{array}\right\} \implies |G(x, y, \widetilde{y}) - G(x, y', \widetilde{y})| < \varepsilon$$

(where the L-bound ensures that the values of G stay away from ∞ and their distance can be measures in the usual Euclidean way). Then, observing that $|\{x \in \Omega : |w_{k_\ell}(x) - w(x)| \geq \delta\}| \leq \frac{1}{\delta}\|w_{k_\ell} - w\|_{1;\Omega}$ tends for $\ell \to \infty$ to 0 by strong convergence, we finally fix $\ell \in \mathbb{N}$ large enough that

$$\widetilde{A}_{3,\ell} := \{x \in \Omega \,:\, |w_{k_\ell}(x) - w(x)| < \delta\}$$

satisfies $\big|\Omega \setminus \widetilde{A}_{3,\ell}\big| < \frac{\varepsilon}{4}$. With these choices and the ℓ fixed in the last step, we introduce

$$A := \widetilde{A}_{0,\ell} \cap \widetilde{A}_1 \cap \widetilde{A}_{2,\ell} \cap \widetilde{A}_{3,\ell}$$

and observe $|\Omega \setminus A| < \varepsilon$. Moreover, for $x \in A$ the construction ensures $x \in A_1$, $|\widetilde{w}_{k_\ell}(x)| < M$, $|w_{k_\ell}(x)| < M$, $|w(x)| < M$, $G(x, w_{k_\ell}(x), \widetilde{w}_{k_\ell}(x)) < L$, $|w_{k_\ell}(x) - w(x)| < \delta$, and the choice of δ then gives $|G(x, w_{k_\ell}(x), \widetilde{w}_{k_\ell}(x)) - G(x, w(x), \widetilde{w}_{k_\ell}(x))| < \varepsilon$. Thus, we have verified the claim (∗∗).

Starting again from an arbitrary $\varepsilon > 0$, we next apply (∗∗) iteratively to find $\ell_1 < \ell_2 < \ldots$ in $\mathbb{N}$ and measurable $A_1, A_2, \ldots \subset \Omega$ such that, for every $m \in \mathbb{N}$, we have $|\Omega \setminus A_m| < 2^{-m}\varepsilon$ and $\big|G\big(\,\cdot\,, w_{k_{\ell_m}}, \widetilde{w}_{k_{\ell_m}}\big) - G\big(\,\cdot\,, w, \widetilde{w}_{k_{\ell_m}}\big)\big| < 2^{-m}\varepsilon$ on A_m. Introducing $B := \bigcap_{m=1}^\infty A_m$, we record $|\Omega \setminus B| < \varepsilon$ and $\lim_{m\to\infty}\big[G\big(\,\cdot\,, w_{k_{\ell_m}}, \widetilde{w}_{k_{\ell_m}}\big) - G\big(\,\cdot\,, w, \widetilde{w}_{k_{\ell_m}}\big)\big] = 0$ uniformly on B. We obtain

$$\lim_{\ell\to\infty} \mathcal{G}[w_{k_\ell}, \widetilde{w}_{k_\ell}] \geq \liminf_{m\to\infty} \int_B G\big(\,\cdot\,, w_{k_{\ell_m}}, \widetilde{w}_{k_{\ell_m}}\big)\,\mathrm{d}x = \liminf_{m\to\infty} \int_B G\big(\,\cdot\,, w, \widetilde{w}_{k_{\ell_m}}\big)\,\mathrm{d}x\,.$$

Next we notice that $\widetilde{G}(x, \widetilde{y}) := \mathbb{1}_B(x)G(x, w(x), \widetilde{y})$ gives a Carathéodory integrand $\widetilde{G}$ with $\widetilde{G}(x, \cdot\,)$ convex for a.e. $x \in \Omega$. Applying the last theorem to the corresponding functional and exploiting its weak lower semicontinuity along the weakly convergent (sub)sequence $\big(w_{k_{\ell_m}}\big)_{m\in\mathbb{N}}$ on the right-hand side of the last estimate, we arrive at

$$\lim_{\ell\to\infty} \mathcal{G}[w_{k_\ell}, \widetilde{w}_{k_\ell}] \geq \int_B G(\,\cdot\,, w, \widetilde{w})\,\mathrm{d}x\,.$$

In order to get back from B to Ω on the right-hand side, we now apply the result of the preceding reasoning to find, for every $i \in \mathbb{N}$, a measurable set $B_i \subset \Omega$ with $|\Omega \setminus B_i| < 2^{-i}$ such that

$$\lim_{\ell\to\infty} \mathcal{G}[w_{k_\ell}, \widetilde{w}_{k_\ell}] \geq \int_{B_i} G(\,\cdot\,, w, \widetilde{w})\,\mathrm{d}x\,.$$

holds. Introducing $\widetilde{B}_i := \bigcap_{j=i}^\infty B_j \subset B_i$ with $|\Omega \setminus \widetilde{B}_i| < 2^{1-i}$, we get that $\mathbb{1}_{\widetilde{B}_i}$ converges, for $i \to \infty$, a.e. on Ω monotonously from below to 1. Therefore, putting together the above and using the monotone convergence theorem in the last step, we arrive at

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{G}[w_k, \widetilde{w}_k] &= \lim_{\ell\to\infty} \mathcal{G}[w_{k_\ell}, \widetilde{w}_{k_\ell}] \\ &\geq \limsup_{i\to\infty} \int_{B_i} G(\,\cdot\,, w, \widetilde{w})\,\mathrm{d}x \geq \lim_{i\to\infty} \int_{\widetilde{B}_i} G(\,\cdot\,, w, \widetilde{w})\,\mathrm{d}x = \mathcal{G}[w, \widetilde{w}]\,.\end{aligned}$$

This finally proves the claimed sequential weak-strong lower semicontinuity of $\mathcal{G}$. □

As its **most important feature**, the weak-strong lower semicontinuity theorem yields a very natural result when **applied to *first-order* functionals**. This result is the culmination point of a long historical development, which ranges from the treatment of specific convex functionals by Tonelli [16] over an early general result by Morrey [12] and the weak-strong semicontinuity approach of De Giorgi [5] to modern and recent refinements.

Corollary (*weak* lower semicontinuity of convex-in-the-gradient 1^{st}-order functionals). *Consider an open set $\Omega \subset \mathbb{R}^n$ and a Carathéodory function $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to [0,\infty]$ such that $F(x,y,\,\cdot\,)\colon \mathbb{R}^{N\times n} \to [0,\infty]$ is convex on $\mathbb{R}^{N\times n}$ for all $(x,y) \in (\Omega\setminus E)\times\mathbb{R}^N$ with a null set $E \subset \Omega$. Then the functional $\mathcal{F}\colon \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N) \to [0,\infty]$, defined by*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,, w, \mathrm{D}w)\,\mathrm{d}x \qquad \text{for } w \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)\,,$$

is sequentially lower semicontinuous on $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ *with the* weak *topology.*

Proof. First assume that Ω is bounded with a Lipschitz boundary $\partial\Omega$ and consider a weakly convergent sequence $w_k \underset{k\to\infty}{\rightharpoonup} w$ in $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$. By Rellich's theorem, we infer strong convergence $w_k \underset{k\to\infty}{\longrightarrow} w$ in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$, and clearly we have weak convergence $\mathrm{D}w_k \underset{k\to\infty}{\rightharpoonup} \mathrm{D}w$ in $\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$. Identifying $\mathbb{R}^{N\times n}$ with $\mathbb{R}^{Nn}$, we can thus apply the weak-strong semicontinuity theorem to deduce

$$\mathcal{F}[w] = \int_\Omega F(\,\cdot\,, w, \mathrm{D}w)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega F(\,\cdot\,, w_k, \mathrm{D}w_k)\,\mathrm{d}x = \liminf_{k\to\infty} \mathcal{F}[w_k]$$

and obtain the claim.

Compact operators. A linear map $L\colon \mathcal{X} \to \mathcal{Y}$ between normed spaces $\mathcal{X}$ and $\mathcal{Y}$ is called compact if the image of the open unit ball in $\mathcal{X}$ under L is relatively compact in $\mathcal{Y}$ (with respect to the norm topology on $\mathcal{Y}$). For compact linear $L\colon \mathcal{X} \to \mathcal{Y}$, weak convergence $x_k \underset{k\to\infty}{\rightharpoonup} x$ in $\mathcal{X}$ implies strong convergence $Lx_k \underset{k\to\infty}{\longrightarrow} Lx$ in $\mathcal{Y}$.

Rellich's theorem. Rellich's theorem asserts the following: For every *bounded* open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and $p \in [1,\infty]$, the inclusion $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N) \to \mathrm{L}^p(\Omega,\mathbb{R}^N)$ is a *compact* linear map. Moreover, in case of zero boundary values, the Lipschitz assumption can be dropped, that is, for every *bounded* open set $\Omega \subset \mathbb{R}^n$ and $p \in [1,\infty)$, the inclusion $\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N) \to \mathrm{L}^p(\Omega,\mathbb{R}^N)$ is a *compact* linear map (where the case $p = \infty$ has been excluded in the second assertion only for the technical reason that a definition of $\mathrm{W}^{1,\infty}_0$ has not been given here).

If Ω is an arbitrary open set, we can find an exhaustion $\Omega = \bigcup_{\ell=1}^\infty \widetilde{\Omega}_\ell$ of Ω, where $\widetilde{\Omega}_\ell \Subset \Omega$ are open sets with Lipschitz boundaries $\partial\widetilde{\Omega}_\ell$ (e.g. each $\widetilde{\Omega}_\ell$ the interior of a finite union of closed axi-parallel cubes) and satisfy $\widetilde{\Omega}_1 \subset \widetilde{\Omega}_2 \subset \widetilde{\Omega}_3 \subset \ldots$. If $w_k \underset{k\to\infty}{\rightharpoonup} w$ converges weakly in $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$, the same is true in $\mathrm{W}^{1,1}(\widetilde{\Omega}_\ell,\mathbb{R}^N)$, and the preceding reasoning ensures

$$\int_{\widetilde{\Omega}_\ell} F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_{\widetilde{\Omega}_\ell} F(\,\cdot\,,w_k,\mathrm{D}w_k)\,\mathrm{d}x \le \liminf_{k\to\infty} \mathcal{F}[w_k]$$

for all $\ell \in \mathbb{N}$. Monotone convergence then implies $\mathcal{F}[w] \le \liminf_{k\to\infty} \mathcal{F}[w_k]$ and gives the claim in the general case. $\square$

Finally, we show that **convexity** is, at least **for zero-order functionals**, not only sufficient but **also necessary for weak lower semicontinuity in $\mathbf{L}^p$ spaces**. While clearly the necessity statement is not needed in order to prove existence of minimizers, it still clarifies the role of convexity as the optimal assumption in order to ensure the required semicontinuity property.

Theorem (necessity of convexity for weak lower semicontinuity in zero-order case). *Consider an open subset Ω of $\mathbb{R}^n$ and a Carathéodory integrand $G\colon \Omega\times\mathbb{R}^N \to [0,\infty]$ with $G(\,\cdot\,,y) \in \mathrm{L}^1(\Omega)$ for all $y \in \mathbb{R}^N$. If the functional $\mathcal{G}$, given by*

$$\mathcal{G}[w] := \int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x \qquad\qquad \textit{for } w \in \mathrm{L}^\infty(\Omega,\mathbb{R}^N)\,,$$

is sequentially weakly$*$ lower semicontinuous on $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$, then $G(x,\,\cdot\,)\colon \mathbb{R}^N \to [0,\infty]$ is convex on $\mathbb{R}^N$ for a.e. $x \in \Omega$.

Remark (on **convexity as an optimal assumption**). Consider a zero-order functional $\mathcal{G}$ with non-negative Carathéodory integrand G and $G(\,\cdot\,,y) \in \mathrm{L}^1(\Omega)$ for $y \in \mathbb{R}^N$ as above. Then, from the above theorem on the necessity and the earlier result on the sufficiency of convexity for weak lower semicontinuity, we have:

$$\begin{aligned}&\mathcal{G} \text{ sequentially weakly$*$ lower semicontinuous on } \mathrm{L}^\infty(\Omega,\mathbb{R}^N)\\ &\qquad\implies G(x,\,\cdot\,) \text{ convex on } \mathbb{R}^N \text{ for a.e. } x\in\Omega\\ &\qquad\qquad\qquad\implies \mathcal{G} \text{ weakly lower semicontinuous on } \mathrm{L}^1(\Omega,\mathbb{R}^N)\,.\end{aligned}$$

However, at least for $|\Omega| < \infty$, weak$*$ convergence in $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$ implies weak convergence in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$, thus weak lower semicontinuity on $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ implies sequential weak$*$ lower

Weak$*$ topology. The weak$*$ topology on the dual $\mathcal{X}^*$ of a normed space $\mathcal{X}$ is the coarsest/weakest topology on $\mathcal{X}^*$ in which all evaluation functionals $\langle\,\cdot\,;x\rangle\colon \mathcal{X}^* \to \mathbb{R}$ with $x \in \mathcal{X}$ are continuous. In other words, sets of the form $\{y^* \in \mathcal{X}^* : |\langle y^*{-}x^*;x_k\rangle| < \varepsilon \text{ for } k=1,2,\ldots,\ell\}$ with $\ell \in \mathbb{N}$, $x_1,x_2,\ldots,x_\ell \in \mathcal{X}$, $\varepsilon > 0$ are weakly open basis neighborhoods of a point $x^* \in \mathcal{X}^*$, and every weakly$*$ open set in $\mathcal{X}^*$ is a union of such sets.

Weak$*$ convergence. The convergence of a sequence $(x_k^*)_{k\in\mathbb{N}}$ in the dual $\mathcal{X}^*$ of a normed space $\mathcal{X}$ to a limit $x^* \in \mathcal{X}^*$ in the weak$*$ topology of $\mathcal{X}^*$ means $\lim_{k\to\infty}\langle x_k^*;x\rangle = \langle x^*;x\rangle$ for all $x \in \mathcal{X}$. It is expressed by writing $x_k \overset{*}{\rightharpoonup} x$ weakly$*$ in $\mathcal{X}^*$.

Weak$*$ convergence in $\mathbf{L}^\infty$ and $\mathbf{W}^{1,\infty}$. Consider a measurable $\Omega \subset \mathbb{R}^n$. Weak$*$ convergence $w_k \overset{*}{\rightharpoonup} w$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$, which is understood as the dual of $\mathrm{L}^1(\Omega,\mathbb{R}^N)$, is characterized by $\lim_{k\to\infty}\int_\Omega w_k \cdot v\,\mathrm{d}x = \int_\Omega w\cdot v\,\mathrm{d}x$ for all $v \in \mathrm{L}^1(\Omega,\mathbb{R}^N)$. Weak$*$ convergence $w_k \overset{*}{\rightharpoonup} w$ in $\mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^N)$ is defined as weak$*$ convergences $w_k \overset{*}{\rightharpoonup} w$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$ together with weak$*$ convergence $\partial_i w_k \overset{*}{\rightharpoonup} \partial_i w$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$ for all $i \in \{1,2,\ldots,n\}$.

semicontinuity on $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$, and the **three statements** are **actually equivalent**. Moreover, (sequential) weak lower semicontinuity of $\mathcal{G}$ on any intermediate space $\mathrm{L}^p(\Omega,\mathbb{R}^N)$, $p\in[1,\infty)$, is also equivalent, and the **convexity** assumption on G is indeed a **necessary and sufficient** criterion **for any of these weak semicontinuity properties**.

The proof the theorem can be based on the following weak convergence lemma:

Lemma. *We write Q for the unit cube $(0,1)^n$ in $\mathbb{R}^n$ and consider $p\in[1,\infty]$ and $w\in\mathrm{L}^p(Q,\mathbb{R}^N)$. If w is Q-periodically extended to $\mathbb{R}^n$ (which means that w is defined a.e. on $\mathbb{R}^n$ by the setting $w(z+x):=w(x)$ for $z\in\mathbb{Z}^n$, $x\in Q$) and $w_k\in\mathrm{L}^p(Q,\mathbb{R}^N)$, $k\in\mathbb{N}$, are given by*

$$w_k(x):=w(kx)\qquad\text{for } x\in Q\,,$$

then with $w_Q:=⨍_Q w\,\mathrm{d}x=\int_Q w\,\mathrm{d}x$ we have

$$\begin{cases} w_k \xrightharpoonup[k\to\infty]{} w_Q \text{ weakly in } \mathrm{L}^p(Q,\mathbb{R}^N) & \text{if } p<\infty\,,\\ w_k \overset{*}{\underset{k\to\infty}{\rightharpoonup}} w_Q \text{ weakly}* \text{ in } \mathrm{L}^\infty(Q,\mathbb{R}^N) & \text{if } p=\infty\,. \end{cases}$$

A proof of the lemma has been discussed in the exercise class.

Proof of the theorem. Set $Q:=(0,1)^n$. For measurable $A\subset\Omega\cap Q$, we first claim that the semicontinuity assumption transfers to the subset in the form

$$w_k \overset{*}{\underset{k\to\infty}{\rightharpoonup}} w \text{ weakly}* \text{ in } \mathrm{L}^\infty(Q,\mathbb{R}^N)\qquad\Longrightarrow\qquad \int_A G(\,\cdot\,,w)\,\mathrm{d}x\le\liminf_{k\to\infty}\int_A G(\,\cdot\,,w_k)\,\mathrm{d}x\,.$$

Indeed, from $w_k \overset{*}{\underset{k\to\infty}{\rightharpoonup}} w$ weakly$*$ in $\mathrm{L}^\infty(Q,\mathbb{R}^N)$ and $A\subset\Omega\cap Q$ we deduce first $\mathbb{1}_Aw_k \overset{*}{\underset{k\to\infty}{\rightharpoonup}} \mathbb{1}_Aw$ weakly$*$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$ and then by assumption $\mathcal{G}[\mathbb{1}_Aw]\le\liminf_{k\to\infty}\mathcal{G}[\mathbb{1}_Aw_k]$. In other words, this means $\int_A G(\,\cdot\,,w)\,\mathrm{d}x+\int_{\Omega\setminus A}G(\,\cdot\,,0)\,\mathrm{d}x\le\liminf_{k\to\infty}\int_A G(\,\cdot\,,w_k)\,\mathrm{d}x+\int_{\Omega\setminus A}G(\,\cdot\,,0)\,\mathrm{d}x$, and the subtraction of $\int_{\Omega\setminus A}G(\,\cdot\,,0)\,\mathrm{d}x$ (which is finite by the L^1 assumption) on both sides gives the first claim.

Now we fix a measurable $A\subset\Omega\cap Q$, $y_1,y_2\in\mathbb{R}^N$, and $\lambda\in[0,1]$. Then we choose a decomposition $Q=Q_1\cup Q_2$ of Q into disjoint measurable sets Q_1 and Q_2 with $|Q_1|=\lambda$ and $|Q_2|=1{-}\lambda$ (which can be taken simply as $Q_1=(0,\lambda)\times(0,1)^{n-1}$, $Q_2=[\lambda,1)\times(0,1)^{n-1}$). We introduce first

$$w:=\mathbb{1}_{Q_1}y_1+\mathbb{1}_{Q_2}y_2\in\mathrm{L}^\infty(Q,\mathbb{R}^N)$$

with $w_Q=|Q_1|y_1{+}|Q_2|y_2=\lambda y_1{+}(1{-}\lambda)y_2$. Then, using the Q-periodic extension of w to $\mathbb{R}^n$, we set $w_k(x):=w(kx)$ for $x\in Q$. By the preceding lemma we have weak$*$ convergence $w_k \overset{*}{\underset{k\to\infty}{\rightharpoonup}} \lambda y_1{+}(1{-}\lambda)y_2$ in $\mathrm{L}^\infty(Q,\mathbb{R}^N)$, and by the semicontinuity discussed above we get

$$\int_A G(x,\lambda y_1{+}(1{-}\lambda)y_2)\,\mathrm{d}x\le\liminf_{k\to\infty}\int_A G(x,w(kx))\,\mathrm{d}x\,.$$

Observing that $w(kx)$ equals y_1 and y_2, respectively, for $kx\in\mathbb{Z}^n{+}Q_1$ and $kx\in\mathbb{Z}^n{+}Q_2$, we rewrite this as

$$\int_A G(x,\lambda y_1{+}(1{-}\lambda)y_2)\,\mathrm{d}x\le\liminf_{k\to\infty}\left[\int_A \mathbb{1}_{\mathbb{Z}^n+Q_1}(kx)G(x,y_1)\,\mathrm{d}x+\int_A \mathbb{1}_{\mathbb{Z}^n+Q_2}(kx)G(x,y_2)\,\mathrm{d}x\right].$$

Since $\mathbb{1}_{\mathbb{Z}^n+Q_1}$ and $\mathbb{1}_{\mathbb{Z}^n+Q_2}$ are the Q-periodic extensions of $\mathbb{1}_{Q_1}$ and $\mathbb{1}_{Q_2}$ on Q, the lemma implies that $x \mapsto \mathbb{1}_{\mathbb{Z}^n+Q_1}(kx)$ and $x \mapsto \mathbb{1}_{\mathbb{Z}^n+Q_2}(kx)$ weak$*$ converge in $\mathrm{L}^\infty(Q)$ to $(\mathbb{1}_{Q_1})_Q = |Q_1| = \lambda$ and $(\mathbb{1}_{Q_2})_Q = |Q_2| = 1{-}\lambda$, respectively. Using these convergences together with the assumption that $G(\,\cdot\,,y_1), G(\,\cdot\,,y_2) \in \mathrm{L}^1(\Omega)$, we end up with

$$\int_A G(x,\lambda y_1{+}(1{-}\lambda)y_2)\,\mathrm{d}x \le \lambda \int_A G(x,y_1)\,\mathrm{d}x + (1{-}\lambda)\int_A G(x,y_2)\,\mathrm{d}x\,.$$

Since A is an arbitrary measurable subset of $\Omega \cap Q$, we conclude

$$G(\,\cdot\,,\lambda y_1{+}(1{-}\lambda)y_2) \le \lambda G(\,\cdot\,,y_1) + (1{-}\lambda)G(\,\cdot\,,y_2) \qquad \text{a.e. on } \Omega\cap Q\,.$$

By translation or an analogous reasoning, one can replace the unit cube Q with a cube $z{+}Q$, $z \in \mathbb{Z}^n$, and obtain the same inequality a.e. on $\Omega \cap (z{+}Q)$. Then, since the countable union $\bigcup_{z\in\mathbb{Z}^n}(\Omega\cap(z{+}Q))$ coincides with Ω up to a null set, we get the inequality a.e. on Ω, and we also get it simultaneously for all $y_1,y_2 \in \mathbb{Q}^n$ and $\lambda \in [0,1]\cap\mathbb{Q}$ on $\Omega\setminus E$ with a common null set E. Relying on continuity of $G(x,\,\cdot\,)$ for all $x \in \Omega\setminus\widetilde{E}$ with another null set $\widetilde{E}$, we finally end up with still the same inequality for all $y_1,y_2 \in \mathbb{R}^n$ and $\lambda\in[0,1]$ on $\Omega\setminus(E\cup\widetilde{E})$. This proves convexity of $G(x,\,\cdot\,)$ for all $x\in\Omega\setminus(E\cup\widetilde{E})$ and in view of $|E\cup\widetilde{E}|=0$ yields the claim. □

Remark. For ***first-order*** **functionals**

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x$$

(with open $\Omega \subset \mathbb{R}^n$ and $\mathbb{R}^N$-valued ws on Ω as usual), it will eventually turn out that **convexity** of F in the gradient variable is **necessary for weak($*$) lower semicontinuity of $\mathcal{F}$ *only in case* $\min\{N,n\}=1$**, which means that the setting is **either one-dimensional or scalar**. In case $\min\{N,n\}\ge 2$, in contrast, there is room for improvement on the semicontinuity results of this section, and indeed we will prove at a later stage that weak lower semicontinuity of $\mathcal{F}$ is still valid if the integrand satisfies a generalized convexity condition known as **quasiconvexity**; see the later Section 5.2

2.3 Coercivity, weak compactness, and existence

We introduce one more notion which is needed to complete the existence program.

Definition (**coercivity**)**.** *Consider a normed space $\mathcal{X}$, a subset $\mathcal{A}$ of $\mathcal{X}$, and a functional $\mathcal{F}\colon \mathcal{A}\to\overline{\mathbb{R}}$ on $\mathcal{A}$. We call $\mathcal{F}$* ***coercive*** *on $\mathcal{A}$, or more precisely* ***$\mathcal{X}$-coercive*** *on $\mathcal{A}$, if it holds*

$$\lim_{\substack{\|w\|_{\mathcal{X}}\to\infty\\ w\in\mathcal{A}}} \mathcal{F}[w] = \infty$$

(*which means by the very definition of the limit that, for every $\varepsilon>0$, there exists some $\delta>0$ such that every $w\in\mathcal{A}$ with $\|w\|_{\mathcal{X}}>\frac{1}{\delta}$ satisfies $\mathcal{F}[w]>\frac{1}{\varepsilon}$*).

Remark (on **characterizations of coercivity**)**.** For a normed space $\mathcal{X}$, a subset $\mathcal{A}$ of $\mathcal{X}$, and a functional $\mathcal{F}\colon\mathcal{A}\to\overline{\mathbb{R}}$, we have:

$\mathcal{F}$ is $\mathcal{X}$-coercive on $\mathcal{A}$.

$\iff$ Every sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ with $\sup\limits_{k\in\mathbb{N}}\mathcal{F}[w_k]<\infty$ satisfies $\sup\limits_{k\in\mathbb{N}}\|w_k\|_{\mathcal{X}}<\infty$.

$\iff$ Every sublevel set $\{w\in\mathcal{A} : \mathcal{F}[w]\le s\}$ with $s\in\mathbb{R}$ is bounded.

The proof is elementary and will be discussed in the exercises.

Thus, coercivity actually ensures boundedness of the sublevel sets of $\mathcal{F}$, while indeed, as discussed in Section 2.1, we need relative compactness of these sublevel sets in order to complete the existence proof. If we involve — what is crucial at this point — the weak topology (on a slightly restricted class of spaces $\mathcal{X}$), however, then the next result shows that closed balls are weakly compact, and as a consequence all bounded sets are weakly relatively compact as required. In this way coercivity, though the notion itself depends only on the norm and does not involve any weak topology whatsoever, gives exactly the required weak compactness condition(s), and on this basis the existence program can then be completed. We now work this out:

Theorem (Banach-Alaoglu theorem, weak compactness).
Weak version: *Every* ***closed ball in a*** **reflexive** ***Banach space*** *is* ***weakly compact****.*
Weak$*$ version: *Every* ***closed ball in a*** **dual** *of a normed space is* ***weakly$*$ compact****.*

Theorem (sequential weak compactness).
Weak version: *Every closed ball in a reflexive Banach space is sequentially weakly compact, which means that every* ***bounded sequence in a reflexive Banach space*** *has a* ***weakly convergent subsequence****.*
Weak$*$ version: *Every closed ball in a dual of a* separable[11] *normed space is sequentially weakly$*$ compact, which means that every* ***bounded sequence in a dual*** *of a* separable *normed space has a* ***weakly$*$ convergent subsequence****.*

We will not discuss the proofs of these compactness statements, which are a matter of functional analysis. We mention, however, that the proof of the sequential statements is more elementary and can essentially be based on the choice of a countable dense subset and a diagonal sequence.

Corollary (Coercivity gives weak compactness.). *Consider a* ***reflexive*** *Banach space $\mathcal{X}$, a subset $\mathcal{A} \subset \mathcal{X}$, and a functional $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$.*

(I) *If $\mathcal{A}$ is weakly closed in $\mathcal{X}$, we have:*

$\mathcal{F}$ *is $\mathcal{X}$-coercive on $\mathcal{A}$.* $\iff$ *Every sublevel set $\{w \in \mathcal{A} : \mathcal{F}[w] \le s\}$ with $s \in \mathbb{R}$ is weakly relatively compact in $\mathcal{A}$, that is, condition (Ia) of Section 2.1 holds for the weak topology.*

(II) *If $\mathcal{A}$ is merely sequentially weakly closed in $\mathcal{X}$, we still have:*

$\mathcal{F}$ *is $\mathcal{X}$-coercive on $\mathcal{A}$.* $\iff$ *Every sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ with $\sup_{k\in\mathbb{N}} \mathcal{F}[w_k] < \infty$ has a weakly convergent subsequence with limit in $\mathcal{A}$, that is, condition (IIa) of Section 2.1 holds for the weak topology.*

Moreover, if $\mathcal{A} \subset \mathcal{X}^*$ lies in the dual $\mathcal{X}^*$ of a separable Banach space $\mathcal{X}$, then $\mathcal{X}^*$-coercivity of $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$ is characterized in the same way with $\mathcal{X}^*$ replacing $\mathcal{X}$ and weakly$*$ replacing weakly everywhere in (I) and (II).

[11] While both weak$*$ compactness and sequential weak compactness do *not* require any separability assumption, somewhat surprisingly, the separability assumption on the predual cannot be dropped from the sequential weak$*$ compactness theorem. The latter can be seen at hand of the functionals $e_k \in \left(\ell^\infty\right)^*$ on the non-separable space ℓ^∞ given by $\langle e_k\,;(x_j)_{j\in\mathbb{N}}\rangle = x_k$. Indeed, $(e_k)_{k\in\mathbb{N}}$ stays in the unit ball of $\left(\ell^\infty\right)^*$, but for every subsequence $(e_{k_\ell})_{\ell\in\mathbb{N}}$, the limit $\lim_{\ell\to\infty}\langle e_{k_\ell}\,;(x_j)_{j\in\mathbb{N}}\rangle$ does not exist for a sequence $(x_j)_{j\in\mathbb{N}} \in \ell^\infty$ with $x_{k_\ell} = (-1)^\ell$ (and x_j arbitrary for $j \notin \{k_\ell : \ell \in \mathbb{N}\}$).

Proof. Since coercivity is characterized by boundedness of sublevel sets (as remarked above), in order to derive the statement (I) we only need to show for $S \subset \mathcal{A}$ the equivalence

$$S \text{ bounded} \iff S \text{ weakly relatively compact in } \mathcal{A}\,.$$

We now prove the forward implication of this. We assume S is bounded and get $S \subset B$ for a suitably large closed ball B in $\mathcal{X}$. By the Banach-Alaoglu theorem, B is weakly compact, and then the weak closure of S in $\mathcal{X}$ is a weakly closed subset of B and thus also weakly compact. Since $\mathcal{A}$ is weakly closed, the weak closure of S in $\mathcal{A}$ is actually the same as its closure in $\mathcal{X}$, and thus S is weakly relatively compact in $\mathcal{A}$. Turning to the backward implication, we assume that S is weakly relatively compact in $\mathcal{A}$. Then, since every fixed $x^* \in \mathcal{X}^*$ is weakly continuous, the image $\{\langle x^*;x\rangle : x \in S\}$ is relatively compact in $\mathbb{R}$ and thus $\sup_{x\in S}|\langle x^*;x\rangle| < \infty$ for all $x^* \in \mathcal{X}^*$. The last property is known as weak boundedness of S and implies its boundedness by the uniform boundedness principle from functional analysis. All in all, we have proved the above equivalence and statement (I).

In order to establish the statement (II), it suffices (since coercivity still means boundedness of sublevel sets and the condition $\sup_{k\in\mathbb{N}} \mathcal{F}[w_k] < \infty$ means that $(w_k)_{k\in\mathbb{N}}$ stays in a sublevel set) to prove for $S \subset \mathcal{A}$ the equivalence

$$S \text{ bounded} \iff \text{Every sequence in } S \text{ has a weakly convergent subsequence with limit in } \mathcal{A}.$$

Here, the forward implication follows by observing that a sequence $(w_k)_{k\in\mathbb{N}}$ in the bounded set S has a weakly convergent subsequence by the sequential Banach-Alaoglu theorem and that, by sequential weak closedness of $\mathcal{A}$, the limit stays in $\mathcal{A}$. For the backward implication, assume that every sequence in S has a weakly convergent subsequence. In particular, since the uniform boundedness principle ensures boundedness of weakly convergent sequences, every sequence in S has a bounded subsequence. This implies boundedness of S (since in every unbounded set there exists a sequence $(w_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \|w_k\|_{k\in\mathbb{N}} = \infty$).

The weak$*$ versions of the statements for $\mathcal{A} \subset \mathcal{X}^*$ can be obtained analogously (where the completeness of the Banach space $\mathcal{X}$ is needed in order to suitably apply the uniform boundedness principle in the proof of the backward implications). □

At this stage, we are finally **left with the task to verify coercivity** of a given (integral) functional. However, this is possible in many relevant cases and usually not too difficult:

Proposition (coercivity criteria). *Consider a non-empty open set Ω in $\mathbb{R}^n$.*

(0) *Assume, for an $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)$-measurable $G\colon \Omega\times\mathbb{R}^N \to \overline{\mathbb{R}}$, that*

$$G(x,y) \ge \gamma|y|^p - \Psi(x) \qquad \text{holds for all } (x,y) \in (\Omega\setminus E)\times\mathbb{R}^N\,,$$

where $\gamma > 0$, $\Psi \in \mathrm{L}^1(\Omega)$, $p \in [1,\infty)$, and a null set $E \subset \Omega$ are fixed. Then, the zero-order functional $\mathcal{G}$ given by

$$\mathcal{G}[w] := \int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x$$

is L^p-coercive on every subset $\mathcal{A}$ of $\mathrm{L}^p(\Omega,\mathbb{R}^N)$.

Uniform boundedness principle. The general uniform boundedness principle says, for a Banach space $\mathcal{X}$, a normed space $\mathcal{Y}$, and $\mathscr{F} \subset \mathcal{L}(\mathcal{X},\mathcal{Y})$, that $\sup_{T\in\mathscr{F}} \|Tx\|_{\mathcal{Y}} < \infty$ for all $x \in \mathcal{X}$ implies $\sup_{T\in\mathscr{F}} \|T\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} < \infty$.

Specifically, if $\mathcal{X}$ is a Banach space and $S \subset \mathcal{X}^*$ is weakly$*$ bounded in the sense of $\sup_{x^*\in S}|\langle x^*;x\rangle| < \infty$ for all $x \in \mathcal{X}$, this applies with $\mathcal{Y} = \mathbb{R}$ and gives boundedness of S.

Moreover, if $\mathcal{X}$ is a normed space and $S \subset \mathcal{X}$ is weakly bounded in the sense of $\sup_{x\in S}|\langle x^*;x\rangle| < \infty$ for all $x^* \in \mathcal{X}^*$, then the image $\mathrm{J}_{\mathcal{X}}(S)$ under the canonical isometric embedding $\mathrm{J}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}^{**}$ is weakly$*$ bounded in $\mathcal{X}^{**}$ and the above with the Banach space $\mathcal{X}^*$ in place of $\mathcal{X}$ gives boundedness of $\mathrm{J}_{\mathcal{X}}(S)$ and thus of S.

(1) *Assume, for an* $\mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^{N\times n})$*-measurable* $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$*, that one of the following conditions holds (where* $\gamma > 0$*,* $p \in [1,\infty)$*,* $\Psi \in \mathrm{L}^1(\Omega)$*, and a null set* $E \subset \Omega$ *are always fixed):*

(a) $\mathcal{A}$ *is an arbitrary subset of* $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ *and*

$$F(x,y,z) \ge \gamma(|y|^p + |z|^p) - \Psi(x) \qquad \textit{for all } (x,y,z) \in (\Omega \setminus E) \times \mathbb{R}^N \times \mathbb{R}^{N\times n},$$

(b) Ω *is a bounded Lipschitz domain,* $\mathcal{A}$ *is an arbitrary subset of* $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$*, and*

$$F(x,y,z) \ge \gamma(|y| + |z|^p) - \Psi(x) \qquad \textit{for all } (x,y,z) \in (\Omega \setminus E) \times \mathbb{R}^N \times \mathbb{R}^{N\times n},$$

(c) Ω *is contained in a strip*[12] *of finite width,* $\mathcal{A}$ *is a subset of a Dirichlet class* $\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N)$ *with* $u_0 \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$*, and*

$$F(x,y,z) \ge \gamma|z|^p - \Psi(x) \qquad \textit{for all } (x,y,z) \in (\Omega \setminus E) \times \mathbb{R}^N \times \mathbb{R}^{N\times n},$$

(d) Ω *is a bounded Lipschitz domain,* $\mathcal{A}$ *is a subset of* $\{w \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N) : |w_\Omega| \le M\}$ *with* $M \in \mathbb{R}$*, and*

$$F(x,y,z) \ge \gamma|z|^p - \Psi(x) \qquad \textit{for all } (x,y,z) \in (\Omega \setminus E) \times \mathbb{R}^N \times \mathbb{R}^{N\times n}.$$

In each of these cases, the first-order functional $\mathcal{F}$ *given by*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,, w, \mathrm{D}w)\,\mathrm{d}x$$

is then $\mathrm{W}^{1,p}$*-coercive on* $\mathcal{A}$*.*

Remarks (on the coercivity criteria)**.**

(1) In case $|\Omega| < \infty$, the proposition can be applied and is often applied with a constant Ψ.

(2) In basic cases with $F(x,y,z) = \widetilde{F}(x,z)$ finite-valued and independent of y, the conditions in (1a) and (1b) can never be satisfied (since the right-hand side grows to ∞, when $|y|$ tends to ∞ with (x,z) fixed, while the left-hand side stays constant). This is the main motivation for considering (1c) and (1d), which are actually designed for such cases.

(3) The proposition provides coercivity in L^p or $\mathrm{W}^{1,p}$ in many reasonable cases. Nonetheless, the above list of cases is far from being complete and could be extended in many ways.

Proof of the proposition, part (0). For $w \in \mathcal{A} \subset \mathrm{L}^p(\Omega,\mathbb{R}^N)$, we estimate

$$\mathcal{G}[w] = \int_\Omega G(\,\cdot\,, w)\,\mathrm{d}x \ge \int_\Omega (\gamma|w|^p - \Psi)\,\mathrm{d}x \ge \gamma\|w\|^p_{\mathrm{L}^p;\Omega} - \|\Psi\|_{\mathrm{L}^1;\Omega} \underset{\|w\|_{\mathrm{L}^p;\Omega}\to\infty}{\longrightarrow} \infty$$

and gain coercivity of $\mathcal{G}$. □

[12] By saying that Ω lies in a strip of finite width we man that there exists a direction vector $\nu \in \mathbb{R}^n$, $|\nu| = 1$ and $a < b$ in $\mathbb{R}$ such that $a < \nu \cdot x < b$ for all $x \in \Omega$, that is, Ω lies in the strip $\{x \in \mathbb{R}^n : a < \nu \cdot x < b\}$ of width $b-a$.

In the **first-order situation** (1), a similar reasoning applies, but this additionally **involves** in all but the simplest cases a **Poincaré inequality**:

Proof of the proposition, part (1). Under the assumption (1a), for $w \in \mathcal{A} \subset \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$, we get

$$\mathcal{F}[w] \geq \gamma \int_\Omega (|w|^p + |\mathrm{D}w|^p)\,\mathrm{d}x - \int_\Omega \Psi\,\mathrm{d}x \geq \gamma\, c(n,p)\|w\|^p_{\mathrm{W}^{1,p};\Omega} - \|\Psi\|_{\mathrm{L}^1;\Omega}\,.$$

Under (1b), for $w \in \mathcal{A} \subset \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$, we have

$$\begin{aligned}\mathcal{F}[w] &\geq \gamma \int_\Omega (|w| + |\mathrm{D}w|^p)\,\mathrm{d}x - \int_\Omega \Psi\,\mathrm{d}x \\ &\geq \gamma(\|\mathrm{D}w\|_{\mathrm{L}^p;\Omega} + |\Omega||w_\Omega|) - \gamma - \|\Psi\|_{\mathrm{L}^1;\Omega} \geq \gamma\, c(n,p,\Omega)\|w\|_{\mathrm{W}^{1,p};\Omega} - \gamma - \|\Psi\|_{\mathrm{L}^1;\Omega}\,,\end{aligned}$$

where in addition to the estimates $|w|_\Omega \geq |w_\Omega|$ and $\|\mathrm{D}w\|^p_{\mathrm{L}^p;\Omega} \geq \|\mathrm{D}w\|_{\mathrm{L}^p;\Omega} - 1$ we also used $\|w\|_{\mathrm{W}^{1,p};\Omega} \leq C(n,p,\Omega)(\|\mathrm{D}w\|_{\mathrm{L}^p;\Omega} + |\Omega||w_\Omega|)$ thanks to the Poincaré inequality for the zero-mean case.

Under (1c), for $w \in \mathcal{A} \subset \mathrm{W}^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$, we find

$$\begin{aligned}\mathcal{F}[w] &\geq \gamma \int_\Omega |\mathrm{D}w|^p\,\mathrm{d}x - \int_\Omega \Psi\,\mathrm{d}x \\ &= \gamma\|\mathrm{D}w\|^p_{\mathrm{L}^p;\Omega} - \|\Psi\|_{\mathrm{L}^1;\Omega} \geq \gamma(c(n,p,\Omega)\|w\|^p_{\mathrm{W}^{1,p};\Omega} - \|u_0\|^p_{\mathrm{W}^{1,p};\Omega}) - \|\Psi\|_{\mathrm{L}^1;\Omega}\,,\end{aligned}$$

where we used that $\|w\|^p_{\mathrm{W}^{1,p};\Omega} \leq C(n,p,\Omega)(\|\mathrm{D}w\|^p_{\mathrm{L}^p;\Omega} + \|u_0\|^p_{\mathrm{W}^{1,p};\Omega})$ thanks to the Poincaré inequality for the zero-boundary-values case.

Under (1d), finally, the functions $w \in \mathcal{A}$ satisfy $|\int_\Omega w\,\mathrm{d}x| \leq M$, and we infer

$$\begin{aligned}\mathcal{F}[w] &\geq \gamma \int_\Omega |\mathrm{D}w|^p\,\mathrm{d}x - \int_\Omega \Psi\,\mathrm{d}x \\ &= \gamma\|\mathrm{D}w\|^p_{\mathrm{L}^p;\Omega} - \|\Psi\|_{\mathrm{L}^1;\Omega} \geq \gamma(c(n,p,\Omega)\|w\|^p_{\mathrm{W}^{1,p};\Omega} - M^p) - \|\Psi\|_{\mathrm{L}^1;\Omega}\,,\end{aligned}$$

where we used that $\|w\|^p_{\mathrm{W}^{1,p};\Omega} \leq C(n,p,\Omega)(\|\mathrm{D}w\|^p_{\mathrm{L}^p;\Omega} + M^p)$ thanks to the Poincaré inequality for the zero-mean case.

In all cases, $\mathrm{W}^{1,p}$-coercivity is then easily read off from the given estimate. □

Finally, we are ready to state the **decisive existence result**, which requires a **coercivity** criterion **and** a **convexity** hypothesis **as** its **principal assumptions**:

Poincaré inequalities. If an open subset Ω of $\mathbb{R}^n$ is contained in a strip of finite width ℓ, the Poincaré inequality in the **zero-boundary-values case**

$$\|w\|_{\mathrm{L}^p;\Omega} \leq C(n,p)\ell\|\mathrm{D}w\|_{\mathrm{L}^p;\Omega} \qquad \text{holds for all } w \in \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)\,.$$

If Ω is a non-empty bounded Lipschitz domain in $\mathbb{R}^n$, the Poincaré inequality in the **zero-mean-value case**

$$\|w - w_\Omega\|_{\mathrm{L}^p;\Omega} \leq C(n,p,\Omega)\|\mathrm{D}w\|_{\mathrm{L}^p;\Omega} \qquad \text{holds for all } w \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)\,.$$

Corollary (general existence theorem)**.** *Fix a non-empty open set $\Omega \subset \mathbb{R}^n$.*

(0) *Consider a Carathéodory integrand $G\colon \Omega\times\mathbb{R}^N \to [0,\infty]$ and a non-empty sequentially weakly closed set $\mathcal{A} \subset \mathrm{L}^p(\Omega,\mathbb{R}^N)$ such that $G(x,\,\cdot\,)$ is convex for a.e. $x \in \Omega$ and such that the assumption of part (0) in the preceding proposition is satisfied with $\boldsymbol{1 < p < \infty}$. Then, for the functional $\mathcal{G}$ given by*
$$\mathcal{G}[w] := \int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x\,,$$
there exists a minimizer *$u \in \mathcal{A}$ with $\mathcal{G}[u] \le \mathcal{G}[w]$ for all $w \in \mathcal{A}$.*

(1) *Consider a Carathéodory integrand $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to [0,\infty]$ and a non-empty sequentially weakly closed set $\mathcal{A} \subset \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ such that $F(x,y,\,\cdot\,)$ is convex for all $(x,y) \in (\Omega\setminus S)\times\mathbb{R}^N$ with a null set S and such that one set of assumptions on F and $\mathcal{A}$ from part (1) of the preceding proposition is satisfied with $\boldsymbol{1 < p < \infty}$. Then, for the functional $\mathcal{F}$ given by*
$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x\,,$$
there exists a minimizer *$u \in \mathcal{A}$ with $\mathcal{F}[u] \le \mathcal{F}[w]$ for all $w \in \mathcal{A}$.*

Proof. The previous proposition guarantees that $\mathcal{G}$ and $\mathcal{F}$ are L^p- and $\mathrm{W}^{1,p}$-coercive on $\mathcal{A}$, respectively. Therefore, the reflexivity of L^p and $\mathrm{W}^{1,p}$ in case $1 < p < \infty$ and statement (II) in the previous corollary "Coercivity gives weak compactness." yield that $\mathcal{G}$ and $\mathcal{F}$ satisfy condition (IIa) of Section 2.1 for the weak topology on $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ and $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, respectively. Moreover, from Section 2.2 we know that $\mathcal{G}$ and $\mathcal{F}$ are sequentially weakly lower semicontinuous on $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ and $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, respectively, that is, condition (IIb) of Section 2.1 is also satisfied. In view of (IIa) and (IIb), the abstract existence theorem of Section 2.1 then applies and gives the claim. □

Remarks (on the general existence theorem and its proof)**.**

(1) Even though the proof has been presented as a very abstract composition of the previously collected ingredients, it is **absolutely important** to keep in mind that the **core reasoning of the direct method** is in fact **quite simple**. We briefly recall this reasoning in the first-order case (1): One assumes $\inf_{\mathcal{A}}\mathcal{F} < \infty$ and starts with a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\lim_{k\to\infty}\mathcal{F}[u_k] = \inf_{\mathcal{A}}\mathcal{F}$. Then coercivity yields boundedness of $(u_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, and weak compactness together with sequential weak closedness of $\mathcal{A}$ gives weak convergence of a subsequence $(u_{k_\ell})_{\ell\in\mathbb{N}}$ to a limit $u \in \mathcal{A}$. Finally, the weak lower semicontinuity of Section 2.2 ensures
$$\mathcal{F}[u] \le \liminf_{\ell\to\infty}\mathcal{F}\big[u_{k_\ell}\big] = \inf_{\mathcal{A}}\mathcal{F}$$
and thus the minimality of u.

(2) In most cases, one can alternatively apply the abstract existence theorem on the basis of (Ia) and (Ib). Indeed, the availability of (Ia) has been provided above in large generality, while (Ib) has been obtained in Section 2.2 for a general class of zero-order functionals and a class of first-order functionals with a stronger convexity assumption. However, for the most general class of first-order functionals with convexity only in the gradient variable, we have checked only (IIb) not (Ib). This is a reason to prefer the sequence-based version (II) of the direct method.

(3) In particular, part (1) of the **general existence theorem applies to quadratic integrals** $\frac{1}{2}\int_\Omega A(\mathrm{D}w,\mathrm{D}w)\,\mathrm{d}x$ (with positive bilinear forms A on $\mathbb{R}^{N\times n}$ which satisfy $A(z,z)\geq\gamma|z|^2$ for all $z\in\mathbb{R}^{N\times n}$ with fixed $\gamma>0$) **and the $\boldsymbol{p}$-energies $\boldsymbol{\mathcal{E}_p}$ with $\boldsymbol{1<p<\infty}$**, at least if **minimization in Dirichlet classes** $\mathrm{W}^{1,2}_{u_0}(\Omega,\mathbb{R}^N)$ and $\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N)$ on bounded open $\Omega\subset\mathbb{R}^n$ is concerned. Indeed, the relevant hypotheses are satisfied, since the Dirichlet classes are sequentially weakly closed by Mazur's lemma, convexity of the integrands follows from the Cauchy-Schwarz and Hölder inequalities, respectively, and the coercivity criterion (1c) applies. Moreover, the existence theorem **covers many variants of these model integrals** (for instance with additional coefficients or lower-order terms) **and also cases with additional side constraints**.

On the contrary, the **existence theorem does *not* apply to integrals** of 1-energy and area type **with merely $\mathbf{W^{1,1}}$-coercivity** at hand, since the **lack of suitable compactness** results prevents the application of the direct method in the Sobolev space $\mathrm{W}^{1,1}$.

(4) In case $|\Omega|<\infty$ the **inequality in the coercivity criteria** (1c) and (1d) **can be weakened** to

$$F(x,y,z)\geq\gamma|z|^p-\Gamma|y|^q-\Psi(x)\qquad\text{for all }(x,y,z)\in(\Omega\setminus E)\times\mathbb{R}^N\times\mathbb{R}^{N\times n},$$

with fixed $q\in(0,p)$ and $\Gamma\in\mathbb{R}$, since this inequality still implies $\mathrm{W}^{1,p}$-coercivity of the integral functional $\mathcal{F}$ on the same classes $\mathcal{A}$ as before (see the exercises for the proof of this claim). In addition, even $q=p$ can be admitted if Γ is small enough, precisely $\Gamma<C_{\mathrm{opt}}^{-p}\gamma$ with the optimal constant C_{opt} in the Poincaré inequality $\|w\|_{\mathrm{L}^p;\Omega}\leq C_{\mathrm{opt}}\|\mathrm{D}w\|_{\mathrm{L}^p;\Omega}$ for $w\in\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$ (case of (1c)) and $\|w-w_\Omega\|_{\mathrm{L}^p;\Omega}\leq C_{\mathrm{opt}}\|\mathrm{D}w\|_{\mathrm{L}^p;\Omega}$ for $w\in\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ (case of (1d)), respectively.

(5) One **can allow in the existence result** and on bounded Ω also in the generalization of Remark (4), **that $\boldsymbol{G}$ and $\boldsymbol{F}$ take values in $\mathbb{R}\cup\{\infty\}$** instead of $[0,\infty]$. (However, some coercivity inequality is still in force and ensures $G_-(\,\cdot\,,w),F_-(\,\cdot\,,w,\mathrm{D}w)\in\mathrm{L}^1(\Omega)$ so that $\mathcal{G}[w],\mathcal{F}[w]\in\mathbb{R}\cup\{\infty\}$ exists for all $w\in\mathcal{A}$.) In fact, non-negativity of the integrand has been assumed only for the application of the weak lower semicontinuity results from Section 2.2, but can be removed from these results under the present assumptions: For instance, if the inequality in (4) holds, the new integrand $\widetilde{F}(x,y,z):=F(x,y,z)+\Gamma|y|^q+\Psi(x)$ is non-negative, and thus $\widetilde{\mathcal{F}}[w]:=\int_\Omega\widetilde{F}(\cdot,w,\mathrm{D}w)\,\mathrm{d}x$ is sequentially weakly lower semicontinuous on $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. However, under the assumptions of either (1c) or (1d) with Ω bounded, Rellich's theorem gives even weak continuity of the term $\int_\Omega|w|^q\,\mathrm{d}x$ on $\mathcal{A}\subset\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Consequently, also $\mathcal{F}[w]=\widetilde{\mathcal{F}}[w]-\Gamma\int_\Omega|w|^q\,\mathrm{d}x-\int_\Omega\Psi\,\mathrm{d}x$ is weakly lower semicontinuous on $\mathcal{A}\subset\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$

Next we show by basic examples that **coercivity and convexity**, the two **principal assumptions** above, are **essentially inevitable in a general existence theory**:

Examples (for **necessity of coercivity and convexity**)**.** We consider, in the **1-dimensional scalar case**, first-order variational integrals $\mathcal{F}[w]=\int_{x_1}^{x_2}F(\cdot,w,w')\,\mathrm{d}x$ on the Dirichlet class $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$, which encodes the **Dirichlet boundary condition** $w(x_1)=y_1$, $w(x_2)=y_2$ with $x_1<x_2$ in $\mathbb{R}$ and $y_1,y_2\in\mathbb{R}$. Specifically, we mention the following cases and examples, which are **treated in detail in the exercises** (see Sheets 1 and 5):

(1) If $\boldsymbol{F(x,y,z)} \mathrel{\widehat{=}} \boldsymbol{F(z)}$ **depends on $\boldsymbol{z}$ only**, we know: For convex $F \in \mathrm{C}^0(\mathbb{R})$, there is always a minimizer of $\int_{x_1}^{x_2} F(w')\,\mathrm{d}x$ in $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$, namely the unique affine function with the given boundary values. So — what is **rather untypical** in view of the subsequent examples — **convexity alone suffices for existence in this basic case**, and no explicit coercivity requirement[13] is needed. However, **in absence of convexity *and* coercivity minimizers may still fail to exist**, as it happens e.g. for $\int_{x_1}^{x_2} \frac{1}{1+w'^2}\,\mathrm{d}x$ on $\mathrm{W}^{1,1}_0((x_1,x_2))$.

(2) The famous **Weierstraß example**[14] from 1869 with integrand of type $F(x,y,z) \mathrel{\widehat{=}} F(x,z)$ is given by

$$\tfrac{1}{2}\int_0^1 x\,w'(x)^2\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,1}((0,1))\,.$$

This integral is convex but ***not* $\mathbf{W^{1,p}}$-coercive** on $\mathrm{W}^{1,p}_{1,0}((0,1))$ for any $p \in [1,\infty]$ and has ***no* minimizer** in $\mathrm{W}^{1,p}_{1,0}((0,1))$ for any $p \in [1,\infty]$ (since the infimum on $\mathrm{W}^{1,p}_{1,0}((0,1))$ can shown to be 0 and cannot be realized). So, this is a basic example for the **failure of existence in the absence of coercivity**.

(3) An example with integrand of type $F(x,y,z) \mathrel{\widehat{=}} F(y,z)$ is given by

$$\int_{x_1}^{x_2} \sqrt{w^2+w'^2}\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,1}((x_1,x_2))\,.$$

This integral is convex and **$\mathbf{W^{1,1}}$-coercive** on $\mathrm{W}^{1,1}_{0,1}((x_1,x_2))$, but ***not* $\mathbf{W^{1,p}}$-coercive** on $\mathrm{W}^{1,p}_{0,1}((x_1,x_2))$ for any $p \in (1,\infty]$, and it has ***no* minimizer** in $\mathrm{W}^{1,p}_{0,1}((x_1,x_2))$ for any $p \in [1,\infty]$ (since the infimum on $\mathrm{W}^{1,p}_{0,1}((x_1,x_2))$ can shown to be 1 and cannot be realized). So, this example shows that **merely $\mathbf{W^{1,1}}$-coercivity does not suffice for existence**.

(4) Another example with integrand of type $F(x,y,z) \mathrel{\widehat{=}} F(y,z)$ and parameter $\Gamma \in \mathbb{R}$ is given by

$$\int_{x_1}^{x_2} \big(w'^2 - \Gamma w^2\big)\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,2}((x_1,x_2))\,.$$

This integral is convex in the w'-variable and satisfies the inequality from Remark (4) above with $q = p = 2$, $\gamma = 1$, the parameter Γ occurring in the integral, and $\Psi \equiv 0$. For the minimization in $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ it is decisive if or not the inequality $\Gamma < C_{\mathrm{opt}}^{-2}$ from Remark (4) is satisfied, where C_{opt} is the optimal constant in the Poincaré inequality $\|w\|_{\mathrm{L}^2((x_1,x_2))} \le C_{\mathrm{opt}}\|w'\|_{\mathrm{L}^2((x_1,x_2))}$ for $w \in \mathrm{W}^{1,2}_0((x_1,x_2))$. Indeed in the 1d situation one can determine C_{opt} by a Fourier expansion method as $C_{\mathrm{opt}} = \frac{x_2-x_1}{\pi}$, and we have:

- In the **subcritical case** $\Gamma < \big(\frac{\pi}{x_2-x_1}\big)^2$, the integral is **$\mathbf{W^{1,2}}$-coercive** on $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$ and **has a minimizer** in $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$.
- In the **supercritical case** $\Gamma > \big(\frac{\pi}{x_2-x_1}\big)^2$, the integral is unbounded from below on $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$, and thus it is ***not* $\mathbf{W^{1,2}}$-coercive** on $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$ and has ***no* minimizer** in $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$.

[13] However, one can show, for convex $F \in \mathrm{C}^0(\mathbb{R})$, that either F is affine or it holds $F(z) \ge \gamma|z| - L - Mz$ for $z \in \mathbb{R}$ with constants $L, M \in \mathbb{R}$ and $\int_{x_1}^{x_2} F(w')\,\mathrm{d}x$ is $\mathrm{W}^{1,1}$-coercive on $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$. So, convexity comes at least close to implying a kind of coercivity in the basic case $F(x,y,z) \mathrel{\widehat{=}} F(z)$.

[14] In fact, the original Weierstraß example is $\frac{1}{2}\int_0^1 x^2\,w'(x)^2\,\mathrm{d}x$ (with x^2 in place of x, but analogous properties).

- In the **critical case** $\Gamma = \big(\frac{\pi}{x_2-x_1}\big)^2$, finally, the integral is ***not* $\mathbf{W^{1,2}}$-coercive** on $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$. **For $\boldsymbol{y_1{+}y_2 \neq 0}$** it is even unbounded from below on $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$ and has ***no* minimizer** in $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$, while **for $\boldsymbol{y_1{+}y_2 = 0}$** it **has minimizers** and actually a 1-dimensional affine space of minimizers in $\mathrm{W}^{1,2}_{y_1,y_2}((x_1,x_2))$.

Proof. In the subcritical case, the verification of coercivity with the help of the mentioned Poincaré inequality is straightforward, and existence then follows.

For the treatment of the other cases we assume, for simplicity of notation, $x_1 = 0$, $x_2 = \pi$, and we abbreviate $\mathcal{F}^\Gamma[w] = \int_0^\pi \big(w'^2 - \Gamma w^2\big)\,\mathrm{d}x$. For an arbitrary $w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ and $\alpha \in \mathbb{R}$, we observe $w{+}\alpha\sin \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ and compute

$$\mathcal{F}^\Gamma[w{+}\alpha\sin] = \mathcal{F}^\Gamma[w] + 2\alpha\int_0^\pi \big(w'\cos - \Gamma w\sin\big)\,\mathrm{d}x + \alpha^2(1{-}\Gamma)\frac{\pi}{2}\,.$$

In the supercritical case $\Gamma > 1$, this gives $\lim_{\alpha\to\pm\infty}\mathcal{F}^\Gamma[w{+}\alpha\sin] = -\infty$, and thus $\mathcal{F}^\Gamma$ is unbounded from below on $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ (which entails all claims in this case). In the critical case $\Gamma = 1$, integration by parts shows $\int_0^\pi \big(w'\cos - w\sin\big)\,\mathrm{d}x = w(0){+}w(\pi) = y_1{+}y_2$, and the above reduces to $\mathcal{F}^\Gamma[w{+}\alpha\sin] = \mathcal{F}^\Gamma[w] + 2\alpha(y_1{+}y_2)$. Hence, for $\pm(y_1{+}y_2) > 0$ we get $\lim_{\alpha\to\mp\infty}\mathcal{F}^1[w{+}\alpha\sin] = -\infty$ and infer that $\mathcal{F}^1$ is unbounded from below also in this case (which again entails all claims). In the remaining case $\Gamma = 1$, $y_1{+}y_2 = 0$, finally, we now prove that $y_1\cos$ with $\mathcal{F}^1[y_1\cos] = 0$ is indeed a minimizer of $\mathcal{F}^1$ in $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$. To this end we still consider an arbitrary $w \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ and choose $\alpha \in \mathbb{R}$ such that $v := w{+}\alpha\sin \in \mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ satisfies $v_{(0,\pi)} = 0$. Under the assumption $v_{(0,\pi)} = 0$, another version of the optimal Poincaré inequality[15] implies $\|v\|_{\mathrm{L}^2((0,\pi))} \le \|v'\|_{\mathrm{L}^2((0,\pi))}$, and this combines with the preceding to $\mathcal{F}^1[w] = \mathcal{F}^1[v] \ge 0$. Since we have $\mathcal{F}^1[y_1\cos{+}\alpha\sin] = \mathcal{F}^1[y_1\cos] = 0$ for all $\alpha \in \mathbb{R}$, this means that all $y_1\cos{+}\alpha\sin$ with $\alpha \in \mathbb{R}$ are minimizers of $\mathcal{F}^1$ in $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$. Since the space of minimizers is unbounded, it also clear that even in this case with existence, $\mathcal{F}^1$ is not $\mathrm{W}^{1,2}$-coercive on $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$. □

(5) The famous **Bolza example** from 1902 with integrand of type $F(x,y,z) \mathrel{\widehat{=}} F(y,z)$ is given by

$$\int_{x_1}^{x_2} \big[(w'^2{-}1)^2 + w^4\big]\,\mathrm{d}x \qquad\qquad \text{for } w \in \mathrm{W}^{1,4}((x_1,x_2))\,.$$

This integral is $\mathrm{W}^{1,4}$-coercive on $\mathrm{W}^{1,4}$, but is ***not* convex** in the w'-variable and has ***no* minimizer** in $\mathrm{W}^{1,4}_0((x_1,x_2))$ (since the infimum on $\mathrm{W}^{1,4}_0((x_1,x_2))$ can be shown to be 0 and cannot be realized). So, this is a basic example for the **failure of existence in the absence of convexity**. It has many variants, in which the role of $z \mapsto (z^2{-}1)^2$ can be taken by another 'double-well potential' with two minimum points, and is also related to physical models for the sailing of a boat into the wind.

2.4 Uniqueness of minimizers

In order to treat the uniqueness problem for minimizers, we first recall a precise definition of strict convexity:

Definition (strictly convex functions). *A function $\mathcal{F}\colon \mathcal{A} \to \mathbb{R}\cup\{\infty\}$ on a subset $\mathcal{A}$ of a real vector space $\mathcal{X}$ is* ***strictly convex*** *on $\mathcal{A}$, if $\mathcal{A}$ is a convex set and the strict convexity inequality*

$$\mathcal{F}[\lambda w{+}(1{-}\lambda)\widetilde{w}] < \lambda\mathcal{F}[w] + (1{-}\lambda)\mathcal{F}[\widetilde{w}] \qquad \textit{holds whenever } \widetilde{w} \neq w \textit{ in } \mathcal{A} \textit{ and } \lambda \in (0,1)\,.$$

[15] In fact, the common root of the Poincaré inequalities relevant here is the **Poincaré-Wirtinger inequality** $\|w\|_{\mathrm{L}^2((-\pi,\pi))} \le \|w'\|_{\mathrm{L}^2((-\pi,\pi))}$ for all $w \in \mathrm{W}^{1,1}_{\mathrm{per}}((-\pi,\pi))$ with $w_{(-\pi,\pi)} = 0$. Applying this inequality to odd extensions one infers $\|w\|_{\mathrm{L}^2((0,\pi))} \le \|w'\|_{\mathrm{L}^2((0,\pi))}$ for all $w \in \mathrm{W}^{1,1}_0((0,\pi))$, and applying it to even extensions one gets $\|w\|_{\mathrm{L}^2((0,\pi))} \le \|w'\|_{\mathrm{L}^2((0,\pi))}$ for all $w \in \mathrm{W}^{1,1}((0,\pi))$ with $w_{(0,\pi)} = 0$. Clearly, one can also pass these inequalities to intervals (x_1,x_2) with $x_1 < x_2$ in $\mathbb{R}$ by scaling.

We remark that, if $\mathcal{F}$ takes the value ∞ at an interior point of $\mathcal{A}$, then evidently $\mathcal{F}$ cannot be strictly convex on $\mathcal{A}$. Since function(al)s with value ∞ are sometimes useful and shall not be ruled out completely, in the sequel we work with the assumption that $\mathcal{F}$ is strictly convex on $\mathcal{A}\cap\{\mathcal{F}<\infty\}$ only (which still implies that $\mathcal{F}$ is non-strictly convex on all of a convex set $\mathcal{A}$, since the weak convexity inequality is trivially satisfied if one of w, $\widetilde{w}$ lies in $\mathcal{A}\cap\{\mathcal{F}=\infty\}$). On the basis of this assumption, the following **simple but central uniqueness principle** holds:

Proposition (uniqueness from strict convexity, abstract version). *Consider a real vector space $\mathcal{X}$. If a function $\mathcal{F}\colon \mathcal{A}\to\mathbb{R}\cup\{\infty\}$ with $\mathcal{F}\not\equiv\infty$ is **strictly convex** on $\mathcal{A}\cap\{\mathcal{F}<\infty\}$, then there is **at most one minimum point** for $\mathcal{F}$ in $\mathcal{A}$.*

Proof. Suppose that $\mathcal{F}$ has two minimum points $\widetilde{u}\neq u$ in $\mathcal{A}$. In view of the assumption $\mathcal{F}\not\equiv\infty$, this means $\mathcal{F}[\widetilde{u}]=\mathcal{F}[u]=\inf_{\mathcal{A}}\mathcal{F}<\infty$. Hence, strict convexity gives $\frac{u+\widetilde{u}}{2}\in\mathcal{A}$ and $\mathcal{F}\big[\frac{u+\widetilde{u}}{2}\big]<\frac{1}{2}\mathcal{F}[u]+\frac{1}{2}\mathcal{F}[\widetilde{u}]=\inf_{\mathcal{A}}\mathcal{F}$. This contradiction completes the proof. □

The abstract principle applies to integral functionals as follows:

Corollary (uniqueness from strict convexity, integral version). *Fix an open $\Omega\subset\mathbb{R}^n$.*

(0) *Consider a convex $\mathcal{A}\subset\mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and an $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)$-measurable $G\colon\Omega\times\mathbb{R}^N\to\mathbb{R}\cup\{\infty\}$ such that*
$$\mathcal{G}[w]:=\int_\Omega G(\,\cdot\,,w)\,\mathrm{d}x\in\mathbb{R}\cup\{\infty\}$$
*exists for all $w\in\mathcal{A}$. If $G(x,\,\cdot\,)$ is **strictly convex** on $\{G(x,\,\cdot\,)<\infty\}$ for a.e. $x\in\Omega$, then $\mathcal{G}$ is strictly convex on $\mathcal{A}\cap\{\mathcal{G}<\infty\}$, and in case $\mathcal{G}\not\equiv\infty$ on $\mathcal{A}$ there is **at most one minimizer** of $\mathcal{G}$ in $\mathcal{A}$.*

(1) *Consider a convex $\mathcal{A}\subset\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and an $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)\otimes\mathcal{B}(\mathbb{R}^{N\times n})$-measurable integrand $F\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}\cup\{\infty\}$ such that*
$$\mathcal{F}[w]:=\int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x\in\mathbb{R}\cup\{\infty\}$$
*exists for all $w\in\mathcal{A}$. If $F(x,\,\cdot\,,\,\cdot\,)$ is **strictly convex** on $\{F(x,\,\cdot\,,\,\cdot\,)<\infty\}$ for a.e. $x\in\Omega$, then $\mathcal{F}$ is strictly convex on $\mathcal{A}\cap\{\mathcal{F}<\infty\}$, and in case $\mathcal{F}\not\equiv\infty$ there is **at most one minimizer** of $\mathcal{F}$ in $\mathcal{A}$.*

Proof. We only deal with the case (1), since the case (0) can be treated analogously. In order to establish the strict convexity claim, we consider $\widetilde{w}\neq w$ in $\mathcal{A}\cap\{\mathcal{F}<\infty\}$. We observe $F(\,\cdot\,,w,\mathrm{D}w)<\infty$, $F(\,\cdot\,,\widetilde{w},\mathrm{D}\widetilde{w})<\infty$ a.e. on Ω and $|\{\widetilde{w}\neq w\}|>0$. For $\lambda\in(0,1)$, we infer by strict convexity
$$F(\,\cdot\,,\lambda(w,\mathrm{D}w){+}(1{-}\lambda)(\widetilde{w},\mathrm{D}\widetilde{w}))\le\lambda F(\,\cdot\,,w,\mathrm{D}w)+(1{-}\lambda)F(\,\cdot\,,\widetilde{w},\mathrm{D}\widetilde{w})\qquad\text{a.e. on }\Omega$$
with the strict inequality '$<$' valid on the non-negligible set $\{\widetilde{w}\neq w\}$. Integrating both sides of the inequality and crucially exploiting $\mathcal{F}[w]<\infty$, $\mathcal{F}[\widetilde{w}]<\infty$, we end up with
$$\mathcal{F}[\lambda w{+}(1{-}\lambda)\widetilde{w}]<\lambda\mathcal{F}[w]+(1{-}\lambda)\mathcal{F}[\widetilde{w}]<\infty\,.$$
In particular, taking into account the convexity of $\mathcal{A}$ we read off $\lambda w{+}(1{-}\lambda)\widetilde{w}\in\mathcal{A}\cap\{\mathcal{F}<\infty\}$. This proves convexity of $\mathcal{A}\cap\{\mathcal{F}<\infty\}$ and strict convexity of $\mathcal{F}$ on $\mathcal{A}\cap\{\mathcal{F}<\infty\}$. Once this is established, the uniqueness claim is clear from the proposition. □

Unfortunately, the **preceding corollary does *not* apply** in the basic case of a first-order variational integral **without zero-order dependence on w**, that is

$$\mathcal{F}[w] = \int_\Omega \widetilde{F}(\,\cdot\,,\mathrm{D}w)\,\mathrm{d}x\,,$$

since in this case the integrand $F(x,y,z) = \widetilde{F}(x,z)$ as a function of $(x,y,z) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ is ***never* strictly convex in y-variable**. Therefore, in the following we also provide a variant of the corollary, which applies to minimization problems for such integrals in Dirichlet classes and is based on the following notion:

Definition (partially strict convexity). *A function $F\colon K \to \mathbb{R}\cup\{\infty\}$ on a subset K of $\mathcal{Y}\times\mathcal{Z}$ with real vector spaces $\mathcal{Y}$ and $\mathcal{Z}$ is **strictly-in-z convex** on K if K is a convex set, the convexity inequality*

$$F(\lambda y{+}(1{-}\lambda)\widetilde{y},\lambda z{+}(1{-}\lambda)\widetilde{z}) \le \lambda F(y,z) + (1{-}\lambda)F(\widetilde{y},\widetilde{z}) \quad \textit{holds for } (y,z),(\widetilde{y},\widetilde{z}) \in K\,,\ \lambda\in[0,1]\,,$$

*and **this inequality is strict whenever $\widetilde{z} \neq z$ and $\lambda \in (0,1)$**.*

Corollary (uniqueness from strict convexity, 2nd integral version). *Consider an open set $\Omega \subset \mathbb{R}^n$, a convex set $\mathcal{A} \subset u_0{+}\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$ with $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $p\in[1,\infty)$, and an $\mathcal{M}^n{\otimes}\mathcal{B}(\mathbb{R}^N){\otimes}\mathcal{B}(\mathbb{R}^{N\times n})$-measurable $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to \mathbb{R}\cup\{\infty\}$ such that*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x \in \mathbb{R}\cup\{\infty\}$$

*exists for all $w\in\mathcal{A}$. If $F(x,\,\cdot\,,\,\cdot\,)$ is **strictly-in-z convex** on $\{F(x,\,\cdot\,,\,\cdot\,)<\infty\}$ for a.e. $x\in\Omega$, then $\mathcal{F}$ is strictly convex on $\mathcal{A}\cap\{\mathcal{F}<\infty\}$, and in case $\mathcal{F}\not\equiv\infty$ there is **at most one minimizer** of $\mathcal{F}$ in $\mathcal{A}$.*

Proof. In order to establish the strict convexity claim, we consider $\widetilde{w}\neq w$ in $\mathcal{A}\cap\{\mathcal{F}<\infty\} \subset u_0{+}\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$, and we first show $|\{\mathrm{D}\widetilde{w}\neq\mathrm{D}w\}|>0$. Indeed, if we had $|\{\mathrm{D}\widetilde{w}\neq\mathrm{D}w\}|=0$, then $\widetilde{w}{-}w$ would equal a constant $\neq 0$ on at least one connected component $\widetilde{\Omega}$ (which is also open in $\mathbb{R}^n$) of Ω. However, we would also have $\widetilde{w}{-}w \in \mathrm{W}^{1,p}_0(\widetilde{\Omega},\mathbb{R}^N)$ and would thus arrive at a contradiction. This proves $|\{\mathrm{D}\widetilde{w}\neq\mathrm{D}w\}|>0$. Observing $F(\,\cdot\,,w,\mathrm{D}w)<\infty$, $F(\,\cdot\,,\widetilde{w},\mathrm{D}\widetilde{w})<\infty$ a.e. on Ω, for $\lambda\in(0,1)$ the assumed convexity then gives

$$F(\,\cdot\,,\lambda(w,\mathrm{D}w){+}(1{-}\lambda)(\widetilde{w},\mathrm{D}\widetilde{w})) \le \lambda F(\,\cdot\,,w,\mathrm{D}w) + (1{-}\lambda)F(\,\cdot\,,\widetilde{w},\mathrm{D}\widetilde{w}) \qquad \text{a.e. on } \Omega\,,$$

and this inequality is strict on the non-negligible set $\{\mathrm{D}\widetilde{w}\neq\mathrm{D}w\}$. Integrating this inequality, strict convexity of $\mathcal{F}$ follows exactly as in the proof of the previous corollary. Once strict convexity of $\mathcal{F}$ is established, the proposition gives uniqueness of its minimizers. □

Remarks (on the uniqueness principles)**.**

(1) The **latter corollary applies to basic model integrals** and shows that they have at most one minimizer **in every Dirichlet class**. For instance, it covers quadratic integrals $\int_\Omega A(\mathrm{D}w,\mathrm{D}w)\,\mathrm{d}x$ (with a positive bilinear forms A on $\mathbb{R}^{N\times n}$) on $\mathrm{W}^{1,2}_{u_0}(\Omega,\mathbb{R}^N)$, the p-energy $\mathcal{E}_p$ with $p>1$ on $\mathrm{W}^{1,p}_{u_0}(\Omega,\mathbb{R}^N)$, and — though we do not have existence for this last case — the scalar non-parametric area $\int_\Omega \sqrt{1{+}|\nabla w|^2}\,\mathrm{d}x$ on $\mathrm{W}^{1,1}_{u_0}(\Omega)$, where the required strict convexity of the integrands can be checked in all three cases with the Cauchy-Schwarz and Hölder inequalities or related elementary computations.

(2) Clearly, **strict convexity** of the integrand F **in the z-variable is inevitable for the uniqueness principles**. This can be seen already in the one-dimensional case at hand of the parametric length integral $\int_{x_1}^{x_2} |w'| \,\mathrm{d}x$ (with $x_1 < x_2$ in $\mathbb{R}$) whose integrand $F(x,y,z) = |z|$ is non-strictly convex in z. Actually, in view of the parametrization invariance of the length this integral has multiple minimizers in every Dirichlet class $\mathrm{W}^{1,1}_{y_1,y_2}((x_1,x_2))$ with $y_2 \neq y_1$ in $\mathbb{R}^N$.

(3) We emphasize that the latter corollary, in which the strictness of convexity regards z only, still requires convexity also in y. Indeed, **also** this **non-strict convexity in y cannot be dropped**, as the integral $\mathcal{F}^\Gamma[w] := \int_0^\pi \big(w'^2 - \Gamma w^2\big) \,\mathrm{d}x$ from Example (4) in the previous section (here for simplicity only on $(0,\pi)$) shows: The integrand $F^\Gamma(x,y,z) = |z|^2 - \Gamma|y|^2$ in this example is strictly convex in z for all $\Gamma \in \mathbb{R}$, convex in y only for $\Gamma \leq 0$, and in the critical case $\Gamma = 1$ there are multiple minimizers of $\mathcal{F}^1$ in $\mathrm{W}^{1,2}_0((0,\pi))$. Indeed, the critical case is also the first one with a chance for non-uniqueness in the sense that the *functional* $\mathcal{F}^\Gamma$ turns out to be strictly convex on $\mathrm{W}^{1,2}_0((0,\pi))$ if and only if $\Gamma < 1$ (and convex if and only if $\Gamma \leq 1$).

In order to check the last claim on $\mathcal{F}^\Gamma$, one looks at the symmetric bilinear form $\mathcal{B}^\Gamma[w,\widetilde{w}] := \int_0^\pi (w'\widetilde{w}' - \Gamma w\widetilde{w}) \,\mathrm{d}x$ on $w, \widetilde{w} \in \mathrm{W}^{1,2}_0((0,\pi))$, which induces the quadratic functional $\mathcal{F}^\Gamma[w] = \mathrm{B}^\Gamma[w,w]$. Thanks to the earlier-mentioned optimal Poincaré inequality on $\mathrm{W}^{1,2}_0((0,\pi))$ one then finds that $\mathcal{B}^\Gamma$ is non-negative and thus $\mathcal{F}^\Gamma$ convex precisely for $\Gamma \leq 1$, while $\mathcal{B}^\Gamma$ is positive and thus $\mathcal{F}^\Gamma$ strictly convex precisely for $\Gamma < 1$.

(4) *If* uniqueness does *not* follow from the above principles based on strict convexity, then — at least in the view of the lecturer — it holds only rarely and can sometimes be quite hard to prove.

2.5 Semi-classical existence theory

For general first-order **integral functionals with merely $\mathbf{W^{1,1}}$-coercivity at hand** (e.g. the 1-energy $\int_\Omega |\mathrm{D}w| \,\mathrm{d}x$ and the non-degenerate 1-energy $\int_\Omega \sqrt{1{+}|\mathrm{D}w|^2} \,\mathrm{d}x$) there is a well-developed existence theory based on weak$*$ compactness in the space $\mathrm{BV}(\Omega,\mathbb{R}^N)$ of functions of bounded variation. However, this theory requires additional background machinery and (at least in the first instance) gives solutions which satisfy a Dirichlet boundary condition only in a certain weakened sense. However, we do not enter into the BV theory here. Rather we explicate an alternative **comparison-principle-based approach**, which, in basic **scalar cases**, allows to bypass this theory and **prove existence directly in the space $\mathbf{W^{1,\infty}(\Omega)}$** (or $\mathrm{C}^{0,1}(\overline{\Omega})$). To this end, we start in fact with a finer discussion of the space $\mathrm{W}^{1,\infty}(\Omega)$:

Remarks and Definitions ($\mathbf{W^{1,\infty}_{(u_0)}(\Omega)}$ and $\mathbf{W^{M}_{(u_0)}(\Omega)}$). Consider an open set $\Omega \subset \mathbb{R}^n$.

(1) We first recall that (a representative of) a function $w \in \mathrm{W}^{1,\infty}(\Omega)$ satisfies

$$|w(y){-}w(x)| \leq \|\nabla w\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \mathrm{d}_\Omega(x,y) \qquad \text{for all } x,y \in \Omega\,,$$

where the inner metric $\mathrm{d}_\Omega(x,y)$ of Ω is the infimum of the lengths of C^1-curves from x to y in Ω, that is, $\mathrm{d}_\Omega(x,y) := \inf\{\int_0^1 |c'| \,\mathrm{d}t : c \in \mathrm{C}^1([0,1],\mathbb{R}^n)\,,\, c([0,1]) \subset \Omega\,,\, c(0) = x\,,\, c(1) = y\}$. In particular, this implies for $w \in \mathrm{W}^{1,\infty}(\Omega)$ ...

- that w is always d_Ω-Lipschitz continuous on Ω and locally Lipschitz continuous on Ω,
- that, in case of a bounded Lipschitz domain Ω, w is Lipschitz continuous on Ω,

- that, in case of a convex Ω, w is Lipschitz continuous on Ω with precisely $\|\nabla w\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}$ as the Lipschitz constant.

(where all Lipschitz properties with no particular metric mentioned are meant with respect to the standard distance on $\mathbb{R}^n$). In particular, if Ω is either a bounded Lipschitz domain or convex, every $w \in \mathrm{W}^{1,\infty}(\Omega)$ extends to a Lipschitz continuous function on $\overline{\Omega}$, and we can understand $\mathrm{W}^{1,\infty}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$.

(2) The **weak$*$ convergence** of a sequence $(w_k)_{k\in\mathbb{N}}$ **in $\mathbf{W^{1,\infty}(\Omega)}$** to a limit $w \in \mathrm{W}^{1,\infty}(\Omega)$ is most conveniently defined (without actually discussing if $\mathrm{W}^{1,\infty}(\Omega)$ itself is a dual space or not) by requiring weak$*$ convergence $w_k \underset{k\to\infty}{\overset{*}{\rightharpoonup}} w$ in $\mathrm{L}^\infty(\Omega) = \big(\mathrm{L}^1(\Omega)\big)^*$ and $\nabla w_k \underset{k\to\infty}{\overset{*}{\rightharpoonup}} \nabla w$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n) = \big(\mathrm{L}^1(\Omega,\mathbb{R}^n)\big)^*$.

(3) In case of $\mathrm{W}^{1,\infty}$ we define the **subspace of functions with zero boundary values** as

$$\mathrm{W}^{1,\infty}_0(\Omega) := \left\{ w \in \mathrm{W}^{1,\infty}(\Omega) \,:\, \lim_{\Omega\ni x\to\partial\Omega\cup\{\infty\}} w(x) = 0 \right\},$$

where the limit is in fact taken for the Lipschitz representative and the addition of $\{\infty\}$ is relevant for unbounded Ω only. It is not difficult to see that functions in $\mathrm{W}^{1,\infty}_0(\Omega)$ are always (i.e. for an arbitrary open set Ω) Lipschitz continuous on Ω and can be extended by zero to Lipschitz continuous functions on all of $\mathbb{R}^n$ with precisely $\|\nabla w\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}$ as the Lipschitz constant. Using this on one hand and Rademacher's theorem on the other hand, one can also write

$$\mathrm{W}^{1,\infty}_0(\Omega) = \left\{ w \in \mathrm{C}^{0,1}(\overline{\Omega}) \,:\, w|_{\partial\Omega} \equiv 0\,,\ \lim_{\Omega\ni x\to\infty} w(x) = 0 \right\}.$$

Once the subspace of functions with zero boundary values is defined we can also introduce, for $u_0 \in \mathrm{W}^{1,\infty}(\Omega)$, the **Dirichlet class**

$$\mathrm{W}^{1,\infty}_{u_0}(\Omega) := u_0 + \mathrm{W}^{1,\infty}_0(\Omega)\,.$$

(4) With the preceding definitions, it follows from the Arzelà-Ascoli theorem (or, what is the same, the case $p=\infty$ of Rellich's theorem) that in case of bounded Ω the subspace $\mathrm{W}^{1,\infty}_0(\Omega)$ and the classes $\mathrm{W}^{1,\infty}_{u_0}(\Omega)$ with $u_0 \in \mathrm{W}^{1,\infty}(\Omega)$ are sequentially weakly$*$ closed in $\mathrm{W}^{1,\infty}(\Omega)$.

(5) Given $M \in [0,\infty]$ and $u_0 \in \mathrm{W}^{1,\infty}(\Omega)$, we also introduce the classes

$$\mathrm{W}^M(\Omega) := \{w \in \mathrm{W}^{1,\infty}(\Omega) \,:\, \|\nabla w\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le M\}\,, \qquad \mathrm{W}^M_{u_0}(\Omega) := \mathrm{W}^M(\Omega) \cap \mathrm{W}^{1,\infty}_{u_0}(\Omega)$$

with bound M on the derivative.

(6) For bounded Ω, $M \in [0,\infty)$, and $u_0 \in \mathrm{W}^{1,\infty}(\Omega)$, it follows from weak$*$ compactness in L^∞, the well-known fact that the gradient is a weak$*$-closed operator, and (4) above that

$\mathbf{W}^{\boldsymbol{M}}_{\boldsymbol{u_0}}\mathbf{(\Omega)}$ is sequentially weakly$*$ compact.

Rademacher's theorem asserts that a Lipschitz continuous function $w\colon \Omega \to \mathbb{R}^N$ on an open subset Ω of $\mathbb{R}^n$ is classically totally differentiable at a.e. point of Ω, that it is also weakly differentiable on Ω, and that its weak derivative is bounded and represented a.e. by the classical one.

In the remainder of this section, we develop the announced existence theory for basic variational integrals

$$\mathcal{F}[w] := \int_\Omega F(\nabla w)\,\mathrm{d}x \in \mathbb{R} \qquad \text{on } \textbf{scalar} \text{ functions } w \in \mathrm{W}^{1,\infty}(\Omega) \tag{$*$}$$

with bounded open $\Omega \subset \mathbb{R}^n$ and continuous $F \colon \mathbb{R}^n \to \mathbb{R}$. In view of the sequential weak$*$ compactness of $\mathrm{W}^M_{u_0}(\Omega)$ (which results from the 'artificially introduced' bound M) we can first apply the direct method as developed in the previous sections:

Proposition (**existence in the restricted class**). *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from ($*$), $M \in [0,\infty)$, $u_0 \in \mathrm{W}^M(\Omega)$, and assume that F is convex on $\mathbb{R}^n$. Then* ***there exists a minimizer*** *of $\mathcal{F}$* ***in*** $\mathbf{W}^{\boldsymbol{M}}_{\boldsymbol{u_0}}\mathbf{(\Omega)}$.

Proof. Since the passage from the integrand F to the new integrand $\max\{F, \min_{\overline{\mathrm{B}_M(0)}} F\}$ preserves convexity and continuity and does not change the values of the functional $\mathcal{F}$ on $\mathrm{W}^M(\Omega)$, we can assume that F is bounded from below on $\mathbb{R}^n$. In fact, after addition of a constant, we can then even assume $F \ge 0$ on $\mathbb{R}^n$. In this situation we record that $\mathrm{W}^M_{u_0}(\Omega)$ is non-empty (it contains u_0) and sequentially weakly$*$ compact (see Remark (6) above) and that $\mathcal{F}$ is sequentially weakly$*$ lower semicontinuous on $\mathrm{W}^{1,\infty}(\Omega) \supset \mathrm{W}^M_{u_0}(\Omega)$ (see Section 2.2 where $F \ge 0$ was assumed). On this basis, the direct method applies and produces the desired minimizer. □

Lemma (**'from restricted back to unrestricted minimizers'**). *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from ($*$), $M \in [0,\infty)$, and assume that F is convex on $\mathbb{R}^n$.* If *a minimizer $u \in \mathrm{W}^M(\Omega)$ of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ satisfies the* **strict** ***inequality*** $\|\boldsymbol{\nabla u}\|_{\mathbf{L}^\infty(\boldsymbol{\Omega},\mathbb{R}^n)} < \boldsymbol{M}$, *then it is* ***also a minimizer*** *of $\mathcal{F}$* ***in*** $\mathbf{W}^{\mathbf{1},\boldsymbol{\infty}}_{\boldsymbol{u}}\mathbf{(\Omega)}$.

Proof. For arbitrary $w \in \mathrm{W}^{1,\infty}_u(\Omega)$ and sufficiently small $\lambda \in (0,1)$ we first record the bound $\|\mathrm{D}u{+}\lambda(\mathrm{D}w{-}\mathrm{D}u)\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le M$ (for which we need in fact $\lambda < \frac{M-\|\mathrm{D}u\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}}{\|\mathrm{D}w-\mathrm{D}u\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}}$). With the help of this bound, we infer $(1{-}\lambda)u + \lambda w = u + \lambda(w{-}u) \in \mathrm{W}^M_u(\Omega)$. The assumed minimality of u in $\mathrm{W}^M_u(\Omega)$ together with the convexity of F and thus $\mathcal{F}$ then gives, still for sufficiently small $\lambda \in (0,1)$,

$$\mathcal{F}[u] \le \mathcal{F}[(1{-}\lambda)u{+}\lambda w] \le (1{-}\lambda)\mathcal{F}[u] + \lambda\mathcal{F}[w]\,.$$

Rearranging terms and dividing by the positive factor λ, we arrive at $\mathcal{F}[u] \le \mathcal{F}[w]$. Since $w \in \mathrm{W}^{1,\infty}_u(\Omega)$ is arbitrary, this is the claim. □

In view of the last lemma, we now aim at verifying the strict inequality $\|\nabla u\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} < M$ for a minimizer $u \in \mathrm{W}^M(\Omega)$ of $\mathcal{F}$ in the restricted class $\mathrm{W}^M_u(\Omega)$. This will be achieved with the help of a comparison principle, which is truly needed only for minimizers but will be recorded even for the more general case of sub- and super-minimizers in the following sense:

Definition (**sub- and super-minimizers**). *Fix an open $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from ($*$), $M \in [0,\infty]$. We call $u \in \mathrm{W}^M(\Omega)$ a* ***sub-minimizer*** *of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ if $\mathcal{F}[u] \le \mathcal{F}[w]$ holds for all $w \in \mathrm{W}^M_u(\Omega)$ with $w \le u$ a.e. on Ω. Analogously, we call $u \in \mathrm{W}^M(\Omega)$ a* ***super-minimizer*** *of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ if $\mathcal{F}[u] \le \mathcal{F}[w]$ holds for all $w \in \mathrm{W}^M_u(\Omega)$ with $w \ge u$ a.e. on Ω.*

Remarks (on sub-minimizers and super-minimizers)**.**

(1) Sub- and and super-minimizer are **made up as analoga to sub-/superharmonic functions** and, more generally, to sub-/super-solutions of PDEs. For instance, following the proof of the Dirichlet principle one finds that a function $u \in \mathrm{W}^{1,\infty}(\Omega)$ is a sub-minimizer of the Dirichlet integral $\mathcal{E}_2$ in $\mathrm{W}^{1,\infty}_u(\Omega) = \mathrm{W}^{\infty}_u(\Omega)$ if and only if it is weakly subharmonic (where the latter means $\Delta u \geq 0$ in $\mathscr{D}'(\Omega)$, that is, $\int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x \leq 0$ for all non-negative $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$).

(2) It turns out that a function $u \in \mathrm{W}^M(\Omega)$ is a minimizer of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ if and only if u is simultaneously a sub-minimizer and a super-minimizer of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$. Indeed, the forward implication of this claim is trivially satisfied, while the backward implication requires a short comparison argument and will be explicated in the exercises.

Lemma. *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from ($*$), $M \in [0,\infty]$, and assume that F is* strictly *convex on $\mathbb{R}^n$. If $u \in \mathrm{W}^M(\Omega)$ is a sub-minimizer of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ and $v \in \mathrm{W}^M(\Omega)$ is a super-minimizer of $\mathcal{F}$ in $\mathrm{W}^M_v(\Omega)$, then we have the **comparison principle***

$$\limsup_{\Omega \ni x \to \partial\Omega} (u(x) - v(x)) \leq 0 \quad \Longrightarrow \quad u \leq v \ \text{ on } \Omega$$

*and the **weak maximum principle***

$$\sup_\Omega (u - v) \leq \limsup_{\Omega \ni x \to \partial\Omega} (u(x) - v(x)) \,.$$

If u and v are even minimizers of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ and $\mathrm{W}^M_v(\Omega)$, respectively, we also have the maximum principle for the modulus $\sup_\Omega |u-v| \leq \limsup_{\Omega \ni x \to \partial\Omega} |u(x)-v(x)|$.

The *proofs* of these principles will be treated in the exercises.

Remarks (on the comparison principle and the maximum principle)**.**

(1) The inequality in the maximum principle is in fact an equality, since the opposite inequality is trivially satisfied.

(2) The above principles apply even in situations with quite irregular domains and boundary data, for instance they include functions u and v on $\Omega = (-1,1)^2 \setminus ((0,1) \times \{0\}) \subset \mathbb{R}^2$ with different boundary values on the opposite sides of the 'cut' $(0,1) \times \{0\}$.

(3) Nevertheless, the above principles are **usually needed for functions $\boldsymbol{u, v \in \mathrm{C}^0(\overline{\Omega})}$ only** (which, as discussed earlier, follows automatically from mild assumptions on Ω). In this standard case, the comparison principle and the maximum principle take the **more common forms**

$$u \leq v \text{ on } \partial\Omega \quad \Longrightarrow \quad u \leq v \text{ on } \Omega \,,$$
$$\sup_\Omega (u-v) \leq \max_{\partial\Omega} (u-v)$$

(where again the last inequality is in fact an equality).

Though not needed explicitly for the existence proof, we record (what in principle is known from Section 2.4, but also comes out in the present context):

Corollary (uniqueness of minimizers). *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from $(*)$, $M \in [0,\infty]$, $u_0 \in \mathrm{W}^{1,\infty}(\Omega)$, and assume that F is* strictly *convex on $\mathbb{R}^n$. Then there is **at most one minimizer** of $\mathcal{F}$ in $\mathrm{W}^M_{u_0}(\Omega)$.*

Proof. If u and v are minimizers of $\mathcal{F}$ in $\mathrm{W}^M_{u_0}(\Omega)$, the maximum principle and the fact that $u-v \in \mathrm{W}^{1,\infty}_0(\Omega)$ yield

$$\sup_\Omega |u-v| \le \limsup_{\Omega \ni x \to \partial\Omega} |u(x)-v(x)| = 0\,.$$

We thus obtain $u = v$ on Ω, which ends the proof. □

With regard to the existence proof, a main conclusion from the maximum principle is:

Lemma ((partial) reduction to the boundary). *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from $(*)$, $M \in [0,\infty]$, and assume that F is* strictly *convex on $\mathbb{R}^n$. Then, if $u \in \mathrm{W}^M(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ is a minimizer of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$, there holds*

$$\sup_{\substack{x,y\in\Omega\\ y\neq x}} \frac{|u(y)-u(x)|}{|y-x|} \le \sup_{q\in\Omega\,,\,p\in\partial\Omega} \frac{|u(q)-u(p)|}{|q-p|}\,.$$

Remarks.

(1) Once more, the inequality in the lemma is in fact an equality, since the opposite inequality is valid for every $u \in \mathrm{C}^0(\overline{\Omega})$.

(2) A closer inspection of the following proof shows that without the assumption $u \in \mathrm{C}^0(\overline{\Omega})$ the conclusion of the lemma stays valid in the form

$$\sup_{\substack{x,y\in\Omega\\ y\neq x}} \frac{|u(y)-u(x)|}{|y-x|} \le \limsup_{\Omega\ni p\to\partial\Omega} \sup_{q\in\Omega} \frac{|u(q)-u(p)|}{|q-p|}\,.$$

Proof of the lemma. We consider arbitrary $x,y \in \Omega$ with $y \neq x$ and abbreviate $\tau := y-x \neq 0$. Setting $u_\tau(z) := u(z+\tau)$ for $z \in \Omega-\tau$, we obtain a minimizer $u_\tau \in \mathrm{W}^M(\Omega-\tau) \cap \mathrm{C}^0(\overline{\Omega-\tau})$ of $\mathcal{F}_\tau[w] := \int_{\Omega-\tau} F(\nabla w)\,\mathrm{d}x$ among $w \in \mathrm{W}^M_{u_\tau}(\Omega-\tau)$. We next introduce $\widetilde{\Omega} := \Omega \cap (\Omega-\tau)$, which clearly contains $x = y-\tau$ and is a non-empty subset of both Ω and the shifted domain $\Omega-\tau$, and we write $\widetilde{\mathcal{F}}[w] := \int_{\widetilde{\Omega}} F(\nabla w)\,\mathrm{d}x$. Then it is not difficult to see that (the restrictions to $\widetilde{\Omega}$ of) both u and u_τ minimize $\widetilde{\mathcal{F}}$ in $\mathrm{W}^M_u(\widetilde{\Omega})$ and $\mathrm{W}^M_{u_\tau}(\widetilde{\Omega})$, respectively. Therefore, the definition of u_τ and the weak maximum principle imply

$$|u(y)-u(x)| = |u_\tau(x)-u(x)| \le \max_{\partial\widetilde{\Omega}} |u_\tau-u|\,.$$

Since $\partial\widetilde{\Omega}$ is non-empty and compact, the maximum on the right-hand side is attained at some $b \in \partial\widetilde{\Omega}$, and we get

$$|u(y)-u(x)| \le |u(b+\tau)-u(b)|\,.$$

After division by $|y-x| = |\tau| > 0$, this can be recast as

$$\frac{|u(y)-u(x)|}{|y-x|} \le \frac{|u(b+\tau)-u(b)|}{|b+\tau-b|}\,.$$

Taking into account $\partial\widetilde{\Omega} \subset \overline{\Omega} \cap (\overline{\Omega}-\tau)$ and $\partial\widetilde{\Omega} \subset \partial\Omega \cup (\partial\Omega-\tau)$ we next observe that we have $b, b+\tau \in \overline{\Omega}$ and in addition $b \in \partial\Omega$ or $b+\tau \in \partial\Omega$. In other words, one of the points b, $b+\tau$ is in $\partial\Omega$ and the other one is in $\overline{\Omega}$ at least. Therefore, we have in fact shown

$$\frac{|u(y)-u(x)|}{|y-x|} \le \sup_{\substack{q\in\overline{\Omega}\,,\,p\in\partial\Omega\\ q\neq p}} \frac{|u(q)-u(p)|}{|q-p|}\,.$$

Since we assumed $u \in \mathrm{C}^0(\overline{\Omega})$, the right-hand side stays unchanged if we pass from $q \in \overline{\Omega}$ to $q \in \Omega$ (and then $q \neq p$ is automatic and need not be required explicitly). After this modification, we finally take the supremum over x and y and arrive at the claim. □

In order to acquire (a very basic class of) useful comparison functions we also provide:

Lemma ('Affine functions minimize basic convex integrals.'). *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from (∗), and assume that F is convex on $\mathbb{R}^n$. Then **every affine function** $a\colon \mathbb{R}^n \to \mathbb{R}$ **minimizes** $\mathcal{F}$ in $\mathrm{W}^{1,\infty}_a(\Omega)$ (and thus also in $\mathrm{W}^M_a(\Omega)$ with $M \ge |\nabla a|$).*

As a preparation for the proof of the lemma we record for open $\Omega \subset \mathbb{R}^n$ that generally

$$\int_\Omega \nabla\varphi \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,1}_0(\Omega)\,. \tag{$**$}$$

Indeed, for $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$, one can take an open ball B with $\operatorname{spt}\varphi \subset B$ and infer from the divergence theorem that $\int_\Omega \partial_i\varphi \,\mathrm{d}x = \int_B \operatorname{div}(\varphi \mathrm{e}_i)\,\mathrm{d}x = 0$ for all $i \in \{1,2,\ldots,n\}$. From this one deduces (∗∗) first for $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega)$, then for arbitrary $\varphi \in \mathrm{W}^{1,1}_0(\Omega)$. In particular, (∗∗) holds for $\varphi \in \mathrm{W}^{1,\infty}_0(\Omega)$ on bounded open Ω, since in this case one has the inclusion $\mathrm{W}^{1,\infty}_0(\Omega) \subset \mathrm{W}^{1,1}_0(\Omega)$ (as approximation of $w \in \mathrm{W}^{1,\infty}_0(\Omega)$ with mollifications of $(w-\varepsilon)_+-(w+\varepsilon)_- \in \mathrm{W}^{1,\infty}_{\mathrm{cpt}}(\Omega)$ shows).

Proof of the lemma. We assume that the open set Ω is non-empty and thus $|\Omega| > 0$, and we observe that ∇a is constant with a value $A \in \mathbb{R}^n$ on $\mathbb{R}^n$. For arbitrary $w \in \mathrm{W}^{1,\infty}_a(\Omega)$, we have $\varphi := w-a \in \mathrm{W}^{1,\infty}_0(\Omega)$, and via (∗), Jensen's inequality for the convex function F, and (∗∗) we get

$$\mathcal{F}[w] = \int_\Omega F(A+\nabla\varphi)\,\mathrm{d}x \ge |\Omega|\, F\bigg(\frac{1}{|\Omega|}\int_\Omega (A+\nabla\varphi)\,\mathrm{d}x\bigg) = |\Omega|\,F(A) = \mathcal{F}[a]\,.$$

This establishes the claimed minimality of a. □

Remark. The last lemma extends to cases with $\mathbb{R}^N$-valued functions, where (∗∗) holds in the form $\int_\Omega \mathrm{D}\varphi\,\mathrm{d}x = 0$ for all $\varphi \in \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)$ and the proof of the lemma works in the same way.

As the final preparatory step in this section, we now **single out and discuss a class of suitable boundary data**, for which the existence program can be completed in a particularly simple way.

Definition (bounded slope condition). *We say that a pair (Ω, u_0) of a bounded open set Ω in $\mathbb{R}^n$ and a function $u_0\colon \partial\Omega \to \mathbb{R}$ satisfies the **bounded slope condition** with constant*

$L \in [0,\infty)$ *if, for every* $x_0 \in \partial\Omega$*, there exist two affine functions* $a^+_{x_0}, a^-_{x_0} \colon \mathbb{R}^n \to \mathbb{R}$ *with the properties*

$$\begin{gathered} a^-_{x_0}(x_0) = u_0(x_0) = a^+_{x_0}(x_0)\,, \\ a^-_{x_0} \le u_0 \le a^+_{x_0} \ \textit{on}\ \partial\Omega\,, \\ |\nabla a^-_{x_0}| \le L\,, \quad |\nabla a^+_{x_0}| \le L\,. \end{gathered}$$

Clearly the gradient of each affine function $\nabla a^\pm_{x_0}$ is constant, its modulus $|\nabla a^\pm_{x_0}|$ gives the maximal slope of $a^\pm_{x_0}$, and thus the required **uniform bound L for these slopes** is actually **responsible for the name** of the bounded slope condition.

Remarks (on the bounded slope condition)**.**

(1) **Necessary criterion:** If (Ω, u_0) satisfies the bounded slope condition with some constant $L \in [0,\infty)$, then, unless u_0 is itself affine, **Ω is necessarily convex**.

Proof. If $\nabla a^+_{x_0} = \nabla a^-_{x_0}$ holds for some $x_0 \in \partial\Omega$, we get $a^+_{x_0} = a^-_{x_0}$ on $\mathbb{R}^n$ (since $a^+_{x_0}(x_0) = a^-_{x_0}(x_0)$ is postulated) and u_0 is affine.

Thus, we can assume $\nabla a^+_{x_0} \ne \nabla a^-_{x_0}$ for all $x_0 \in \partial\Omega$. Then we claim

$$\Omega = \bigcap_{x_0 \in \partial\Omega} \{a^+_{x_0} > a^-_{x_0}\}\,, \tag{$*{*}*$}$$

and once this is verified, Ω is an intersection of half-spaces and thus convex as claimed. It remains to check ($***$). To this end we observe that $a^+_{x_0} \ge a^-_{x_0}$ on $\partial\Omega$ implies first $a^+_{x_0} \ge a^-_{x_0}$ on Ω (since each point of the bounded Ω can be written as convex combination of two points from $\partial\Omega$) and then even $a^+_{x_0} > a^-_{x_0}$ on Ω (since equality $a^+_{x_0}(z) = a^-_{x_0}(z)$ at $z \in \Omega$ together with $\nabla a^+_{x_0} \ne \nabla a^-_{x_0}$ would yield the contradiction that $a^+_{x_0} < a^-_{x_0}$ on a part of each neighborhood of z). This shows the inclusion '$\subset$' in ($***$). To establish the opposite inclusion, we consider $x \in \mathbb{R}^n$ with $x \notin \Omega$. We fix an arbitrary $z \in \Omega$ and observe that the closed line segment from x to z contains some $x_0 \in \partial\Omega$. If we had $a^+_{x_0}(x) > a^-_{x_0}(x)$, then with the information $a^+_{x_0}(z) > a^+_{x_0}(z)$ from the previous reasoning we would get $a^+_{x_0}(x_0) > a^+_{x_0}(x_0)$ as well. Since this contradicts one requirement of the bounded slope condition, we thus have $x \notin \{a^+_{x_0} > a^-_{x_0}\}$ and $x \notin \bigcap_{x_0\in\partial\Omega}\{a^+_{x_0} > a^-_{x_0}\}$. This gives '$\subset$' in ($***$) and completes the proof. ☐

(2) **Sufficient criterion:** If a bounded open $\boldsymbol{\Omega}$ in $\mathbb{R}^n$ is **uniformly convex** and it holds $\boldsymbol{u_0 \in \mathrm{C}^2(\mathbb{R}^n)}$, then (Ω, u_0) (or more precisely $(\Omega, u_0|_{\partial\Omega})$) **satisfies the bounded slope condition** with some constant $L \in [0,\infty)$.

Idea of proof. The proof can be reduced to the case of the boundary point $x_0 = 0$ with $u_0(0) = 0$, $\nabla u_0(0) = 0$. In this situation, the uniform convexity assumption implies the existence of an $\varepsilon > 0$ such that $x_n \ge \varepsilon|\overline{x}|^2$ holds for all $x = (\overline{x}, x_n) \in \overline{\Omega}$ with $|x| \ll 1$. By Taylor's theorem we have $|u_0(x)| \le C(|\overline{x}|^2 + x_n^2)$ for $|x| \ll 1$, and altogether we can conclude $|u_0(x)| \le \frac{C}{\varepsilon} x_n$ for $x \in \overline{\Omega}$ with $|x| \ll 1$. This shows that $a^\pm_0(x) := \pm L x_n$ with $L = \frac{C}{\varepsilon}$ satisfies the requirements of the bounded slope condition in a neighborhood of 0, and a finer analysis of the reasoning shows that the size of the neighborhood can be controlled. Possibly enlarging L, one can ensure the requirements also away from 0. ☐

Uniformly convex functions. Consider $\varepsilon > 0$. A function $f \colon K \to \mathbb{R}$ on a bounded convex set $K \subset \mathbb{R}^n$ is called ε-uniformly convex if the function $K \to \mathbb{R}$, $x \mapsto f(x) - \varepsilon|x|^2$ is convex on K.

Uniformly convex sets. A bounded convex open set Ω in $\mathbb{R}^n$ is called uniformly convex if there exists some $\varepsilon > 0$ such that Ω coincides locally near every point of $\partial\Omega$ with the rotated supergraph of an ε-uniformly convex function (defined on a bounded convex set in $\mathbb{R}^{n-1}$). If Ω has a C^2 boundary, uniform convexity is equivalent with the condition that all principal curvatures (with respect to the inward unit normal) of $\partial\Omega$ at all points of $\partial\Omega$ are larger than a fixed positive constant.

(3) The sufficient criterion in (2) is quite sharp with regard to the assumptions on both Ω and u_0. This is demonstrated by the non-sufficiency of the following slightly weaker conditions (with corresponding examples in dimension 2):

- If Ω is convex and $u_0 \in \mathrm{C}^\infty(\mathbb{R}^n)$, it does not suffice for the bounded slope condition. Indeed, for $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\,,\, x_2 > 0\}$ with flat boundary portion $(-1,1)\times\{0\}$ and $u_0(x) = x_1^2$, there exists no suitable affine function a_0^+.
- If Ω is strictly convex and $u_0 \in \mathrm{C}^\infty(\mathbb{R}^n)$, it does not suffice for the bounded slope condition. Indeed, for $\Omega = \{x \in \mathbb{R}^2 : x_1^4 < x_2 < 1\}$ and $u_0(x) = x_1^2$, there exists no suitable affine function a_0^+ (essentially since $u_0(x) = \sqrt{x_2}$ for $x \in \partial\Omega$ with $|x| \ll 1$).
- If Ω is uniformly convex and $u_0 \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ with any fixed $\alpha \in (0,1)$, it does not suffice for the bounded slope condition. Indeed, for $\Omega = \{x \in \mathbb{R}^2 : x_1^2 < x_2 < 1\}$ and $u_0(x) = |x_1|^{1+\alpha}$, there exists no suitable affine function a_0^+ (essentially since $u_0(x) = x_2^{\frac{1+\alpha}{2}}$ for $x \in \partial\Omega$ with $|x| \ll 1$ and $\frac{1+\alpha}{2} < 1$).
  Alternatively, the last example also works with the ball $\Omega = \{x \in \mathbb{R}^2 : x_1^2+(x_2-1)^2 < 1\}$ and $u_0(x) = |x_1|^{1+\alpha}$.

(4) **If (Ω, u_0) satisfies the bounded slope condition** with constant $L \in [0,\infty)$, then the pointwise infimum $\overline{u}_0 := \inf_{x_0\in\partial\Omega} a_{x_0}^+$ of the affine functions $a_{x_0}^+$ from the definition extends u_0 to all of $\mathbb{R}^n$. Moreover, since all $a_{x_0}^+$ are in particular Lipschitz with constant L, also $\overline{u}_0$ is Lipschitz with constant L on $\mathbb{R}^n$. In particular, identifying u_0 with its extension $\overline{u}_0|_\Omega$ and relying on Rademacher's theorem, we **can understand $u_0 \in \mathrm{W}^M(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$** and $\mathrm{W}^M_{u_0}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$ **for all $M \ge L$**.

Alternatively, one can arrive at the same conclusions by considering $\sup_{x_0\in\partial\Omega} a_{x_0}^-$.

At this stage, the existence result targeted at can be stated and then quickly proved.

Theorem (existence under the bounded slope condition). *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, $F \in \mathrm{C}^0(\mathbb{R}^n)$, $\mathcal{F}$ from $(*)$, $u_0\colon \partial\Omega \to \mathbb{R}$, $L \in [0,\infty)$, and assume that F is convex on $\mathbb{R}^n$. If (Ω, u_0) satisfies the bounded slope condition with constant L, then **there exists a minimizer** of $\mathcal{F}$ **in $\mathrm{W}^{1,\infty}_{u_0}(\Omega)$**.*

Proof. We first implement the proof under the additional assumption that F is strictly convex on $\mathbb{R}^n$. We fix $M > L$ and record $u_0 \in \mathrm{W}^M(\Omega)$ (and thus $\mathrm{W}^M_{u_0}(\Omega) \neq \emptyset$) according to the preceding Remark (4). On this basis we employ the initial proposition of this section to find a minimizer u of $\mathcal{F}$ in the restricted class $\mathrm{W}^M_{u_0}(\Omega)$, where in view of $\mathrm{W}^M_{u_0}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$ the boundary condition $u = u_0$ on $\partial\Omega$ holds in the classical sense. Thus $u|_\Omega$ is bounded by $a_p^\pm$ from the bounded slope condition in the sense that

$$a_p^- \le u \le a_p^+ \qquad \text{on } \partial\Omega$$

for every fixed $p \in \partial\Omega$. Since u is a minimizer of $\mathcal{F}$ in $\mathrm{W}^M_u(\Omega)$ and by a previous lemma the affine functions $a_p^\pm$ are minimizers $\mathcal{F}$ in $\mathrm{W}^M_{a_p^\pm}(\Omega)$, the comparison principle applies and gives

$$a_p^- \le u \le a_p^+ \qquad \text{on } \Omega$$

for every fixed $p \in \partial\Omega$. As the bounded slope condition also requires $a_p^\pm(p) = u_0(p) = u(p)$, we infer

$$a_p^-(q) - a_p^-(p) \le u(q) - u(p) \le a_p^+(q) - a_p^+(p) \qquad \text{for all } q \in \Omega\,,\, p \in \partial\Omega\,.$$

Finally, we involve the bound $|\nabla a_p^\pm| \le L$, which means that $a_p^\pm$ is Lipschitz continuous on $\mathbb{R}^n$ with constant $\le L$. Applying this on both the left-hand and right-hand side, we infer

$$-L|q-p| \le u(q) - u(p) \le L|q-p| \qquad \text{for all } q \in \Omega\,,\ p \in \partial\Omega\,.$$

or in other words

$$\sup_{q\in\Omega\,,\,p\in\partial\Omega} \frac{|u(q)-u(p)|}{|q-p|} \le L\,.$$

The reduction-to-the-boundary lemma improves this to

$$\sup_{\substack{x,y\in\Omega\\ y\neq x}} \frac{|u(y)-u(x)|}{|y-x|} \le L\,,$$

which means that u is Lipschitz continuous on Ω with Lipschitz constant $\le L$. From this one can deduce, either by using that ∇u is a.e. a classical derivative thanks to Rademacher's theorem or by a more mollification argument, that

$$\|\nabla u\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le L\,.$$

Since we have taken $M > L$, an earlier lemma brings us back from restricted to unrestricted minimality, that is, it ensures that u is a minimizer of $\mathcal{F}$ also in $\mathrm{W}^{1,\infty}_{u_0}(\Omega) = \mathrm{W}^{1,\infty}_{u}(\Omega)$. This establishes the claim under the additional strict convexity assumption.

In order to complete the proof for merely convex F, we introduce, for $\varepsilon > 0$, the regularizations $F_\varepsilon \in \mathrm{C}^0(\mathbb{R}^n)$, which are given by $F_\varepsilon(z) := F(z)+\varepsilon|z|^2$ for $z \in \mathbb{R}^n$ and are strictly convex on $\mathbb{R}^n$. By the first part of the proof, for every $\varepsilon > 0$ there exists some $u_\varepsilon \in \mathrm{W}^L_{u_0}(\Omega)$ which minimizes $\mathcal{F}_\varepsilon$ in $\mathrm{W}^{1,\infty}_{u_0}(\Omega)$. As observed earlier, $\mathrm{W}^L_{u_0}(\Omega)$ is sequentially weakly$*$ compact, and thus there is a positive null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $u_{\varepsilon_k} \xrightarrow[k\to\infty]{*} u$ converges weakly$*$ in $\mathrm{W}^{1,\infty}(\Omega)$ to some limit $u \in \mathrm{W}^L_{u_0}(\Omega) \subset \mathrm{W}^{1,\infty}_{u_0}(\Omega)$. The using in turn weak$*$ lower semicontinuity of $\mathcal{F}$ (which is available thanks to convexity of F and the results of Section 2.2), the minimality of u_{ε_k} for $\mathcal{F}_{\varepsilon_k}$, and $\lim_{k\to\infty} \varepsilon_k \int_\Omega |\nabla w|^2\,\mathrm{d}x = 0$, we conclude

$$\mathcal{F}[u] \le \liminf_{k\to\infty} \mathcal{F}[u_{\varepsilon_k}] \le \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}[u_{\varepsilon_k}] \le \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}[w] = \mathcal{F}[w]$$

for all $w \in \mathrm{W}^{1,\infty}_{u_0}(\Omega)$. Thus u minimizes $\mathcal{F}$ in $\mathrm{W}^{1,\infty}_{u_0}(\Omega)$, and we have completed the existence program also in the general case. □

Remarks (on the existence theorem and the semi-classical theory as a whole)**.**

(1) The semi-classical existence theory **allows to solve certain variational problems with *scalar* functions which cannot be treated directly** with the methods of Sections 2.1, 2.2, 2.3. For instance, it applies to the scalar 1-energy $\int_\Omega |\nabla w|\,\mathrm{d}x$ and the scalar non-parametric area integral $\int_\Omega \sqrt{1+|\nabla w|^2}\,\mathrm{d}x$ with a Dirichlet boundary condition (provided that the boundary datum satisfies a bounded slope condition, of course).

(2) The semi-classical theory can alternatively be implemented by working with the space of Lipschitz functions on Ω and the subspace of functions with Lipschitz constant $\le M$ in place of the spaces $\mathrm{W}^{1,\infty}(\Omega)$ and $\mathrm{W}^M(\Omega)$. In the end, this leads to the same existence theorem (which is clear, since on convex Ω the spaces coincide up to passage to representatives).

In these notes, however, after the considerations of the previous sections the above implementation with Sobolev functions appears quite natural (and it also leads to slightly more general versions of some of the auxiliary lemmas).

(3) **The affine functions in the bounded slope condition play the role of barriers** as commonly used in the theory of PDEs, and in fact the above method can be extended to more general domains and boundary data without the bounded slope condition *if* one can only ensure the existence of barriers with similar properties (but no longer necessarily affine). In general, the construction of barriers is a non-trivial matter and can be achieved (only) under severely refined assumptions on the integrand F (in contrast to the case above where solely convexity of F was needed); see, for instance, [11, Sections 1.3, 1.4].

Chapter 3

Euler-Lagrange Equations

3.1 The case without constraints (or with linear ones only)

In principle the **first-order criteria for minimum and maximum points**, as they are known from calculus for functions in n variables, **apply also to functionals on ∞-dimensional spaces**. In order to make this precise in wide generality, one considers, for a real vector space $\mathcal{X}$ and $\varepsilon > 0$, **(special) variations**

$$(w+t\varphi)_{t\in(-\varepsilon,\varepsilon)}$$

of an element $w \in \mathcal{X}$ in direction $\varphi \in \mathcal{X}$ and uses **directional derivatives**, for which the following terminology is common:

Definition (first variation). *Consider a real vector space $\mathcal{X}$ and functional $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$ on a subset $\mathcal{A}$ of $\mathcal{X}$. Suppose, for $w \in \mathcal{A}$ and $\varphi \in \mathcal{X}$, that we have $w+t\varphi \in \mathcal{A}$ and $|\mathcal{F}[w+t\varphi]| < \infty$ for $|t| \ll 1$ and that the directional derivative*

$$\delta\mathcal{F}[w;\varphi] := \partial_\varphi \mathcal{F}[w] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}[w+t\varphi]$$

*exists in $\mathbb{R}$. Then $\delta\mathcal{F}[w;\varphi]$ is called the **first variation of $\mathcal{F}$** at w in direction φ.*

Remark and Definition (admissible variations)**.** One says that a variation $(w+t\varphi)_{t\in(-\varepsilon,\varepsilon)}$ is **$\mathcal{A}$-admissible** if it satisfies $w+t\varphi \in \mathcal{A}$ for $|t| \ll 1$ (as required for the existence of $\delta\mathcal{F}[w;\varphi]$).

With this terminology we state the first-order criterion as follows:

Proposition (first-order criterion for minimizers). *Fix a real vector space $\mathcal{X}$ and $u \in \mathcal{A} \subset \mathcal{X}$.*

(I) ***Necessary criterion:*** *If u is a minimizer of a functional $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$ in $\mathcal{A}$, then*

$$\delta\mathcal{F}[u;\varphi] = 0 \text{ holds for all } \varphi \in \mathcal{X} \text{ such that } \delta\mathcal{F}[u;\varphi] \text{ exists.}$$

(II) ***Sufficient criterion:*** *Suppose that $\mathcal{A}$ is convex with $\mathcal{A} \subset u+\mathcal{V}$ for some vector subspace $\mathcal{V}$ of $\mathcal{X}$ and that $\mathcal{F}\colon \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ is convex on $\mathcal{A}$. If*

$$\delta\mathcal{F}[u;\varphi] \text{ exists with } \delta\mathcal{F}[u;\varphi] = 0 \text{ for all } \varphi \in \mathcal{V}\,,$$

then u is a minimizer of $\mathcal{F}$ in $\mathcal{A}$.

Proof. To prove (I) consider a minimizer u of $\mathcal{F}$ in $\mathcal{A}$. Whenever $\delta\mathcal{F}[u;\varphi]$ exists, the definition requires $u+t\varphi \in \mathcal{A}$ and $|\mathcal{F}[u+t\varphi]| < \infty$ for $|t| \ll 1$ so that $t \mapsto \mathcal{F}[u+t\varphi]$ is defined and $\mathbb{R}$-valued on an interval around 0 with a minimum at 0. The first-order criterion for minimum points of single-variable functions then gives $\delta\mathcal{F}[u;\varphi] = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{F}[u+t\varphi] = 0$.

In order to establish (II) we assume $\mathcal{A} \subset u+\mathcal{V}$ and $\delta\mathcal{F}[u;\varphi] = 0$ for all $\varphi \in \mathcal{V}$. For arbitrary $w \in \mathcal{A}$ and $t \in [0,1]$, the convexity of $\mathcal{A}$ and $\mathcal{F}$ guarantees $u+t(w-u) = (1-t)u+tw \in \mathcal{A}$ and

$$\mathcal{F}[u+t(w-u)] \le \mathcal{F}[u]+t(\mathcal{F}[w]-\mathcal{F}[u])\,.$$

Furthermore, in view of the assumptions we have $w-u \in \mathcal{V}$, and we get

$$0 = \delta\mathcal{F}[u;w-u] = \lim_{t\searrow 0}\frac{\mathcal{F}[u+t(w-u)] - \mathcal{F}[u]}{t} \le \mathcal{F}[w] - \mathcal{F}[u]\,.$$

Hence we have shown $\mathcal{F}[u] \le \mathcal{F}[w]$ for all $w \in \mathcal{A}$. □

Remark. It is common to call u an **extremal** of $\mathcal{F}$ if $\delta\mathcal{F}[u;\varphi] = 0$ holds for suitably many φ. However, this is not a precise definition, and the terminology is also misleading insofar that extremals are merely critical points of $\mathcal{F}$ and in general (i.e. without convexity or concavity of $\mathcal{F}$) cannot be expected to be local minimum or maximum points.

For integral functionals, the first variation can be computed explicitly:

Proposition (first variation and Euler equation for integral functionals). *Given an open set Ω in $\mathbb{R}^n$ and an $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)\otimes\mathcal{B}(\mathbb{R}^{N\times n})$-measurable $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$, set*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x$$

for $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ (whenever the integral exists in $\overline{\mathbb{R}}$).

(I) ***First-variation formula:*** *For $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1(\Omega)$ and $\varphi \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, there exist $\mathcal{F}[u+t\varphi] \in \mathbb{R}$ with $|t| \ll 1$ and*

$$\delta\mathcal{F}[u;\varphi] = \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi\big]\,\mathrm{d}x \in \mathbb{R}$$

provided *there exist a majorant $\Phi \in \mathrm{L}^1(\Omega)$, a null set $E \subset \Omega$ and $\varepsilon > 0$ such that, for $t \in (-\varepsilon,\varepsilon)$, the integrand $F(x,y,z)$ is totally differentiable in $(y,z) \in \mathbb{R}^N\times\mathbb{R}^{N\times n}$ at all points $\xi_t(x) := (x,u(x)+t\varphi(x),\mathrm{D}u(x)+t\mathrm{D}\varphi(x))$, $x \in \Omega\setminus E$, with derivative bounds $|\nabla_z F(\xi_t(x))\cdot\mathrm{D}\varphi(x)| \le \Phi(x)$ and $|\nabla_y F(\xi_t(x))\cdot\varphi(x)| \le \Phi(x)$.*

(II) ***Euler(-Lagrange) equation:*** *Assume that the requirements of (I) are met for fixed $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$. Then we have $\nabla_z F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$ and $\nabla_y F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and the first-order criterion*

$$\delta\mathcal{F}[u;\varphi] = 0 \text{ for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$$

is satisfied **if and only if**[1] $\operatorname{div}\big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\big]$ *exists weakly in $\mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with*

$$-\operatorname{div}\big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\big] + \nabla_y F(\,\cdot\,,u,\mathrm{D}u) \equiv 0 \qquad \text{on } \Omega\,. \tag{EL}$$

[1]The weak divergence $\operatorname{div} W \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ of a matrix field $W \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$ is taken row-wise, that is, in the sense of $(\operatorname{div} W)_i = \operatorname{div}(W_{i1},W_{i2},\ldots,W_{in})$ for $i \in \{1,2,\ldots,N\}$. As a result, the weak divergence is characterized by the equality $\int_\Omega W\cdot\mathrm{D}\varphi\,\mathrm{d}x = -\int_\Omega(\operatorname{div} W)\cdot\varphi\,\mathrm{d}x$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$.

One calls the equation (EL) the **Euler(-Lagrange)** equation of the integral functional $\mathcal{F}$ and the partial differential operator $\mathrm{E}_{\mathcal{F}}$, defined by

$$\mathrm{E}_{\mathcal{F}} u := -\operatorname{div}\big[\nabla_z F(\,\cdot\,, u, \mathrm{D}u)\big] + \nabla_y F(\,\cdot\,, u, \mathrm{D}u)\,,$$

the **Euler(-Lagrange) operator** of $\mathcal{F}$.

Proof of (I). For $t \in (-\varepsilon, \varepsilon)$ and $x \in \Omega \setminus E$, we compute with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\xi_t(x)) = \nabla_z F(\xi_t(x)) \cdot \mathrm{D}\varphi(x) + \nabla_y F(\xi_t(x)) \cdot \varphi(x)$$

and then from the assumptions obtain the uniform L^1-bound

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t} F(\xi_t(x))\Big| \le 2\Phi(x)\,.$$

In particular, this implies $|F(\xi_t(x)) - F(\xi_0(x))| \le t \sup_{s\in(0,t)} \big|\frac{\mathrm{d}}{\mathrm{d}s} F(\xi_s(x))\big| \le 2t\Phi(x)$, and in view of $F(\xi_0) = F(\,\cdot\,, u, \mathrm{D}u) \in \mathrm{L}^1(\Omega)$ we infer $F(\xi_t) \in \mathrm{L}^1(\Omega)$, that is, $F[u+t\varphi] \in \mathbb{R}$ exists for $t \in (-\varepsilon, \varepsilon)$. On the basis of these observations we **may compute by exchange of differentiation and integration**

$$\delta\mathcal{F}[u;\varphi] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_\Omega F(\xi_t(x))\,\mathrm{d}x = \int_\Omega \nabla_z F(\xi_0(x)) \cdot \mathrm{D}\varphi(x) + \nabla_y F(\xi_0(x)) \cdot \varphi(x)\,\mathrm{d}x \in \mathbb{R}\,.$$

In view of $\xi_0(x) = (x, u(x), \mathrm{D}u(x))$ this is the claimed formula. □

Proof of (II). We first prove the $\mathrm{L}^1_{\mathrm{loc}}$-integrability of $\nabla_z F(\,\cdot\,, u, \mathrm{D}u)$ and $\nabla_y F(\,\cdot\,, u, \mathrm{D}u)$, which clearly follows once we establish $\partial_{z_{ij}} F(\,\cdot\,, u, \mathrm{D}u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\partial_{y_i} F(\,\cdot\,, u, \mathrm{D}u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ for all $i \in \{1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots, n\}$. To verify this, we first record from the assumptions of (I) with $t = 0$ that $(\nabla_z F(\,\cdot\,, u, \mathrm{D}u) \cdot \mathrm{D}\varphi)_\pm \in \mathrm{L}^1(\Omega)$ and $(\nabla_y F(\,\cdot\,, u, \mathrm{D}u) \cdot \varphi)_\pm \in \mathrm{L}^1(\Omega)$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$. For compact $K \subset \Omega$, we then choose $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ with $\mathrm{D}\varphi \equiv \mathrm{e}_{ij}$ on K and $\varphi \equiv \mathrm{e}_i$ on K, respectively, to infer $(\partial_{z_{ij}} F(\,\cdot\,, u, \mathrm{D}u))_\pm \in \mathrm{L}^1(K)$ and $(\partial_{y_i} F(\,\cdot\,, u, \mathrm{D}u))_\pm \in \mathrm{L}^1(K)$. This implies the claimed integrability.

Now we turn to the proof of the claimed equivalence. In view of (I), the first-order criterion $\delta\mathcal{F}[u;\varphi] = 0$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ can be rewritten in the form

$$\int_\Omega \nabla_z F(\,\cdot\,, u, \mathrm{D}u) \cdot \mathrm{D}\varphi\,\mathrm{d}x = -\int_\Omega \nabla_y F(\,\cdot\,, u, \mathrm{D}u) \cdot \varphi\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)\,,$$

and this means precisely that $\operatorname{div}\big[\nabla_z F(\,\cdot\,, u, \mathrm{D}u)\big] = \nabla_y F(\,\cdot\,, u, \mathrm{D}u)$ exists weakly on Ω. □

Remarks (on the Euler equation)**.**

(1) In a way, the proposition identifies the **first derivative of** the **integral functional** $\mathcal{F}$ as the **differential operator** $\mathrm{E}_{\mathcal{F}}$ and the **first-order criterion for minimizers** of $\mathcal{F}$ as the second-order **differential equation** (EL). In case $N = 1$, (EL) is a *single* equation, in case $N \ge 2$ it is a *system* of N equations. In case $n = 1$ it is an *ordinary* differential equation, in case $n \ge 2$ a *partial* differential equation. In general, (EL) is *non-linear* (but, if $\nabla_z F$ is C^1, expansion of the divergence shows that (EL) is generally *quasi-linear*, i.e. linear in $\mathrm{D}^2 u$).

(2) The **connection between an integral functional and its Euler equation is crucially based on the fundamental lemma in the calculus of variations** (that is, the lemma which says that $w \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $\int_\Omega w \cdot \varphi\,\mathrm{d}x = 0$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ satisfies $w \equiv 0$ a.e. on Ω). The lemma is usually used to establish uniqueness of weak derivatives and the weak divergence and thus plays an implicit rule in the above proof. Most crucially, however, the lemma ensures, if $\nabla_z F$ is C^1, that C^2 weak solutions to (EL) are also classical solutions.

(3) The proposition is **stated for *weakly* differentiable** u and thus (potentially) applies to minimizers u in Sobolev spaces, as they are provided by the existence theory of Chapter 2. (In contrast, for the integrand F, the existence of classical derivatives $\nabla_z F$, $\nabla_y F$ has been assumed. However, since F is *not* obtained from an existence theory but rather given, this is also reasonable and can be checked in many concrete cases.)

(4) A **zero-order functional** $\mathcal{G}[w] := \int_\Omega G(\,\cdot\,, w)\,\mathrm{d}x$ (with suitable measurable G) is connected, in an analogous way, with its **Euler equation** $\nabla_y G(\,\cdot\,, u) \equiv 0$ a.e. on Ω. However, the information in this equation (which is not a differential equation, as no derivatives of the unknown u occur) is often rather obvious and **less interesting**. Indeed, in the unconstrained case one expects anyway that $u \in \mathrm{L}^p(\Omega, \mathbb{R}^N)$ minimizes $\mathcal{G}$ in $\mathrm{L}^p(\Omega, \mathbb{R}^N)$ if and only if $u(x)$ minimizes $G(x, \cdot\,)$ for a.e. $x \in \Omega$, while the above Euler equation merely shows, for a minimizer u of $\mathcal{G}$ in $\mathrm{L}^p(\Omega, \mathbb{R}^N)$, that $u(x)$ is a critical point of $G(x, \cdot\,)$ for a.e. $x \in \Omega$.

Next we show that the **assumptions of the proposition** and particularly the assumption on the existence of the majorant Φ for quantities involving u (which is clearly difficult to verify without explicit knowledge of u) **can be obtained from growth assumptions on the integrand *F*** alone. In fact, the **subsequent theorem achieves this for minimization in Dirichlet classes** in two slightly different settings. The first one needs weaker localized assumptions and gives the necessity of the Euler equation in the weak/distributional sense with $\mathrm{C}^\infty_{\mathrm{cpt}}$ test functions. The second one comes with somewhat stronger global assumptions, ensures that $\mathrm{W}^{1,p}_0$ test functions can be used, and also gives the sufficiency of the Euler equation in convex situations:

Theorem (on **growth conditions and the Euler equation in $\mathbf{W^{1,p}}$**). *Consider an open set Ω in $\mathbb{R}^n$ and an $\mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^{N\times n})$-measurable $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$, which is totally differentiable in (y, z) at all points of $(\Omega \setminus E) \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ with a null set $E \subset \Omega$. Then, for the integral functional given by*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,, w, \mathrm{D}w)\,\mathrm{d}x$$

(*whenever the integral exists in* $\overline{\mathbb{R}}$), *we have the following assertions*:

(I) *For $p \in [1, \infty]$, consider $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $F(\,\cdot\,, u, \mathrm{D}u) \in \mathrm{L}^1(\Omega)$, and assume that $\nabla_z F$ and $\nabla_y F$ satisfy the growth condition*

$$|\nabla_z F(x, y, z)| + |\nabla_y F(x, y, z)| \le \begin{cases} \Psi(x) + C|z|^p + C|y|^{p^*}, & \text{if } p \le n \\ \Psi(x) + C|z|^p + b(|y|)\,, & \text{if } n < p < \infty \\ \Psi(x) + b(|z|) + b(|y|)\,, & \text{if } p = \infty \end{cases}$$

for all $(x, y, z) \in (\Omega \setminus E) \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ with $\Psi \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, $C \in [0, \infty)$, $p^* := \frac{np}{n-p}$ if $p < n$, any exponent $p^* \in [1, \infty)$ if $p = n$, and locally bounded $b\colon [0, \infty) \to [0, \infty)$. Then $\delta\mathcal{F}[u; \varphi]$ exists and is given by the first-variation formula for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$. Moreover, if u is a minimizer of $\mathcal{F}$ in $u + \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$, then u weakly solves the Euler equation (EL).

(II) *For $p \in [1,\infty]$, consider $u \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ with $F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1(\Omega)$, and assume that Ω is a bounded Lipschitz domain and $\nabla_z F$ and $\nabla_y F$ satisfy the growth conditions*

$$|\nabla_z F(x,y,z)| \le \begin{cases} \Psi(x)^{1/p'} + C|z|^{p/p'} + C|y|^{p^*/p'}, & \text{if } p \le n \\ \Psi(x)^{1/p'} + C|z|^{p/p'} + b(|y|)\,, & \text{if } n < p < \infty \\ \Psi(x) + b(|z|) + b(|y|)\,, & \text{if } p = \infty \end{cases},$$

$$|\nabla_y F(x,y,z)| \le \begin{cases} \Psi(x)^{1/(p^*)'} + C|z|^{p/(p^*)'} + C|y|^{p^*/(p^*)'}, & \text{if } p \le n \\ \Psi(x) + C|z|^{p} + b(|y|)\,, & \text{if } n < p < \infty \\ \Psi(x) + b(|z|) + b(|y|)\,, & \text{if } p = \infty \end{cases}$$

for all $(x,y,z) \in (\Omega\setminus E)\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ with $\Psi \in \mathrm{L}^1(\Omega)$ and C, p^*, b as in (I), where we have abbreviated $p' := \frac{p}{p-1} \in [1,\infty]$, $(p^*)' := \frac{p^*}{p^*-1} \in (1,\infty]$ (and understand $1/1' = 0$, $0^0 = 1$). Then $\delta\mathcal{F}[u;\varphi]$ exists and is given by the first-variation formula for all $\varphi \in \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$. Moreover, we have

$$u \text{ minimizes } \mathcal{F} \text{ in } \mathrm{W}^{1,p}_u(\Omega,\mathbb{R}^N) \implies \delta\mathcal{F}[u;\varphi] = 0 \text{ holds for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$$
$$\iff \text{(EL) holds for } u\,,$$

and, if $F(x,\,\cdot\,)$ is convex on $\mathbb{R}^N\times\mathbb{R}^{N\times n}$ for all $x \in \Omega\setminus E$, the first '$\Longrightarrow$' is also an '$\Longleftrightarrow$'.

Here the exponents in (II) can be written in alternative ways, and most importantly one can simplify $\frac{p}{p'} = p{-}1$, $\frac{p^*}{(p^*)'} = p^*{-}1$. However, the above statement is intended to showcase that they occur for a reason and in a systematic way, as will now be further clarified.

Remarks (on the growth conditions)**.**

(1) **Model examples** of variational integrals and some corresponding Euler equations **have been discussed in Chapter 1** (though at that stage the precise connection and the term Euler equation were not yet available). In the basic cases and many variants the growth conditions are satisfied with an obvious choice of the exponent p and both the existence theory and the above theorem apply in $\mathrm{W}^{1,p}$. In particular, this is true in virtually all cases with integrand F independent of y, while in y-dependent cases the validity of growth conditions and the Euler equation can sometimes be an issue.

(2) The **growth conditions** in the theorem **are made up in a way that** allows to deduce,

- in (I), that $\boldsymbol{\nabla_z F(\,\cdot\,,w,\mathrm{D}w),\ \nabla_y F(\,\cdot\,,w,\mathrm{D}w)}$ **are** $\mathbf{L}^1_{\mathbf{loc}}$ on Ω for all $w \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ (uses Sobolev's embedding on compact $K \subset \Omega$ with smooth ∂K)
- in (II), that $\boldsymbol{\nabla_z F(\,\cdot\,,w,\mathrm{D}w){\cdot}\mathrm{D}\varphi,\ \nabla_y F(\,\cdot\,,w,\mathrm{D}w){\cdot}\varphi}$ **are** $\mathbf{L}^1$ on Ω for all $w,\varphi \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ (where, e.g. for the $\nabla_z F$-term, one first gets $|\nabla_z F(\,\cdot\,,w,\mathrm{D}w)| \in \mathrm{L}^{p'}(\Omega)$ and then relies on Hölder's inequality).

Clearly, these integrability properties come along with quantitative estimates, which are also needed in order to ensure the validity of the first-variation formula (see the proof below).

(3) ***If $\nabla_z F$ and $\nabla_y F$ exist and are continuous*** on $\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$, the **growth condition** in (I) is automatically satisfied[2] **with $p=\infty$**. If one has a minimizer u is in $\mathrm{W}^{1,\infty}_{(\mathrm{loc})}(\Omega,\mathbb{R}^N)$ (as provided, in some cases, by the semi-classical existence theory of Section 2.5), then this perfectly suffices to obtain the Euler equation for u. In most cases, however, the existence theory provides minimizers in a Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ with $p<\infty$ only, and then the growth condition with $p=\infty$ is not sufficient for the Euler equation, but rather one needs p-growth with the same $p<\infty$.

(4) A closer look at the involved exponents shows that the growth requirements in (II) are stronger than those in (I) but are still comparably general and often satisfied. What is more restrictive in practice, is the convexity requirement in (y,z), which is needed for the backward implication in (II) and thus the sufficiency of the Euler equation.

(5) If one can a-priori restrict the class of competitors to a class of *bounded* $\mathrm{W}^{1,p}$-functions, then in case $p\le n$ one can take into account the boundedness and thus improve on the (exponents in) the growth assumptions needed for the Euler equation.

Proof of (I) *in the theorem.* We first consider the case $p\le n$. For the given $u\in\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, arbitrary $\varphi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$, and $t\in\mathbb{R}$, we abbreviate once more $\xi_t:=(\,\cdot\,,u{+}t\varphi,\mathrm{D}u{+}t\mathrm{D}\varphi)$. For $|t|\le 1$, the growth condition gives

$$\begin{aligned}|\nabla_z F(\xi_t)|+|\nabla_y F(\xi_t)| &\le \Psi + C|\mathrm{D}u{+}t\mathrm{D}\varphi|^p + C|u{+}t\varphi|^{p^*}\\ &\le \Psi + 2^{p-1}C\big(|\mathrm{D}u|^p{+}|\mathrm{D}\varphi|^p\big)+2^{p^*-1}C\big(|u|^{p^*}{+}|\varphi|^{p^*}\big)=:\Lambda\in\mathrm{L}^1_{\mathrm{loc}}(\Omega)\,,\end{aligned}$$

where $|u|^{p^*}\in\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ results from the local Sobolev embedding $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\subset\mathrm{L}^{p^*}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. Setting $\Phi:=\Lambda(|\mathrm{D}\varphi|{+}|\varphi|)$, we infer $|\nabla_z F(\xi_t)\cdot\mathrm{D}\varphi|\le\Phi$, $|\nabla_y F(\xi_t)\cdot\varphi|\le\Phi$ for $|t|\le 1$ with $\Phi\in\mathrm{L}^1(\Omega)$ (since $|\mathrm{D}\varphi|{+}|\varphi|$ is bounded and compactly supported). With the majorant Φ at hand, we may apply (II) in the last proposition, and this gives the Euler equation (EL) in case u minimizes $\mathcal{F}$ in $u{+}\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$.

In case $n<p<\infty$, from Sobolev's embedding we get $u\in\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\subset\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and thus $\sup_{|t|\le1}b(|u{+}t\varphi|)\le\sup_{s\le|u|+|\varphi|}b(s)\in\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ (since also b is locally bounded). For $|t|\le 1$, we then have

$$\begin{aligned}|\nabla_z F(\xi_t)|+|\nabla_y F(\xi_t)| &\le \Psi + C|\mathrm{D}u{+}t\mathrm{D}\varphi|^p + b(|u{+}t\varphi|)\\ &\le \Psi + 2^{p-1}C\big(|\mathrm{D}u|^p{+}|\mathrm{D}\varphi|^p\big)+\sup\nolimits_{s\le|u|+|\varphi|}b(s)=:\Lambda\in\mathrm{L}^1_{\mathrm{loc}}(\Omega)\,,\end{aligned}$$

and the remainder of the argument works as before.

The adaption of the reasoning to the case $p=\infty$ with two 'b-terms' is straightforward. □

Proof of (II) *in the theorem.* Starting once more with the case $p\le n$, we justify the first-variation formula for the given $u\in\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ and arbitrary $\varphi\in\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Indeed,

[2]Indeed, already from local boundedness of $\nabla_z F$, $\nabla_y F$ on $\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ one obtains the '∞-growth' condition as follows. Introduce $\Omega_M:=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)\ge 1/M\,,\,|x|\le M\}$, $K_M:=\{(y,z)\in\mathbb{R}^N\times\mathbb{R}^{N\times n}:|z|{+}|y|\le M\}$ for $M\in\mathbb{N}$ and $\Omega_0:=K_0:=\emptyset$. Then set $\Psi(x):=\sup_{\Omega_M\times K_M}\big(|\nabla_z F|{+}|\nabla_y F|\big)<\infty$ for $x\in\Omega_M\setminus\Omega_{M-1}$ and $b(t):=\sup_{\Omega_M\times K_M}\big(|\nabla_z F|{+}|\nabla_y F|\big)<\infty$ for $t\in\big(\frac{M-1}{2},\frac{M}{2}\big]$, $b(0):=b(0+)$. For arbitrary $(x,y,z)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ there is a smallest $M\in\mathbb{N}$ with $(x,y,z)\in\Omega_M\times K_M$. Now either $x\in\Omega_M\setminus\Omega_{M-1}$ or $M{-}1<|z|{+}|y|\le M$ holds (in case $M=1$ with '$\le$' in place of '$<$'). In the first case, the above choices ensure $|\nabla_z F(x,y,z)|{+}|\nabla_y F(x,y,z)|\le\Psi(x)$, in the second one they give $|\nabla_z F(x,y,z)|{+}|\nabla_y F(x,y,z)|\le b\big(\frac{|z|+|y|}{2}\big)\le b(|z|){+}b(|y|)$. In all cases, it thus comes out that $|\nabla_z F(x,y,z)|{+}|\nabla_y F(x,y,z)|\le\Psi(x){+}b(|z|){+}b(|y|)$ holds, and this is the claimed growth condition.

setting again $\xi_t := (\,\cdot\,, u+t\varphi, \mathrm{D}u+t\mathrm{D}\varphi)$, for $|t| \le 1$ we have

$$\begin{aligned}|\nabla_z F(\xi_t)| &\le \Psi^{1/p'} + C|\mathrm{D}u+t\mathrm{D}\varphi|^{p/p'} + C|u+t\varphi|^{p^*/p'}\\ &\le \Psi^{1/p'} + 2^{p/p'}C\big(|\mathrm{D}u|^{p/p'}+|\mathrm{D}\varphi|^{p/p'}\big) + 2^{p^*/p'}C\big(|u|^{p^*/p'}+|\varphi|^{p^*/p'}\big) =: \Lambda_z \in \mathrm{L}^{p'}(\Omega)\,,\end{aligned}$$

where we exploited that $|u|^{p^*}, |\varphi|^{p^*} \in \mathrm{L}^1(\Omega)$ by Sobolev's embedding. Analogously, we get $|\nabla_y F(\xi_t)| \le \Lambda_y$ for $|t| \le 1$ with $\Lambda_y \in \mathrm{L}^{(p^*)'}(\Omega)$. In conclusion, we have $|\nabla_z F(\xi_t)\raisebox{0.2ex}{\scriptsize$\bullet$}\mathrm{D}\varphi| \le \Phi$, $|\nabla_z F(\xi_t)\raisebox{0.2ex}{\scriptsize$\bullet$}\varphi| \le \Phi$ for $\Phi := \Lambda_z|\mathrm{D}\varphi|+\Lambda_y|\varphi| \in \mathrm{L}^1(\Omega)$ (where $\Lambda_z \in \mathrm{L}^{p'}(\Omega)$, $|\mathrm{D}\varphi| \in \mathrm{L}^p(\Omega)$ implies $\Lambda_z|\mathrm{D}\varphi| \in \mathrm{L}^1(\Omega)$ and $\Lambda_y \in \mathrm{L}^{(p^*)'}(\Omega)$, $|\varphi| \in \mathrm{L}^{p^*}(\Omega)$ implies $\Lambda_y|\varphi| \in \mathrm{L}^1(\Omega)$ by Hölder's inequality). On this basis we may apply (I) in the last proposition (with $\mathcal{A} = \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$) and get the existence of

$$\delta\mathcal{F}[u;\varphi] = \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi\big]\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)\,. \quad (*)$$

From this formula, the integrabilities $|\nabla_z F(\,\cdot\,,u,\mathrm{D}u)| \in \mathrm{L}^{p'}(\Omega)$, $|\nabla_y F(\,\cdot\,,u,\mathrm{D}u)| \in \mathrm{L}^{(p^*)'}(\Omega)$, and Sobolev's embedding we then obtain that $\delta\mathcal{F}[u;\,\cdot\,]$ is continuous on $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. For $p > n$, the majorant Φ, the formula $(*)$, and the continuity property can be obtained, similar as in the last proof, by technical modification of the reasoning (which we do not explicate here). As a side benefit, we record that $F(\,\cdot\,,w,\mathrm{D}w) \in \mathrm{L}^1(\Omega)$ holds and thus $\mathcal{F}[w] \in \mathbb{R}$ exists for all $w \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Indeed, setting $\varphi := w-u$ and still using $\xi_t = (\,\cdot\,, u+t\varphi, \mathrm{D}u+t\mathrm{D}\varphi)$, we see

$$\begin{aligned}|F(\,\cdot\,,w,\mathrm{D}w)-F(\,\cdot\,,u,\mathrm{D}u)| = |F(\xi_1)-F(\xi_0)| &\le \sup_{t\in[0,1]}\Big|\frac{\mathrm{d}}{\mathrm{d}t}F(\xi_t)\Big|\\ &= \sup_{t\in[0,1]}|\nabla_z F(\xi_t)\raisebox{0.2ex}{\scriptsize$\bullet$}\mathrm{D}\varphi+\nabla_y F(\xi_t)\raisebox{0.2ex}{\scriptsize$\bullet$}\varphi| \le 2\Phi \in \mathrm{L}^1(\Omega)\,.\end{aligned}$$

This implies $F(\,\cdot\,,w,\mathrm{D}w) \in \mathrm{L}^1(\Omega)$, since $F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1(\Omega)$ is assumed.

Finally, we turn to the main conclusions of (II). From the deduction of $(*)$ it is clear that (II) from the last proposition applies and a minimizing u satisfies $\delta\mathcal{F}[u;\varphi] = 0$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$, which is nothing but the Euler equation (EL). Moreover, in case $p < \infty$ at least, $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ is norm-dense in $\mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$, and thus, by continuity, having $\delta\mathcal{F}[u;\varphi] = 0$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ is equivalent with having $\delta\mathcal{F}[u;\varphi] = 0$ for all $\varphi \in \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$. For $p = \infty$, the norm-density is no longer at hand, but the resulting equivalence is still true, since, for each $\varphi \in \mathrm{W}^{1,\infty}_0(\Omega,\mathbb{R}^N)$, there are approximations $\varphi_k \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ such that φ_k converge uniformly to φ and $\mathrm{D}\varphi_k$ are uniformly bounded and converge a.e. to $\mathrm{D}\varphi$. At this stage it just remains to establish the additional implication giving the sufficiency of (EL) in case F is convex in (y,z). However, since $\mathcal{F}$ is then a finite-valued convex functional on $\mathrm{W}^{1,p}_u(\Omega,\mathbb{R}^N)$, this implication comes straightforwardly from part (II) of the first proposition in this section (applied with $\mathrm{W}^{1,p}_u(\Omega,\mathbb{R}^N)$, $\mathcal{V} = \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$). □

Outlook (on related topics)**.**

(1) Clearly, one may also consider **fully unconstrained minimization**, that is, minimization in all of $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ without requiring any constraint or boundary condition. In this case, it turns out that the Euler equation (EL) is just a part of the necessary first-order criterion for minimizers. Indeed, the full necessary criterion for a minimizer u of $\mathcal{F}$ in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ reads

(under technical assumptions which ensure $\nabla_z F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^{N\times n})$, $\nabla_y F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^{(p^*)'}(\Omega,\mathbb{R}^N)$, for instance)

$$\delta\mathcal{F}[u;\varphi] = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)\,,$$

where the point is that φ runs in $\mathrm{W}^{1,p}$ not $\mathrm{W}^{1,p}_0$. Clearly, $\varphi \in \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$ are included, and therefore the Euler equation (EL) is a part of this condition, but additionally it also contains

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\boldsymbol{\cdot}\mathrm{D}\varphi + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\boldsymbol{\cdot}\varphi\big]\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}^1(\overline{\Omega})\,.$$

In case of $\nabla_z F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{C}^0(\overline{\Omega})$ and a bounded C^1-domain Ω, one can use a version of the divergence theorem to integrate by parts. Also using (EL) in form of $\operatorname{div}\big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\big] = \nabla_y F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1(\Omega,\mathbb{R}^N)$, this results in

$$\int_{\partial\Omega} \nabla_z F(\,\cdot\,,u,\mathrm{D}u)\nu_\Omega\boldsymbol{\cdot}\varphi\,\mathrm{d}\mathcal{H}^{n-1} = 0 \qquad \text{for all } \varphi \in \mathrm{C}^1(\overline{\Omega})$$

with the outward unit normal ν_Ω to Ω on $\partial\Omega$. The fundamental lemma of the calculus of variations (in a version for vector measures, for instance) then gives

$$\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\nu_\Omega \equiv 0 \qquad \text{on } \partial\Omega\,. \tag{$**$}$$

Hence, unconstrained minimization **automatically enforces** the condition ($**$), which is called a **conormal boundary condition**. For variants $F(x,y,z) = \frac{1}{2}|z|^2 - V(x)\boldsymbol{\cdot}z + \widetilde{F}(x,y)$ of the Dirichlet integrand (with a matrix field $V \in \mathrm{C}^0(\overline{\Omega})$), this condition reduces to the more usual Neumann boundary condition $\partial_{\nu_\Omega}u = V\nu_\Omega$ on $\partial\Omega$ (where $\partial_{\nu_\Omega}u = \psi$ with arbitrary $\mathbb{R}^N$-valued ψ can be realized by taking $V = \psi\otimes\overline{\nu_\Omega}$ with an extension $\overline{\nu_\Omega}$ of ν_Ω to $\overline{\Omega}$).

(2) Another central topic in the calculus of variations, again parallel to ideas from finite-dimensional calculus, are **second-order criteria for *local* minimizers or maximizers** of functionals $\mathcal{F}$ in terms of the **second variation**

$$\delta^2\mathcal{F}[u;\varphi] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}\mathcal{F}[u{+}t\varphi]\,.$$

Since we mostly focus on global rather than local minimizers, in this lecture we refrain from discussing such criteria any further.

3.2 Convex constraints and obstacle problems

In the sequel we make a first step towards the treatment of admissible classes $\mathcal{A} \subset \mathcal{X}$ which are defined with **non-linear constraints** and hence are *not* affine (sub)spaces of $\mathcal{X}$. In general, the **special variations $(w{+}t\varphi)_{t\in(-\varepsilon,\varepsilon)}$** of $w \in \mathcal{A}$ from Section 3.1 are then **of little use**, since $(w{+}t\varphi)_{t\in(-\varepsilon,\varepsilon)}$ is $\mathcal{A}$-admissible only for very few directions $\varphi \in \mathcal{X}$ (and in strongly non-linear situations even for no non-zero directions φ at all). Therefore, it is a **basic guiding principle** in the calculus of variations that **non-linear constraints require the usage of more general variations than $(w{+}t\varphi)_{t\in(-\varepsilon,\varepsilon)}$.**

However, the guiding principle will take full effect only in the subsequent Section 3.3, while for the treatment of only *convex* classes $\mathcal{A}$ it will be enough to generalize the notion of variation only marginally: In fact, for a real vector space $\mathcal{X}$ and $\varepsilon > 0$, we now use **one-sided variations**

$$(w+t\varphi)_{t\in[0,\varepsilon)}$$

of an element $w \in \mathcal{X}$ in direction $\varphi \in \mathcal{X}$, where the parameter runs in the one-sided interval $[0,\varepsilon)$ rather than in the both-sided interval $(-\varepsilon,\varepsilon)$. If the variation is $\mathcal{A}$-admissible for some $\mathcal{A} \subset \mathcal{X}$, that is, we have $w+t\varphi \in \mathcal{A}$ for $0 \le t \ll 1$, we denote the corresponding one-sided directional derivative of $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$ by

$$\delta^+\mathcal{F}[w;\varphi] := \partial^+_\varphi\mathcal{F}[w] := \lim_{t\searrow 0}\frac{\mathcal{F}[w+t\varphi]-\mathcal{F}[w]}{t}$$

(whenever this exists in $\mathbb{R}$). We emphasize, however, that the one-sided derivative $\delta^+\mathcal{F}[w;\varphi]$ simply coincides with the both-sided derivative $\delta\mathcal{F}[w;\varphi]$ whenever the latter exists (possibly after suitable extension of $\mathcal{F}$ outside $\mathcal{A}$). Thus, the **usage of $\boldsymbol{\delta^+\mathcal{F}}$**, though technically convenient in the subsequent statement, **can be avoided in most relevant situations**, where in fact both-sided (directional) differentiability is at hand.

Now we provide a necessary criterion and a sufficient criterion for minimizers, which both stay very close to the criteria at the beginning of Section 3.1.

Proposition (**first-order criterion for cases with convex constraints**)**.** *Fix a real vector space $\mathcal{X}$ and $u \in \mathcal{A} \subset \mathcal{X}$.*

(I) ***Necessary criterion*: *If u is a minimizer of a functional $\mathcal{F}\colon \mathcal{A} \to \overline{\mathbb{R}}$ in $\mathcal{A}$, then we have***

$$\delta^+\mathcal{F}[u;w-u] \ge 0 \textit{ whenever } \delta^+\mathcal{F}[u;w-u] \textit{ exists for } w \in \mathcal{A}\,.$$

(II) ***Sufficient criterion*: *Suppose that $\mathcal{A}$ is convex and $\mathcal{F}\colon \mathcal{A} \to \mathbb{R}\cup\{\infty\}$ is convex on $\mathcal{A}$. If***

$$\delta^+\mathcal{F}[u;w-u] \textit{ exists with } \delta^+\mathcal{F}[u;w-u] \ge 0 \textit{ for all } w \in \mathcal{A}\,,$$

then u is a minimizer of $\mathcal{F}$ in $\mathcal{A}$.

Proof. The proof follows the corresponding one in Section 3.1.

For (I) we argue: If u minimizes and $\delta^+\mathcal{F}[u;\varphi]$ exists for $\varphi = w-u$, then $\mathcal{F}[u+t\varphi]$ is defined and $\ge\mathcal{F}[u]$ for $0 \le t \ll 1$. Thus, we get $\delta\mathcal{F}[u;\varphi] = \lim_{t\searrow 0}\frac{\mathcal{F}[u+t\varphi]-\mathcal{F}[u]}{t} \ge 0$.

For (II) we can repeat the reasoning for the corresponding (II) in Section 3.1, which still works in the same way with only $\delta^+\mathcal{F}[u;w-u] \ge 0$ instead of $\delta\mathcal{F}[u;w-u] = 0$ at hand. $\square$

Remarks (on the preceding criteria)**.**

(1) It is not restrictive to consider $\delta^+\mathcal{F}[u;w-u]$ only for directions $w-u$ with $w \in \mathcal{A}$.
(Indeed, for $\delta^+\mathcal{F}[u;\varphi]$ to be defined we have required $u+t\varphi \in \mathcal{A}$ for $0 < t \ll 1$. If we fix such a t and set $w = u+t\varphi \in \mathcal{A}$, we get $\varphi = \frac{1}{t}(w-u)$ and $\delta^+\mathcal{F}[u;\varphi] = \frac{1}{t}\delta^+\mathcal{F}[u;w-u]$. Therefore, $\delta^+\mathcal{F}[u;\varphi]$ always coincides, up to a positive factor, with $\delta^+\mathcal{F}[u;w-u]$ for some $w \in \mathcal{A}$.)

(2) The usage of $\delta^+\mathcal{F}$ rather than $\delta\mathcal{F}$ in the preceding criteria is, in practice, only of marginal importance. The more decisive point is indeed that the **criteria only yield/require the *in*equality $\boldsymbol{\delta^+\mathcal{F}[u;w-u] \ge 0}$**.

(3) It is **crucial for** the **existence theory** of Chapter 2 that one minimizes a functional in sequentially weakly closed $\mathcal{A}$ in reflexive $\mathcal{X}$. If $\mathcal{A}$ is also convex, $\mathcal{A}$ is sequentially weakly closed in $\mathcal{X}$ if and only if $\boldsymbol{\mathcal{A}}$ **is closed** in $\mathcal{X}$, and it **may well happen that a minimizer** $\boldsymbol{u}$ **is in** $\boldsymbol{\partial\mathcal{A}}$. It is **in this case** that **one-sided variations are truly needed** and one may hope at best for $\delta\mathcal{F}[u;w{-}u] \ge 0$ in general, while for u in $\operatorname{int}(\mathcal{A})$ one can usually apply the interior criteria of Section 3.1 and get even $\delta\mathcal{F}[u;w{-}u] = 0$.

In case of a first-order integral functional

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x \qquad \text{for } w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$$

the abstract criterion has a more concrete interpretation. In order to address this we first record that the first-variation formula holds for one-sided directional derivatives in the form

$$\delta^+\mathcal{F}[u;\varphi] = \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\boldsymbol{\cdot}\mathrm{D}\varphi + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\boldsymbol{\cdot}\varphi\big]\,\mathrm{d}x$$

under the same assumptions as in Section 3.1 (obvious, since $\delta^+\mathcal{F}[u;\varphi]$ equals $\delta\mathcal{F}[u;\varphi]$ if the latter exists) and in fact even if the same assumptions are only imposed for $t \in [0,\varepsilon)$ rather than $t \in (-\varepsilon,\varepsilon)$ (analogous proof). If these assumptions are satisfied for fixed $u \in \mathcal{A} \subset \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and all test functions $\varphi = w{-}u$ with $w \in \mathcal{A}$, we infer that the first-order criterion

$$\delta^+\mathcal{F}[u;w{-}u] \ge 0 \qquad \text{for all } w \in \mathcal{A}$$

can be equivalently recast, for the above integral functional $\mathcal{F}$, in form of the ***variational inequality***

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\boldsymbol{\cdot}\mathrm{D}(w{-}u) + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\boldsymbol{\cdot}(w{-}u)\big]\,\mathrm{d}x \ge 0 \qquad \text{for all } w \in \mathcal{A}\,. \tag{vI}$$

This inequality can thus seen as a **sort-of Euler equation in the presence of convex constraints**.

Finer interpretations of the variational inequality can be given in the specific case of the ***obstacle problem*** for scalar functions. We thus take $N=1$ for the remainder of this section (which means that we can write ∇u and $\partial_y F$ in place of $\mathrm{D}u$ and $\nabla_y F$) and first consider $\psi\colon \Omega \to \mathbb{R}$ and $u_0 \in \mathrm{W}^{1,p}(\Omega)$ on open $\Omega \subset \mathbb{R}^n$ with $p \in (n,\infty]$ or $p = n = 1$. Then we introduce the **admissible class** for the obstacle problem **with obstacle** $\boldsymbol{\psi}$ **and boundary datum** $\boldsymbol{u_0}$ as

$$K^p_{u_0,\psi} := \{w \in \mathrm{W}^{1,p}_{u_0}(\Omega) \,:\, w \ge \psi \text{ on } \Omega\}\,. \tag{$*$}$$

Here the competitors w have a continuous representative (by the Sobolev embedding and the assumption that $p > n$ or $p = n = 1$), and this representative is used to give a pointwise meaning to the inequality $w \ge \psi$ even for completely arbitrary ψ. Moreover, it is straightforward to check that $K^p_{u_0,\psi}$ is a closed convex subset of $\mathrm{W}^{1,p}(\Omega)$, and as a consequence the minimization of reasonable integral functionals in the class $K^p_{u_0,\psi}$ is covered by our earlier existence theory.

Whenever the variational inequality (vI) holds for $u \in \mathcal{A} = K^p_{u_0,\psi}$, then specifically, for every non-negative $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, the competitor $w = u{+}\varphi$ is admissible in (vI), and therefore (vI) implies

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\boldsymbol{\cdot}\mathrm{D}\varphi + \partial_y F(\,\cdot\,,u,\nabla u)\varphi\big]\,\mathrm{d}x \ge 0 \qquad \text{for all non-negative } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,,$$

which is nothing but the (distributional) **supersolution property**

$$\mathrm{E}_{\mathcal{F}}u \geq 0 \qquad \text{in } \mathscr{D}'(\Omega)$$

for the Euler-Lagrange operator $\mathrm{E}_{\mathcal{F}}u := -\operatorname{div}\big[\nabla_z F(\,\cdot\,,u,\nabla u)\big] + \partial_y F(\,\cdot\,,u,\nabla u)$. The next theorem, however, fully identifies the meaning of the variational inequality in the obstacle case and shows that the supersolution property is *only a part* of the information contained:

Theorem (Euler equation for the obstacle problem, case $p > n$). *Consider a bounded open set Ω in $\mathbb{R}^n$, an* upper *semicontinuous obstacle $\psi\colon \Omega \to \mathbb{R}\cup\{-\infty\}$, a boundary datum $u_0 \in \mathrm{W}^{1,p}(\Omega)$ with either $p \in (n,\infty]$ or $p = n = 1$, and $u \in K^p_{u_0,\psi}$ (for $K^p_{u_0,\psi}$ defined in (∗)) with $\nabla_z F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^n)$, $\partial_y F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^1(\Omega)$. Then u solves the* ***variational inequality***

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u) + \partial_y F(\,\cdot\,,u,\nabla u)(w-u)\big]\,\mathrm{d}x \geq 0 \qquad \textit{for all } w \in K^p_{u_0,\psi}$$

if and only if *u satisfies both*

$$\begin{cases} \mathrm{E}_{\mathcal{F}}u \geq 0 \textit{ in } \mathscr{D}'(\Omega) & (\textbf{\textit{supersolution property on full domain}})\,, \\ \mathrm{E}_{\mathcal{F}}u \equiv 0 \textit{ in } \mathscr{D}'(\{u>\psi\}) & (\textbf{\textit{solution property on non-contact set}})\,. \end{cases}$$

We stress that the formulation of this theorem is only reasonable as a consequence of the Sobolev embedding. Indeed, boundedness of $w-u$ (thanks to $\mathrm{W}^{1,p}_0(\Omega) \hookrightarrow \mathrm{C}^0_\mathrm{b}(\Omega)$) ensures that the variational inequality is well-defined for all competitors w. Much more crucially, however, continuity of u together with the assumed upper semicontinuity of ψ **guarantees** that $\boldsymbol{\{u>\psi\}}$ **is** an **open** subset of Ω and the distributional solution property makes sense at all. We also mention that we have started to apply common terminology according to which the relatively closed set $\{u=\psi\}$ of points in which the obstacle ψ is touched by the solution u is called **contact set** or **coincidence set**, while the open set $\{u>\psi\}$ is known as the **non-contact set**.

Remarks (on the Euler equation for the obstacle problem in case $p > n$)**.**

(1) In the theorem, one can actually replace $\nabla_z F(\,\cdot\,,u,\nabla u)$ and $\partial_y F(\,\cdot\,,u,\nabla u)$ by an vector field $V \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^n)$ and an arbitrary function $g \in \mathrm{L}^1(\Omega)$ and correspondingly $\mathrm{E}_{\mathcal{F}}u$ by the distribution $-\operatorname{div} V+g$. This explains why in the above statement no hypothesis for F itself have been specified.

(2) The theorem provides a full interpretation of the variational inequality — which, as we recall, is the basic necessary criterion for minimizers — in case of the obstacle problem: The **solutions of the variational inequality are supersolutions** to the standard/unconstrained Euler-Lagrange equation **on the full domain $\boldsymbol{\Omega}$ and are true solutions** to this equation **on the non-contact set $\boldsymbol{\{u > \psi\}}$**. This confirms and makes precise the very plausible expectation that the presence of the obstacle takes effect on the first-order criterion only on the contact set $\{u=\psi\}$.

Distributions. The space $\mathscr{D}'(\Omega,\mathbb{R}^N)$ of $\mathbb{R}^N$-valued distributions on open $\Omega \subset \mathbb{R}^n$ is the space of all $\mathbb{R}$-linear functionals $\mathrm{C}^\infty_\mathrm{cpt}(\Omega,\mathbb{R}^N) \to \mathbb{R}$ which are continuous along $\mathscr{D}$-convergent sequences. By identifying $u \in \mathrm{L}^1_\mathrm{loc}(\Omega,\mathbb{R}^N)$ with $T_u \in \mathscr{D}'(\Omega,\mathbb{R}^N)$ given by $\langle T_u;\varphi\rangle := \int_\Omega u\cdot\varphi\,\mathrm{d}x$ for $\varphi \in \mathrm{C}^\infty_\mathrm{cpt}(\Omega,\mathbb{R}^N)$, one understands $\mathrm{L}^1_\mathrm{loc}(\Omega,\mathbb{R}^N) \subset \mathscr{D}'(\Omega,\mathbb{R}^N)$. In the scalar case $N=1$, as usual one writes $\mathscr{D}'(\Omega)$ for $\mathscr{D}'(\Omega,\mathbb{R})$.

Derivatives in the sense of distributions do always exist. Specifically, *every* vector field $\mathrm{W} \in \mathrm{L}^1_\mathrm{loc}(\Omega,\mathbb{R}^n)$ has a **distributional divergence** $\operatorname{div} W \in \mathscr{D}'(\Omega)$ given by $\langle \operatorname{div} W;\varphi\rangle := -\int_\Omega W\cdot\nabla\varphi\,\mathrm{d}x$ for $\varphi \in \mathrm{C}^\infty_\mathrm{cpt}(\Omega)$.

One says that a scalar distribution $T \in \mathscr{D}'(\Omega)$ is a **non-negative distribution** and writes $T \geq 0$ if $\langle T;\varphi\rangle \geq 0$ holds for all non-negative $\varphi \in \mathrm{C}^\infty_\mathrm{cpt}(\Omega)$.

(3) The condition $K^p_{u_0,\psi} \neq \emptyset$, which is clearly necessary for $u \in K^p_{u_0,\psi}$ to exist and thus implicitly assumed above, implies that the obstacle ψ and the boundary datum u_0 satisfy[3] the **compatibility condition** $\limsup_{\Omega \ni x \to \partial\Omega}(\psi(x){-}u_0(x)) \leq 0$. If Ω has the $\mathrm{W}^{1,p}$-extension property, u_0 has a unique $\mathrm{C}^0(\overline{\Omega})$ representative, and the compatibility condition means that there is an upper semicontinuous extension of ψ to $\overline{\Omega}$ with $\psi \leq u_0$ on $\partial\Omega$.

(4) In the case $\mathcal{F} = \mathcal{E}_2$ of the **Dirichlet integral** the obstacle problem and the corresponding variational inequality yield a **model for the shape of an elastic membrane** (given by the graph of a minimizer/solution u) **which is spanned over an obstacle** (given by the graph of ψ) and is clamped at the boundary (in a manner specified by u_0).

Proof of the forward implication in the theorem. Assume that u satisfies the variational inequality. Then it has already been explained that the usage of competitors $w = u{+}\varphi$ with $0 \leq \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ yields the supersolution property $\mathrm{E}_{\mathcal{F}} u \geq 0$ in $\mathscr{D}'(\Omega)$. In order to verify the solution property on the non-contact set $\{u{>}\psi\}$, consider now $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\{u{>}\psi\})$, which we extend by 0 to all of Ω. From continuity of u and upper semicontinuity of ψ we infer that the $u{-}\psi$ is lower semicontinuous and thus attains its positive minimum on the compact set $\operatorname{spt}\varphi \subset \{u{-}\psi{>}0\}$. This, together with the boundedness of φ, gives the existence of a constant $\varepsilon > 0$ such that $u{-}\psi \geq \varepsilon|\varphi|$ holds on $\operatorname{spt}\varphi$ and then on Ω. Rearranging the inequality, we have $u{\pm}\varepsilon\varphi \geq \psi$ on Ω and thus $u{\pm}\varepsilon\varphi \in K^p_{u_0,\psi}$. Testing the variational inequality with the competitors $u{\pm}\varepsilon\varphi$, we get

$$\pm\varepsilon \int_{\{u>\psi\}} \big[\nabla_z F(\,\cdot\,,u,\nabla u) \cdot \nabla\varphi + \partial_y F(\,\cdot\,,u,\nabla u)\varphi\big] \,\mathrm{d}x \geq 0\,.$$

Since this holds for both choices of sign, we clearly get

$$\int_{\{u>\psi\}} \big[\nabla_z F(\,\cdot\,,u,\nabla u) \cdot \nabla\varphi + \partial_y F(\,\cdot\,,u,\nabla u)\varphi\big] \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\{u{>}\psi\})\,,$$

which is the claimed solution property $\mathrm{E}_{\mathcal{F}} u = 0$ in $\mathscr{D}'(\{u{>}\psi\})$. □

Remark (on an **ad-hoc extension to the case $p \leq n$).** In the case $p \leq n$, the same method yields *some version* of the forward implication in the theorem at least. Indeed, if one slightly modifies the definition (∗) of $K^p_{u_0,\psi}$ by requiring $w \geq \psi$ only a.e. on Ω, and if $u \in K^p_{u_0,\psi}$ with $p \in [1,n]$, $\nabla_z F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^n)$, $\partial_y F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^{(p^*)'}(\Omega)$ solves the variational inequality, one may still conclude $\mathrm{E}_{\mathcal{F}} u \geq 0$ in $\mathscr{D}'(\Omega)$ and $\mathrm{E}_{\mathcal{F}} u \equiv 0$ in $\mathscr{D}'(\{u \gtrapprox \psi\})$, where the notation

$$\{u \gtrapprox \psi\} := \{x \in \Omega \,:\, \exists\varepsilon,\delta > 0\colon u \geq \psi{+}\varepsilon \text{ on } \mathrm{B}_\delta(x)\}$$

is used for the open set $\{u \gtrapprox \psi\}$ on which u and ψ are locally bounded away from each other. This, however, is only an ad-hoc solution, since in general $\{u \gtrapprox \psi\}$ is only a part of the non-contact set and one cannot get the backward implication in this way. The 'technically correct' extension of the Euler equation to the case $p \leq n$ rather needs some more background machinery, as will be explained below.

For proving the backward implication in the theorem, the following technical lemma on $\mathrm{W}^{1,p}_0$ zero boundary values and positive/negative parts will be useful. While the statement seems very plausible, it is *not that* immediate from the definition of $\mathrm{W}^{1,p}_0(\Omega)$, $p < \infty$, as the closure of $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$.

[3]Indeed, the compatibility condition follows from the obstacle condition $\psi \leq u$ on Ω, since the Sobolev embedding enforces $u{-}u_0 \in \mathrm{C}^0_0(\Omega)$ and thus $\lim_{\Omega \ni x \to \partial\Omega}(u(x){-}u_0(x)) = 0$.

Lemma. *Consider an open $\Omega \subset \mathbb{R}^n$ and $v \in \mathrm{W}_0^{1,p}(\Omega)$ with $p \in (n,\infty]$ or $p = n = 1$. Then there exist $\varphi_k \in \mathrm{C}^\infty_{\mathrm{cpt}}(\{v>0\})$ with $0 \le \varphi_1 \le \varphi_2 \le \varphi_3 \le \ldots$ on Ω such that we have the convergences*

$$\varphi_k \underset{k\to\infty}{\longrightarrow} v_+ \text{ uniformly on } \Omega\,, \qquad \nabla\varphi_k \underset{k\to\infty}{\longrightarrow} \nabla v_+ \begin{cases} \text{in } \mathrm{L}^p(\Omega,\mathbb{R}^n)\,, & \text{if } p<\infty \\ \text{a.e. on } \Omega \text{ and uniformly bounded}\,, & \text{if } p=\infty \end{cases}.$$

In particular, for every $v \in \mathrm{W}_0^{1,p}(\Omega)$ with $p > n$ or $p = n = 1$ one has $v_\pm \in \mathrm{W}_0^{1,p}(\{\pm v>0\}) \subset \mathrm{W}_0^{1,p}(\Omega)$.

We remark that $v \in \mathrm{W}_0^{1,p}(\Omega)$ implies $v_\pm \in \mathrm{W}_0^{1,p}(\Omega)$ even for arbitrary $p \in [1,\infty]$ with a somewhat simpler proof than the following one. The above approximations from below and the possibility to assert even $v_\pm \in \mathrm{W}_0^{1,p}(\{\pm v>0\})$ on *open* sets $\{\pm v>0\}$, however, draw strongly on the Sobolev embedding into continuous functions in the case $p > n$.

Proof of the lemma. First assume $p < \infty$. By the chain rule, $v_k := (v-2^{-k})_+ \ge 0$ are weakly differentiable functions, and it is a standard matter to verify $v_k \underset{k\to\infty}{\longrightarrow} v_+$ uniformly on Ω and $\nabla v_k \underset{k\to\infty}{\longrightarrow} \nabla v_+$ in $\mathrm{L}^p(\Omega,\mathbb{R}^n)$. In view of the inclusion $\mathrm{W}_0^{1,p}(\Omega) \subset \mathrm{C}_0^0(\Omega)$ (which comes with the Morrey-Sobolev embedding), v is continuous with $\lim_{\Omega\ni x\to\partial\Omega\cup\{\infty\}} v(x) = 0$, and thus the v_k are continuous with $\operatorname{spt} v_k \Subset \{v_{k+1} > 2^{-k-2}\} \subset \{v>0\}$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, by convergence properties of mollifications we may choose a radius $\delta_k > 0$ such that the mollification $\varphi_k := (v_k)_{\delta_k} \ge 0$ satisfies $\varphi_k \in \mathrm{C}^\infty_{\mathrm{cpt}}(\{v_{k+1} > 2^{-k-2}\})$, $\sup_\Omega |\varphi_k - v_k| \le 2^{-k-3}$, and $\|\nabla\varphi_k - \nabla v_k\|_{\mathrm{L}^p;\Omega} \le k^{-1}$. These properties ensure $\varphi_k \le \varphi_{k+1}$ on Ω for all $k \in \mathbb{N}$ (as indeed, on $\{v_{k+1} > 2^{-k-2}\}$ we have $v_k \le v_{k+1} - 2^{-k-2}$ and $\varphi_k \le v_k + 2^{-k-3} \le v_{k+1} - 2^{-k-3} \le \varphi_{k+1}$, while outside $\operatorname{spt}\varphi_k$ we simply have $\varphi_k = 0 \le \varphi_{k+1}$). Moreover, recalling the convergences of v_k, we also get $\varphi_k \underset{k\to\infty}{\longrightarrow} v_+$ uniformly on Ω and $\nabla\varphi_k \underset{k\to\infty}{\longrightarrow} \nabla v_+$ in $\mathrm{L}^p(\Omega,\mathbb{R}^n)$.

In the case $p = \infty$, the reasoning needs to be changed only with regard to the gradient convergence. Indeed, one gets $\nabla v_k \underset{k\to\infty}{\longrightarrow} \nabla v_+$ a.e. on Ω, and the condition on $\nabla\varphi_k$ can be taken as $\|\nabla\varphi_k - \nabla v_k\|_{\mathrm{L}^1;\Omega} \le k^{-1}$ (even for unbounded Ω, as $v_k \in \mathrm{W}^{1,\infty}_{\mathrm{cpt}}(\Omega)$ has $\nabla v_k \in \mathrm{L}^1(\Omega,\mathbb{R}^n)$). Possibly passing to a subsequence, we find $\nabla\varphi_k - \nabla v_k \underset{k\to\infty}{\longrightarrow} 0$ and $\nabla\varphi_k \underset{k\to\infty}{\longrightarrow} \nabla v_+$ a.e. on Ω. Finally, the mentioned uniform bound is simply given by $\|\nabla\varphi_k\|_{\mathrm{L}^\infty;\Omega} \le \|\nabla v_k\|_{\mathrm{L}^\infty;\Omega} \le \|\nabla v_+\|_{\mathrm{L}^\infty;\Omega}$. □

We now return to the main line of argument:

Proof of the backward implication in the theorem. Assume that u satisfies $\mathrm{E}_{\mathcal{F}}u \ge 0$ in $\mathscr{D}'(\Omega)$ and $\mathrm{E}_{\mathcal{F}}u = 0$ in $\mathscr{D}'(\{u>\psi\})$, that is,

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla\varphi + \partial_y F(\,\cdot\,,u,\nabla u)\varphi\big]\,\mathrm{d}x \ge 0 \qquad \text{for all non-negative } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$$

with equality in case $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\{u>\psi\})$. Then the inequality remains valid[4] for non-negative test functions $\varphi \in \mathrm{W}_0^{1,p}(\Omega)$, and we still get equality in case $\varphi \in \mathrm{W}_0^{1,p}(\{u>\psi\})$. To verify the variational inequality, we now consider an arbitrary competitor $w \in K^p_{u_0,\psi}$, which in particular satisfies $w-u \in \mathrm{W}_0^{1,p}(\Omega)$. For the positive part of $w-u$, the lemma gives $0 \le (w-u)_+ \in \mathrm{W}_0^{1,p}(\Omega)$, and the above supersolution property yields

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u)_+ + \partial_y F(\,\cdot\,,u,\nabla u)(w-u)_+\big]\,\mathrm{d}x \ge 0\,.$$

The treatment of the negative part of $w-u$ is slightly more subtle: By the lemma we have $(w-u)_- \in \mathrm{W}_0^{1,p}(\{w<u\})$, and in view of $w \ge \psi$ on Ω we get $\{w<u\} \subset \{u>\psi\}$ and $(w-u)_- \in \mathrm{W}_0^{1,p}(\{u>\psi\})$. Therefore, the solution property even yields

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u)_- + \partial_y F(\,\cdot\,,u,\nabla u)(w-u)_-\big]\,\mathrm{d}x = 0$$

[4]The passage to $\mathrm{W}_0^{1,p}$ test functions is possible, since $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ is dense in $\mathrm{W}_0^{1,p}(\Omega)$ with respect to the convergence used in the preparatory lemma and the left-hand side of the inequality is suitably continuous thanks to the Sobolev embedding $\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$ and the integrability assumptions $\nabla_z F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^n)$, $\partial_y F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^1(\Omega)$. Moreover, the preparatory lemma shows that non-negative $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ functions are dense among non-negative $\mathrm{W}_0^{1,p}(\Omega)$ functions in the same sense, i.e. also non-negativity can be preserved.

Subtracting the integral equality for $(w-u)_+$ from the integral inequality for $(w-u)_+$, we arrive at the claimed variational inequality for $w-u = (w-u)_+ - (w-u)_-$. $\square$

Next we discuss the Euler equation for the obstacle problem also in case $p \le n$. Since $\mathrm{W}^{1,p}$ functions need no longer be continuous in this case, some more background machinery is needed to describe the finer 'pointwise' behavior of such functions and suitable classes of 'negligible' sets. A basic tool in this regard is the following concept:

Definition (p-capacity). *For $p \in [1,\infty)$, the (Sobolev) **p-capacity** of a set $A \subset \mathbb{R}^n$ is defined as*

$$\mathrm{Cap}_p(A) := \inf\left\{ \int_{\mathbb{R}^n} \big(|w|^p + |\nabla w|^p\big)\,\mathrm{d}x \,:\, w \in \mathrm{W}^{1,p}(\mathbb{R}^n)\,,\ w \ge 1 \text{ a.e. near } A \right\} \in [0,\infty]\,,$$

where $w \ge 1$ a.e. near A is meant to indicate $w \ge 1$ a.e. on an open neighborhood of A.

In this definition, one may also add the requirement $0 \le w \le 1$ a.e. on $\mathbb{R}^n$ without changing the infimum, since, for $w \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ with $w \ge 1$ a.e. near A, the cut-off $\widehat{w} := \min\{w_+, 1\}$ is still in $\mathrm{W}^{1,p}(\mathbb{R}^n)$ with with $\widehat{w} \ge 1$ a.e. near A, $0 \le \widehat{w} \le 1$ a.e. on $\mathbb{R}^n$, *and* $\int_{\mathbb{R}^n} \big(|\widehat{w}|^p + |\nabla \widehat{w}|^p\big)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \big(|w|^p + |\nabla w|^p\big)\,\mathrm{d}x$.

Remarks (on p-capacities)**.**

(1) By definition, $\mathrm{Cap}_p(A)$ is the minimum value in the obstacle problem for the functional $w \mapsto \int_{\mathbb{R}^n} \big(|w|^p + |\nabla w|^p\big)\,\mathrm{d}x$ with obstacle given by the characteristic function $\mathbb{1}_A$ (at least if one disregards the subtlety that the condition $w \ge 1$ is required a.e. near A).

(2) A slightly modified variant $\widetilde{\mathrm{Cap}}_p$ of p-capacity is defined with just $\int_{\mathbb{R}^n} |\nabla w|^p\,\mathrm{d}x$ in place of $\int_{\mathbb{R}^n} \big(|w|^p + |\nabla w|^p\big)\,\mathrm{d}x$. In case $p < n$ or $p = n = 1$ this does not change much in the sense that one can show $\mathrm{const}(n,p,d)\mathrm{Cap}_p(A) \le \widetilde{\mathrm{Cap}}_p(A) \le \mathrm{Cap}_p(A)$ for $A \subset \mathbb{R}^n$ with $\mathrm{diam}(A) \le d < \infty$ (where the right-hand inequality is trivial, while the left-hand one results from cut-off and Sobolev's embedding). For $1 \ne p \ge n$, in contrast, $\widetilde{\mathrm{Cap}}_p$ differs from Cap_p in the essential and undesirable way that $\widetilde{\mathrm{Cap}}_p$ vanishes on all bounded sets in $\mathbb{R}^n$. Thus, even though we consider mostly $p \le n$ and the real difference occurs only in the subcase $p = n$, we prefer to work with Cap_p here.

(3) The p-capacity Cap_p is formally an outer measure (i.e. a merely σ-subadditive measure) on $\mathbb{R}^n$. However, this measure has very few measurable sets and does not behave like more usual measures. For instance, one can show $\mathrm{Cap}_p(\overline{A}) = \mathrm{Cap}_p(A) < \infty$ but still $\mathrm{Cap}_p(\partial A) > 0$ for smooth bounded domains A in $\mathbb{R}^n$.

(A proof of the outer measure property is given in [8, Chapter 4.7, Theorem 1] for $\widetilde{\mathrm{Cap}}_p$ with $p < n$, but works for Cap_p in a very similar way.)

(4) **Zero sets for Cap_p** are trivial in case $p > n$ and **related to $(n-p)$-dimensional sets** in case $p < n$. More precisely, for $A \subset \mathbb{R}^n$, it holds

$$\begin{aligned} &\text{in case } p > n \text{ or } p = n = 1: && \mathrm{Cap}_p(A) = 0 \iff A = \emptyset\,,\\ &\text{in case } p < n: && A \text{ is } \mathcal{H}^{n-p}\text{-}\sigma\text{-finite} \implies \mathrm{Cap}_p(A) = 0\,,\\ &\text{in case } p \le n: && \mathrm{Cap}_p(A) = 0 \implies \dim_{\mathcal{H}}(A) \le n-p\,. \end{aligned}$$

Indeed, for our purposes capacity zero sets will be more relevant than capacity itself, and in this sense the first statement shows that Cap_p with $p > n$ is not of much interest.

(For the derivation of these properties compare with [8, Chapter 4.7, Theorems 3 and 4].)

Similar to properties which hold up to a null set for a measure one may also consider properties which hold up to a capacity zero set. Basically the only difference is that, in the capacity case, one prefers to use the word 'quasi' instead of 'almost':

Definition (Cap$_p$-quasi everywhere properties). *A property (which depends on a variable from a subset of $\mathbb{R}^n$) is said to hold* Cap_p*-quasi everywhere or at* Cap_p*-quasi every point, in short* Cap_p*-q.e., if it holds with the exception of a* Cap_p *zero set.*

Definition (Cap$_p$-quasi (semi)continuity). *A function $\psi\colon \Omega \to \overline{\mathbb{R}}$ on a subset Ω of $\mathbb{R}^n$ is called* Cap_p*-quasi continuous (or* Cap_p*-quasi lower/upper semicontinuous) on Ω if, for every $\varepsilon > 0$, there exists a relatively open subset A of Ω with $\mathrm{Cap}_p(A) < \varepsilon$ such that $\psi|_{\Omega\setminus A}$ is continuous (or lower/upper semicontinuous).*

Lemma (on **quasi continuous representatives of Sobolev functions**). *Consider an open $\Omega \subset \mathbb{R}^n$ and $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ with $p \in [1,\infty)$. Then,* Cap_p*-quasi every point in Ω is an* L^p*-Lebesgue point for u, and the Lebesgue representative u^* of u is* Cap_p*-quasi continuous on Ω.*

For a proof of the lemma compare with [8, Chapter 4.8, Theorem 1].

We remark that in case $p > n$ or $p = n = 1$, both definitions and the lemma trivialize: Cap_p-quasi everywhere is then the same as everywhere, quasi (semi)continuity is (semi)continuity, and the lemma is a direct consequence of the Sobolev embedding. As indicated above, however, our focus is now on the non-trivial case $p \le n$.

With the preceding concepts and results at hand, we can discuss the Euler equation for the obstacle problem also in case $p \le n$. To this end it is natural to define the admissible class with a quasi everywhere obstacle constraint for the Lebesgue representative as

$$K^p_{u_0,\psi} := \{w \in \mathrm{W}^{1,p}_{u_0}(\Omega) \,:\, w^* \ge \psi \text{ holds } \mathrm{Cap}_p\text{-q.e. on } \Omega\} \tag{$**$}$$

(which for $p > n$ or $p = n = 1$ reduces to the earlier definition ($*$)). The statement then reads as follows:

Theorem (Euler equation for the obstacle problem, $\underline{\text{case } p \le n}$). *Consider a bounded open set Ω in $\mathbb{R}^n$, an* upper *semicontinuous obstacle $\psi\colon \Omega \to \mathbb{R}\cup\{-\infty\}$, a boundary datum $u_0 \in \mathrm{W}^{1,p}(\Omega)$ with $p \in [1,n]$, and $u \in K^p_{u_0,\psi}$ (for $K^p_{u_0,\psi}$ defined in ($**$)) with $\nabla_z F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^{p'}(\Omega,\mathbb{R}^n)$, $\partial_y F(\,\cdot\,,u,\nabla u) \in \mathrm{L}^{(p^*)'}(\Omega)$ (where as usual we understand $p^* := \frac{np}{n-p}$ for $p < n$, while p^* is arbitrary in $[1,\infty)$ for $p = n$). Then u solves the* ***variational inequality***

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u) + \partial_y F(\,\cdot\,,u,\nabla u)(w-u)\big]\,\mathrm{d}x \ge 0 \qquad \textit{for all } w \in K^p_{u_0,\psi}$$

One calls $x_0 \in \Omega$ an **L^p-Lebesgue point** for $u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, $p \in [1,\infty)$, on open $\Omega \subset \mathbb{R}^n$, if there exists some $y \in \mathbb{R}^N$ with $\lim_{r\searrow 0} ⨍_{\mathrm{B}_r(x_0)} |u-y|^p\,\mathrm{d}x = 0$. The (necessarily unique) value y is the Lebesgue value of u at x_0, and the mapping u^* which associates to Lebesgue points the corresponding Lebesgue values is known as the **Lebesgue representative** of u. Given $u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, a differentiation theorem for measures yields that the set of non-L^p-Lebesgue points for u on Ω is negligible for the n-dimensional Lebesgue measure and that u^* is indeed an a.e. defined representative of u. For Sobolev functions u, the above lemma goes beyond this and shows that the set of non-L^p-Lebesgue points for u is indeed negligible even in a stronger capacity sense.

if and only if *there exists a non-negative Radon measure* μ *on* Ω *with the absolute-continuity property* $\mathrm{Cap}_p(A) = 0 \implies \mu(A) = 0$ *for* $A \in \mathcal{B}(\Omega)$ *such that*

$$\mathrm{E}_{\mathcal{F}}u = \mu \textit{ in } \mathscr{D}'(\Omega) \qquad\qquad \textit{and} \qquad\qquad \mu(\{u^* > \psi\}) = 0\,.$$

(*where* μ *is identified with the distribution* $\varphi \mapsto \int_\Omega \varphi \, \mathrm{d}\mu$).

Remarks (on the Euler equation for the obstacle problem in case $p \le n$)**.**

(1) We emphasize that in the present case we **can*not* use anymore** the earlier definition ($*$) of the admissible class $K^p_{u_0,\psi}$ with ***everywhere* constraint** $w \ge \psi$ on Ω, since in general neither w nor w^* nor any other representative is naturally pointwisely defined on all of Ω. Clearly, this could be overcome by simply defining the admissible class with the $\mathcal{L}^n$-almost everywhere constraint $w \ge \psi$ a.e. on Ω. However, if we would then state the theorem only for $\mathcal{L}^n$-almost everywhere constraints, admittedly it would appear conceptually simpler, but it would also be much weaker. Indeed, if $\psi = \mathbb{1}_S$ is the characteristic function of a set $S \subset \Omega$ with $|S| = 0$, $\mathrm{Cap}_p(S) > 0$ (for instance, a regular hypersurface S), then ψ coincides $\mathcal{L}^n$-a.e. but not Cap_p-q.e. with the zero function. Thus, the class $K^p_{u_0,\psi}$ defined in ($**$) with the $\mathbf{Cap}_{\boldsymbol{p}}$**-q.e. constraint 'sees'** this kind of **thin obstacles**, while the alternative definition with $\mathcal{L}^n$-a.e. constraint does not. All in all, the $\mathbf{Cap}_{\boldsymbol{p}}$**-q.e. constraint** is weaker than an everywhere constraint but stronger than an $\mathcal{L}^n$-a.e. constraint and turns out to be the optimal notion in the sense that it allows to impose the **finest possible constraint on general** $\mathbf{W}^{1,p}$ **functions**.

(2) The absolute-continuity property $\mathrm{Cap}_p(A) = 0 \implies \mu(A) = 0$ ensures that, even though u^* is naturally defined only Cap_p-q.e., the expression $\mu(\{u^* > \psi\})$ makes perfect sense.

(3) The conditions

$$\mathrm{E}_{\mathcal{F}}u = \mu \text{ in } \mathscr{D}'(\Omega) \qquad\qquad \text{and} \qquad\qquad \mu(\{u^* > \psi\}) = 0\,.$$

in the theorem **still express** a **supersolution property on the full domain** and a **solution property on a non-contact set**. Indeed, since μ is non-negative, one immediately reads off the supersolution property $\mathrm{E}_{\mathcal{F}}u \ge 0$ in $\mathscr{D}'(\Omega)$. Moreover, the vanishing of μ on the non-contact set $\{u^* > \psi\}$ can be understood as a way of expressing '$\mathrm{E}_{\mathcal{F}}u \equiv 0$ on $\{u^* > \psi\}$' (even though $\{u^* > \psi\}$ need not be open and therefore this property cannot be understood in a purely distributional fashion).

(4) In fact, the theorem can be extended to Cap_p-quasi upper semicontinuous obstacles ψ, but the proof of this more general version requires additional Cap_p-q.e. approximation results and will not be discussed here.

Proof of the theorem. We start with some preliminary steps.

STEP 1: We show that the validity of the variational inequality for u implies the existence of a non-negative Radon measure μ on Ω such that $\mathrm{E}_{\mathcal{F}}u = \mu$ in $\mathscr{D}'(\Omega)$, that is,

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla\varphi + \partial_y F(\,\cdot\,,u,\nabla u)\varphi\big]\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}\mu \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,. \qquad (+)$$

As before, the variational inequality implies $\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla\varphi+\partial_y F(\,\cdot\,,u,\nabla u)\varphi\big]\,\mathrm{d}x \ge 0$ for $0 \le \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, i.e. $\langle T;\varphi\rangle := \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla\varphi+\partial_y F(\,\cdot\,,u,\nabla u)\varphi\big]\,\mathrm{d}x$ for $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$

defines a non-negative distribution $T \in \mathscr{D}'(\Omega)$. We next establish, for arbitrary $\psi \in \mathrm{C}^0_{\mathrm{cpt}}(\Omega)$, the auxiliary identity

$$\inf\{\langle T;\varphi\rangle : \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,,\ \varphi \geq \psi \text{ on } \Omega\} = \sup\{\langle T;\varphi\rangle : \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,,\ \varphi \leq \psi \text{ on } \Omega\}$$

Indeed, '$\geq$' is immediate from non-negativity and linearity of T. To check '$\leq$', we fix a non-negative $\varphi_0 \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\varphi_0 \equiv 1$ on an open neighborhood U of $\operatorname{spt}\psi$. Then, for every $\varepsilon > 0$, mollification of $\psi-\frac{\varepsilon}{2}$ and cut-off in $U\setminus\operatorname{spt}\psi$ yield some $\varphi_\varepsilon \in \mathrm{C}^\infty_{\mathrm{cpt}}(U)$ such that $\psi-\varepsilon \leq \varphi_\varepsilon \leq \psi$ on U and hence $\varphi_\varepsilon \leq \psi \leq \varphi_\varepsilon+\varepsilon\varphi_0$ on Ω. We deduce $\inf\{\ldots\} \leq \langle T;\varphi_\varepsilon+\varepsilon\varphi_0\rangle = \langle T;\varphi_\varepsilon\rangle+\varepsilon\langle T;\varphi_0\rangle \leq \sup\{\ldots\}+\varepsilon\langle T;\varphi_0\rangle$ and obtain '$\leq$', since $\varepsilon > 0$ is arbitrary and φ_0 does not depend on ε. We now define $\langle \overline{T};\psi\rangle$ with $\psi \in \mathrm{C}^0_{\mathrm{cpt}}(\Omega)$ as the coinciding value of the above supremum and infimum. Thanks to the coincidence, it is then straightforward to check that $\overline{T}\colon \mathrm{C}^0_{\mathrm{cpt}}(\Omega) \to \mathbb{R}$ is finite-valued and linear. Moreover, the non-negativity of T easily implies that $\overline{T}$ is still non-negative and coincides with T on $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$. By the Riesz representation theorem for such functionals, there exists a non-negative Radon measure μ on Ω such that $\langle \overline{T};\psi\rangle = \int_\Omega \psi\,\mathrm{d}\mu$ for all $\psi \in \mathrm{C}^0_{\mathrm{cpt}}(\Omega)$. Specifically, we have $\langle T;\varphi\rangle = \int_\Omega \varphi\,\mathrm{d}\mu$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, and this is the claim.

Step 2: For a non-negative Radon measure μ on Ω given by (+), we establish the absolute-continuity property $\mathrm{Cap}_p(A) = 0 \implies \mu(A) = 0$ for $A \in \mathcal{B}(\Omega)$.

First we consider a compact $A \subset \Omega$ with $\mathrm{Cap}_p(A) = 0$, and exploiting compactness we fix a non-negative $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\eta \equiv 1$ on A. For arbitrary $\varepsilon > 0$, the definition of p-capacity yields some $w_\varepsilon \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ with $w_\varepsilon \geq 1$ a.e. on an open $O_\varepsilon \supset A$ and $\|w_\varepsilon\|_{\mathrm{W}^{1,p};\mathbb{R}^n} < \varepsilon$. Possibly replacing w_ε with its positive part, we assume $e_\varepsilon \geq 0$ a.e. on $\mathbb{R}^n$. In a next step, which draws once more on the assumption that A is compact and thus $\operatorname{dist}(A,\mathbb{R}^n \setminus O_\varepsilon) > 0$ holds, possibly replacing w_ε with its mollification and decreasing O_ε, we can assume $w_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)$. Then $\eta w_\varepsilon \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ satisfies $\eta w_\varepsilon \geq 0$ on Ω and $\eta w_\varepsilon \equiv 1$ on A. Via the equation (+), and the Sobolev(-Poincaré) inequality, we can thus estimate

$$\begin{aligned}
\mu(A) \leq \int_\Omega \eta w_\varepsilon\,\mathrm{d}x &= \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(\eta w_\varepsilon) + \partial_y F(\,\cdot\,,u,\nabla u)\eta w_\varepsilon\big]\,\mathrm{d}x \\
&\leq \mathrm{const}\Big[\|\nabla(\eta w_\varepsilon)\|_{\mathrm{L}^p;\Omega} + \|\eta w_\varepsilon\|_{\mathrm{L}^{p^*};\Omega}\Big] \\
&\leq \mathrm{const}\|\nabla(\eta w_\varepsilon)\|_{\mathrm{L}^p;\Omega} \\
&\leq \mathrm{const}(\eta)\|w_\varepsilon\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} \leq \mathrm{const}(\eta)\varepsilon\,,
\end{aligned}$$

where const changes from line to line and depends on n, p, Ω, the $\mathrm{L}^{p'}$-norm of $\nabla_z F(\,\cdot\,,u,\nabla u)$, and the $\mathrm{L}^{(p^*)'}$-norm of $\partial_y F(\,\cdot\,,u,\nabla u)$ (in addition to the indicated dependence on η). Since $\varepsilon > 0$ is arbitrary and η does not depend on ε, this means $\mu(A) = 0$.

For arbitrary $A \in \mathcal{B}(\Omega)$, the inner regularity $\mu(A) = \sup\{\mu(K) : K \text{ compact} \subset A\}$ of the Radon measure μ yields the same conclusion.

A version of the **Riesz representation theorem** asserts that every *non-negative* linear functional T on $\mathrm{C}^0_{\mathrm{cpt}}(\Omega)$ (with open $\Omega \subset \mathbb{R}^n$ or more generally with a locally-compact separable metric space Ω) can be represented in the form $\langle T;\varphi\rangle = \int_\Omega \varphi\,\mathrm{d}\mu$ for all $\varphi \in \mathrm{C}^0_{\mathrm{cpt}}(\Omega)$ with a unique non-negative Radon measure μ on Ω.

We emphasize that this variant of the representation theorem differs from another well-known variant for functionals $T \in \mathrm{C}^0_0(\Omega,\mathbb{R}^N)^*$ on the one hand through the non-negativity hypothesis, on the other hand insofar that *no* explicit continuity assumption on T is needed (but rather it turns out in the proof that, locally on Ω, continuity follows automatically).

STEP 3: We check that (+) suitably extends to Sobolev test functions φ. Precisely, we show that the validity of (+) with a non-negative Radon measure μ on Ω implies

$$\int_\Omega \left[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla\varphi+\partial_y F(\,\cdot\,,u,\nabla u)\varphi\right]\mathrm{d}x=\int_\Omega \varphi^*\,\mathrm{d}\mu \qquad \text{for all } \varphi\in \mathrm{W}^{1,p}_0(\Omega)\,, \qquad (++)$$

where the right-hand term is well-defined in view of the lemma on quasi-continuous representatives and the result of Step 2.

We first establish (++) in case φ is additionally bounded, i.e. for $\varphi\in \mathrm{W}^{1,p}_0(\Omega)\cap \mathrm{L}^\infty(\Omega)$. To this end, we aim at constructing, for $k\in\mathbb{N}$, approximations $\varphi_k\in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\sup_\Omega|\varphi_k|\le \|\varphi\|_{\mathrm{L}^\infty;\Omega}$ such that $\lim_{k\to\infty}\varphi_k=\varphi$ in $\mathrm{W}^{1,p}(\Omega)$ and $\lim_{k\to\infty}\varphi_k(x)=\varphi^*(x)$ whenever $x\in\Omega$ is an L^p-Lebesgue point for φ. One possible construction of such φ_k proceeds as follows. One fixes, for all $k\in\mathbb{N}$, cut-off functions $\eta_k\in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ such that $\eta_k\equiv 1$ on $\{x\in\Omega:\mathrm{dist}(x,\mathbb{R}^n\setminus\Omega)>k^{-1}\}$ and $0\le\eta_k\le 1$ on Ω. Moreover, from the definition of $\mathrm{W}^{1,p}_0(\Omega)$ one obtains, for $\ell\in\mathbb{N}$, approximations $\widetilde{\varphi}_\ell\in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ such that $\lim_{\ell\to\infty}\widetilde{\varphi}_\ell=\varphi$ in $\mathrm{W}^{1,p}(\Omega)$. Possibly applying a cut-off and mollification procedure, one can additionally achieve $\sup_\Omega|\widetilde{\varphi}_\ell|\le\|\varphi\|_{\mathrm{L}^\infty;\Omega}$ for all $\ell\in\mathbb{N}$. For mollifications $\overline{\varphi}_\varepsilon$ of φ with mollification radius $\varepsilon>0$, one moreover knows $\sup_\Omega|\overline{\varphi}_\varepsilon|\le \|\varphi\|_{\mathrm{L}^\infty;\Omega}$ for all $\varepsilon>0$, $\lim_{\varepsilon\searrow 0}\overline{\varphi}_\varepsilon=\varphi$ in $\mathrm{W}^{1,p}(\Omega)$, and $\lim_{\varepsilon\searrow 0}\overline{\varphi}_\varepsilon(x)=\varphi^*(x)$ whenever $x\in\Omega$ is an L^p-Lebesgue point for φ. Choosing suitable subsequences $\ell_k\to\infty$ and $\varepsilon_k\searrow 0$, one can now achieve $\lim_{k\to\infty}\big(\|\overline{\varphi}_{\varepsilon_k}-\widetilde{\varphi}_{\ell_k}\|_{\mathrm{L}^p;\mathrm{spt}\,\eta_k}\sup_\Omega|\nabla\eta_k|\big)=0$ and then straightforwardly check the claimed convergence properties for $\varphi_k:=\eta_k\overline{\varphi}_{\varepsilon_k}+(1-\eta_k)\widetilde{\varphi}_{\ell_k}\in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$. Once the approximations φ_k are at hand, we use the preceding lemma and the absolute-continuity property from Step 2 to conclude that $\lim_{k\to\infty}\varphi_k=\varphi^*$ converges in fact Cap_p-q.e. and thus also μ-a.e. on Ω. With this knowledge, we use (+) for φ_k and pass to the limit $k\to\infty$ on both sides. The left-hand sides converge in view of the $\mathrm{W}^{1,p}$-convergence $\varphi_k\to\varphi$ and the integrability assumptions $\nabla_z F(\,\cdot\,,u,\nabla u)\in \mathrm{L}^{p'}(\Omega,\mathbb{R}^n)$, $\partial_y F(\,\cdot\,,u,\nabla u)\in \mathrm{L}^{(p^*)'}(\Omega)$ to the the same integral term with φ. For the right-hand sides, using the μ-a.e. convergence and the uniform bound for $|\varphi_k|$, we can apply the dominated convergence theorem to deduce $\lim_{k\to\infty}\int_\Omega\varphi_k\,\mathrm{d}\mu=\int_\Omega\varphi^*\,\mathrm{d}\mu$. Therefore, we end up with (++) for $\varphi\in \mathrm{W}^{1,p}_0(\Omega)\cap \mathrm{L}^\infty(\Omega)$.

In order to further extend (++) from $\varphi\in \mathrm{W}^{1,p}_0(\Omega)\cap \mathrm{L}^\infty(\Omega)$ to arbitrary $\varphi\in \mathrm{W}^{1,p}_0(\Omega)$, we can use a considerably simpler approximation procedure: First considering a non-negative $\varphi\in \mathrm{W}^{1,p}_0(\Omega)$, we now set $\varphi_k:=\min\{\varphi,k\}\in \mathrm{W}^{1,p}_0(\Omega)\cap \mathrm{L}^\infty(\Omega)$. Then on the left-hand side (++) we may pass from φ_k to φ by dominated convergence, on the right-hand side, using $\lim_{k\to\infty}\varphi_k^*=\varphi^*$ in L^p-Lebesgue points for φ^* and thus μ-a.e., we may pass from φ_k^* to φ^* by monotone convergence. Arguing this way we obtain (++) for non-negative $\varphi\in \mathrm{W}^{1,p}_0(\Omega)$. For arbitrary $\varphi\in \mathrm{W}^{1,p}_0(\Omega)$, (++) then follows simply by decomposition $\varphi=\varphi_+-\varphi_-$.

STEP 4: We establish the forward implication in the theorem.

We assume that u satisfies the variational inequality. Then in view of Step 1 and Step 2 we have (+) for a non-negative Radon measure μ with the stated absolute-continuity property, and it only remains to prove $\mu(\{u^*>\psi\})=0$. To achieve this, we first consider an arbitrary competitor $w\in K^p_{u_0,\psi}$ with $w\le u$ a.e. on Ω. Then we have $w^*\le u^*$ in every L^p-Lebesgue point, thus we get $\mathrm{Cap}_p(\{w^*>u^*\})=0$ and then also $\mu(\{w^*>u^*\})=0$. Together with (++) and the variational inequality this leads to

$$\int_{\{u^*>w^*\}}(w^*-u^*)\,\mathrm{d}\mu = \int_\Omega (w^*-u^*)\,\mathrm{d}\mu$$
$$= \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u)+\partial_y F(\,\cdot\,,u,\nabla u)(w-u)\big]\,\mathrm{d}x \ge 0\,.$$

From this we read off that we necessarily have $\mu(\{u^*>w^*\})=0$. In case $\psi\in K^p_{u_0,\psi}$ we can now simply conclude by taking $w=\psi$, but in our general setting ψ need neither be a $\mathrm{W}^{1,p}$ function nor have the right boundary values, and thus we cannot get trough that simply. To solve this, we consider an arbitrary compact $S\subset\Omega$, and in addition we choose a compact $S'\subset\Omega$ with $S\subset\mathrm{int}(S')$ and some non-negative $\eta\in\mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{int}(S'))$ with $\eta\equiv 1$ on S. Then we use a procedure sometimes called Moreau-Yosida approximation: In case $\psi\not\equiv-\infty$, for all $k\in\mathbb{N}$ and $x\in S'$, we define $\psi_k(x):=\sup_{y\in S'}\big[\psi(y)-k|x-y|\big]$ (and in the exceptional case $\psi:\equiv-\infty$ we agree on $\psi_k\equiv -k$ instead). We record — without going into the details of the comparably straightforward verifications — that the compactness of S' and the upper semicontinuity of ψ lead to $\sup_{S'}\psi<\infty$ and the following properties of ψ_k: Every ψ_k with fixed $k\in\mathbb{N}$ is Lipschitz continuous on S' (with Lipschitz constant k), there holds $\psi_1\ge\psi_2\ge\psi_3\ge\ldots$ on S', and $\lim_{k\to\infty}\psi_k=\psi$ converges pointwisely on S'. Furthermore we set $w_k:=\eta\min\{\psi_k,u\}+(1-\eta)u$ and observe that in this way we indeed obtain a function $w_k\in K^p_{u_0,\psi}$ with $w_k\le u$ a.e. on Ω. Therefore, the previous reasoning for the competitor w applies for w_k and yields $\mu(\{u^*>w_k^*\})=0$. Since, by construction, $w_k^*\le\psi_k$ holds Cap_p-q.e. and μ-a.e. on S, this implies specifically $\mu(S\cap\{u^*>\psi_k\})=0$. Via the pointwise convergence of ψ_k to ψ, we conclude $\mu(S\cap\{u^*>\psi\})=0$. But then, since S is an arbitrary compact subset of Ω, we finally arrive at the claim $\mu(\{u^*>\psi\})=0$.

Step 5: We establish the backward implication in the theorem.

Assume that we have $\mathrm{E}_{\mathcal{F}}u=\mu$ in $\mathscr{D}'(\Omega)$ or in other words (+) and also $\mu(\{u^*>\psi\})=0$. Then, for arbitrary $w\in K^p_{u_0,\psi}$, we get $w-u\in\mathrm{W}^{1,p}_0(\Omega)$, and Step 3 yields

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u)+\partial_y F(\,\cdot\,,u,\nabla u)(w-u)\big]\,\mathrm{d}x = \int_\Omega (w^*-u^*)\,\mathrm{d}\mu\,,$$

where we have used that $(w-u)^*=w^*-u^*$ holds Cap_p-q.e. and thanks to Step 2 also μ-a.e. on Ω. For the right-hand side we get

$$\int_\Omega (w^*-u^*)\,\mathrm{d}\mu \ge \int_{\{w^*<u^*\}}(w^*-u^*)\,\mathrm{d}\mu = 0$$

since $w^*\ge\psi$ holds Cap_p-q.e. and μ-a.e. and thus $\mu(\{u^*>\psi\})=0$ implies $\mu(\{w^*<u^*\})=0$. Altogether we end up with

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\nabla u)\cdot\nabla(w-u)+\partial_y F(\,\cdot\,,u,\nabla u)(w-u)\big]\,\mathrm{d}x \ge 0\,,$$

which is the targeted variational inequality. □

3.3 Isoperimetric or holonomic constraints

We recall the **guiding principle** that **non-linear constraints require more general variations** than those of the special from $(w+t\varphi)_{t\in(-\varepsilon,\varepsilon)}$. This principle will now take full effect, and thus we explicitly coin a general notion of variation:

Definition (**general (admissible) variations**). *Consider a subset $\mathcal{X}$ of a topological vector space and a family $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ of elements of $\mathcal{X}$ with $\varepsilon>0$. If the mapping $t\mapsto u_t$, defined on a possibly smaller neighborhood of 0, is a* $\mathbf{C^1}$ ***curve*** *in $\mathcal{X}$, then $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ is called a* ***(general) variation of*** $\boldsymbol{u := u_0}$ ***in*** $\boldsymbol{\mathcal{X}}$ ***in direction*** $\boldsymbol{\varphi := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u_t = \lim_{t\to 0}\frac{u_t-u}{t}}$. *If, for given $\mathcal{A}\subset\mathcal{X}$, the variation satisfies $u_t\in\mathcal{A}$ for $|t|\ll 1$, it is called* ***$\mathcal{A}$-admissible***.

We emphasize that, for a C^1 curve $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ in $\mathrm{W}^{1,p}_{(\mathrm{loc})}(\Omega,\mathbb{R}^N)$, the four quantities u_t, $\mathrm{D}u_t$, $\frac{\mathrm{d}}{\mathrm{d}t}u_t$, $\mathrm{D}\frac{\mathrm{d}}{\mathrm{d}t}u_t$ are all continuous $\mathrm{L}^p_{(\mathrm{loc})}$-valued functions of $t\in(-\varepsilon,\varepsilon)$. Moreover, as the convergence $\lim_{h\to 0}\frac{u_{t+h}-u_t}{h}=\frac{\mathrm{d}}{\mathrm{d}t}u_t$ in $\mathrm{W}^{1,p}_{(\mathrm{loc})}(\Omega,\mathbb{R}^N)$ comprises the convergence $\lim_{h\to 0}\frac{\mathrm{D}u_{t+h}-\mathrm{D}u_t}{h}=\mathrm{D}\frac{\mathrm{d}}{\mathrm{d}t}u_t$ in $\mathrm{L}^p_{(\mathrm{loc})}(\Omega,\mathbb{R}^{N\times n})$, we infer that $(\mathrm{D}u_t)_{t\in(-\varepsilon,\varepsilon)}$ is a C^1 curve in $\mathrm{L}^p_{(\mathrm{loc})}(\Omega,\mathbb{R}^{N\times n})$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t=\mathrm{D}\frac{\mathrm{d}}{\mathrm{d}t}u_t \qquad\qquad \text{in } \mathrm{L}^p_{(\mathrm{loc})}(\Omega,\mathbb{R}^{N\times n})\,,$$

where the left-hand $\frac{\mathrm{d}}{\mathrm{d}t}$ is taken in $\mathrm{W}^{1,p}_{(\mathrm{loc})}(\Omega,\mathbb{R}^N)$, while the right-hand $\frac{\mathrm{d}}{\mathrm{d}t}$ is taken in $\mathrm{L}^p_{(\mathrm{loc})}(\Omega,\mathbb{R}^{N\times n})$. In the sequel, we prefer to use the latter notation for the mixed derivative.

In the setting of general variations, we still have:

Proposition (**first-variation formula for general variations**). *For open $\Omega\subset\mathbb{R}^n$ and $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)\otimes\mathcal{B}(\mathbb{R}^{N\times n})$-measurable $F\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\overline{\mathbb{R}}$, set*

$$\mathcal{F}[w]:=\int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x$$

(whenever this exists in $\overline{\mathbb{R}}$). Then, for $u\in\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $F(\,\cdot\,,u,\mathrm{D}u)\in\mathrm{L}^1(\Omega)$ and a variation $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ of u in $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ in direction $\varphi\in\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{F}[u_t]=\int_\Omega\big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi+\nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi\big]\,\mathrm{d}x\in\mathbb{R}$$

provided that, *for a null set E and $|t|\ll 1$, the integrand $F(x,y,z)$ is totally differentiable in $(y,z)\in\mathbb{R}^N\times\mathbb{R}^{N\times n}$ at all points $(x,u_t(x),\mathrm{D}u_t(x))$ with $x\in\Omega\setminus E$, for $|t|\ll 1$ one has $F(\,\cdot\,,u_t,\mathrm{D}u_t)\in\mathrm{L}^1(\Omega)$, the derivative (as $\mathrm{L}^1(\Omega)$-valued curve) $\frac{\mathrm{d}}{\mathrm{d}t}F(\,\cdot\,,u_t,\mathrm{D}u_t)$ exists and is continuous at $t=0$ with $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}F(\,\cdot\,,u_t,\mathrm{D}u_t)=\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi+\nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi$.*

Proof. Under the assumptions made, the auxiliary expression

$$H(t):=F(\,\cdot\,,u_t,\mathrm{D}u_t)\in\mathrm{L}^1(\Omega)$$

is differentiable in t, $|t|\ll 1$, and satisfies

$$\lim_{t\to 0}H'(t)=H'(0)=\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi+\nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi \qquad\qquad \text{in } \mathrm{L}^1(\Omega)\,.$$

With the help of the standard derivative estimate for the $\mathrm{L}^1(\Omega)$-valued differentiable curve $t\mapsto H(t)-H'(0)t$, we conclude

$$\begin{aligned}\left|\frac{\mathcal{F}[u_t]-\mathcal{F}[u]}{t}-\int_\Omega H'(0)\,\mathrm{d}x\right|&=\left|\int_\Omega\frac{H(t)-H'(0)t-H(0)}{t}\,\mathrm{d}x\right|\\&\le\left\|\frac{H(t)-H'(0)t-H(0)}{t}\right\|_{\mathrm{L}^1(\Omega)}\le\sup_{(-|t|,|t|)}\|H'-H'(0)\|_{\mathrm{L}^1(\Omega)}\underset{t\to 0}{\longrightarrow}0\,.\end{aligned}$$

This proves $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{F}[u_t]=\int_\Omega H'(0)\,\mathrm{d}x$, which is the claim. ☐

Remarks and Definitions (on the first-variation formula for general variations)**.**

(1) **If $\nabla_z F$, $\nabla_y F$ satisfy growth conditions** of the type already discussed in Section 3.1, then **the assumptions of the preceding proposition can be ensured** for all variations in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$.

More precisely, in full technical detail, we have:

Lemma (on growth conditions and the general first-variation formula)**.** *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, that $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ is Carathéodory with $F(x,\,\cdot\,,\,\cdot\,)\in \mathrm{C}^1(\mathbb{R}^N\times\mathbb{R}^{N\times n})$ for all $x\in\Omega\setminus E$ with a null set $E\subset\Omega$, and that $\nabla_z F$, $\nabla_y F$ satisfy the growth conditions*

$$|\nabla_z F(x,y,z)|\le \begin{cases}\Psi(x)^{1/p'}+C|z|^{p/p'}+C|y|^{p^*/p'}\,, & \text{if } p\le n\\ \Psi(x)^{1/p'}+C|z|^{p/p'}+b(|y|)\,, & \text{if } n<p<\infty\\ \Psi(x)+b(|z|)+b(|y|)\,, & \text{if } p=\infty\end{cases}\,,$$

$$|\nabla_y F(x,y,z)|\le \begin{cases}\Psi(x)^{1/(p^*)'}+C|z|^{p/(p^*)'}+C|y|^{p^*/(p^*)'}\,, & \text{if } p\le n\\ \Psi(x)+C|z|^{p}+b(|y|)\,, & \text{if } n<p<\infty\\ \Psi(x)+b(|z|)+b(|y|)\,, & \text{if } p=\infty\end{cases}$$

for all $(x,y,z)\in(\Omega\setminus E)\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ with $\Psi\in\mathrm{L}^1(\Omega)$, $C\in[0,\infty)$, $p^*:=\frac{np}{n-p}$ if $p<n$, any exponent $p^*\in[1,\infty)$ if $p=n$, and locally bounded $b\colon[0,\infty)\to[0,\infty)$. Consider $u\in\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ with $F(\,\cdot\,,u,\mathrm{D}u)\in\mathrm{L}^1(\Omega)$ and a variation $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ of u in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ in direction $\varphi\in\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$. Then, $F(\,\cdot\,,u_t,\mathrm{D}u_t)\in\mathrm{L}^1(\Omega)$ is differentiable in t, $|t|\ll 1$ (as $\mathrm{L}^1(\Omega)$-valued curve) with derivative given by the chain-rule formula

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\,\cdot\,,u_t,\mathrm{D}u_t)=\nabla_z F(\,\cdot\,,u_t,\mathrm{D}u_t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t+\nabla_y F(\,\cdot\,,u_t,\mathrm{D}u_t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}u_t\qquad \text{in } \mathrm{L}^1(\Omega) \text{ for } |t|\ll 1\,,$$

and we have the continuity properties

$$\lim_{t\to 0}\nabla_z F(\,\cdot\,,u_t,\mathrm{D}u_t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t=\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi\,,\quad \lim_{t\to 0}\nabla_y F(\,\cdot\,,u_t,\mathrm{D}u_t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}u_t=\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi\quad \text{in } \mathrm{L}^1(\Omega)\,.$$

In particular, all requirements of the preceding proposition are ensured and the first-variation formula applies.

Proof in case $p\le n$ (the other cases being similar). With the abbreviations

$$H(t):=F(\,\cdot\,,u_t,\mathrm{D}u_t)\,,\qquad G_z(t):=\nabla_z F(\,\cdot\,,u_t,\mathrm{D}u_t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t\,,\qquad G_y(t):=\nabla_y F(\,\cdot\,,u_t,\mathrm{D}u_t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}u_t$$

(specifically $G_z(0)=\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot\mathrm{D}\varphi$, $G_y(0)=\nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi$) we aim at proving that, for $|t|\ll 1$, the quantity $H(t)\in\mathrm{L}^1(\Omega)$ has derivative

$$H'(t)=G_z(t)+G_y(t)\qquad \text{in } \mathrm{L}^1(\Omega)\tag{$*$}$$

and that G_z, G_y are continuous at 0.

In order to establish continuity of G_z at 0, we first consider a null sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}\setminus\{0\}$ such that the convergences $u_{t_k}\to u$, $\mathrm{D}u_{t_k}\to\mathrm{D}u$, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_k}\mathrm{D}u_t\to\mathrm{D}\varphi$ for $k\to\infty$ are valid a.e. on Ω. By continuity of $\nabla_z F$ in (y,z), these convergences imply $\lim_{k\to\infty}G_z(t_k)=G_z(0)$ a.e. on Ω. To obtain the same convergence in $\mathrm{L}^1(\Omega)$, we estimate with the growth condition for $\nabla_z F$ and Young's inequality

$$|G_z(t)|\le\big(\Psi^{1/p'}+C|\mathrm{D}u_t|^{p/p'}+C|u_t|^{p^*/p'}\big)\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t\Big|\le \mathrm{const}(n,p,C)\Big[\Psi+|\mathrm{D}u_t|^p+|u_t|^{p^*}+\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t\Big|^p\Big]\qquad \text{a.e. on } \Omega\,.$$

We recall $\Psi\in\mathrm{L}^1(\Omega)$, and, taking into account that $(u_t)_{|t|\ll 1}$ is a variation of u in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, we infer from the definition and the Sobolev embedding that the $\mathrm{L}^1(\Omega)$-valued quantities $|\mathrm{D}u_t|^p$, $|u_t|^{p^*}$, $|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}u_t|^p$ depend continuously on t with $|t|\ll 1$. Therefore, the below-mentioned variant of the dominated convergence theorem gives $\lim_{k\to\infty}G_z(t_k)=$

The following **variant of** the **dominated convergence** theorem is proved in the exercises: Consider a measure space $(\Omega,\mathcal{A},\mu)$ and $f_k,f\in\mathrm{L}^1(\Omega,\mathbb{R}^N;\mu)$, $g_k,g\in\mathrm{L}^1(\Omega;\mu)$ such that $|f_k|\le g_k$ holds μ-a.e. on Ω for all $k\in\mathbb{N}$ and the convergences $\lim_{k\to\infty}f_k=f$, $\lim_{k\to\infty}g_k=g$ are valid μ-a.e. on Ω. Then $\lim_{k\to\infty}\int_\Omega g_k\,\mathrm{d}\mu=\lim_{k\to\infty}\int_\Omega g\,\mathrm{d}\mu$ implies $\lim_{k\to\infty}f_k\,\mathrm{d}\mu=\int_\Omega f\,\mathrm{d}\mu$.

$G_z(0)$ also in $\mathrm{L}^1(\Omega)$. At this stage, to conclude even $\lim_{t\to 0} G_z(t) = G_z(0)$ in $\mathrm{L}^1(\Omega)$, it suffices to ensure that every null sequence in $\mathbb{R}\setminus\{0\}$ contains a subsequence $(t_k)_{k\in\mathbb{N}}$ with the above convergence properties. However, this is at hand, since the required convergences hold as L^p-convergences by the definition of the variation, and a standard measure theory result then gives them a.e. along a subsequence.

The continuity of G_y at 0 follows analogously (by using $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_k} u_t \to \varphi$ instead of $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_k} \mathrm{D}u_t \to \mathrm{D}\varphi$) and we do not repeat the relevant arguments in detail.

In order to derive the formula $(*)$, we start with an auxiliary observation: Consider some $x\in\Omega\setminus E$ and convergent sequences $\omega_k\to\omega$ in $\mathbb{R}^N\times\mathbb{R}^{N\times n}$, $0\neq t_k\to 0$ in $\mathbb{R}$ with $\frac{\omega_k-\omega}{t_k}\to\zeta$ for $k\to\infty$. Then it is a routine matter to verify with the total differentiability of F in (y,z) the convergence $\frac{F(x,\omega_k)-F(x,\omega)}{t_k}\to\nabla_{(y,z)}F(x,\omega)\cdot\zeta$ for $k\to\infty$. For the main part of the reasoning, consider a null sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}\setminus\{0\}$ such that the convergences $u_{t_k}\to u$, $\lim_{k\to\infty}\mathrm{D}u_{t_k}\to\mathrm{D}u$, $\frac{u_{t_k}-u}{t_k}\to\varphi$, $\frac{\mathrm{D}u_{t_k}-\mathrm{D}u}{t_k}\to\mathrm{D}\varphi$ hold a.e. on Ω. Then the auxiliary observation yields

$$\lim_{k\to\infty}\frac{H(t_k)-H(0)}{t_k} = G_z(0)+G_y(0) \qquad \text{a.e. on } \Omega$$

In order to carry over this convergence to $\mathrm{L}^1(\Omega)$, we estimate with the standard estimate $|F(x,\widetilde{\omega})-F(x,\omega)|\le \sup_{\lambda\in[0,1]}|\nabla_{(y,z)}F(x,\lambda\widetilde{\omega}+(1-\lambda)\omega)|\,|\widetilde{\omega}-\omega|$, the growth conditions for $\nabla_z F$ and $\nabla_y F$, and Young's inequality

$$\begin{aligned}\left|\frac{H(t)-H(0)}{t}\right| &\le \Big[\Psi^{1/p'}+C(|\mathrm{D}u_t|+|\mathrm{D}u|)^{p/p'}+C(|u_t|+|u|)^{p^*/p'}\Big]\Big|\frac{\mathrm{D}u_t-\mathrm{D}u}{t}\Big|\\ &\quad+\Big[\Psi^{1/(p^*)'}+C(|\mathrm{D}u_t|+|\mathrm{D}u|)^{p/(p^*)'}+C(|u_t|+|u|)^{p^*/(p^*)'}\Big]\Big|\frac{u_t-u}{t}\Big|\\ &\le \mathrm{const}(n,p,C)\bigg[\Psi+|\mathrm{D}u_t|^p+|\mathrm{D}u|^p+|u_t|^{p^*}+|u|^{p^*}+\Big|\frac{\mathrm{D}u_t-\mathrm{D}u}{t}\Big|^p+\Big|\frac{u_t-u}{t}\Big|^{p^*}\bigg].\end{aligned}$$

As a side benefit, since we assumed $H(0)=F(\,\cdot\,,u,\mathrm{D}u)\in\mathrm{L}^1(\Omega)$ and the right-hand side of this estimate is in $\mathrm{L}^1(\Omega)$, for $|t|\ll 1$, we can read off $H(t)\in\mathrm{L}^1(\Omega)$. Moreover, the right-hand side converges, for $t\to 0$, to $\mathrm{const}(n,p,C)\big[\Psi+2|\mathrm{D}u|^p+2|u|^{p^*}+|\mathrm{D}\varphi|^p+|\varphi|^{p^*}\big]$ in $\mathrm{L}^1(\Omega)$, and along the subsequence $(t_k)_{k\in\mathbb{N}}$ we have a.e. convergence to the same limit. Therefore, dominated convergence (again the mentioned variant) applies once more and gives $\lim_{k\to\infty}\frac{H(t_k)-H(0)}{t_k}=G_z(0)+G_y(0)$ also in $\mathrm{L}^1(\Omega)$. Reasoning with subsequences and the definition of the variation as before, we even obtain

$$\lim_{t\to 0}\frac{H(t)-H(0)}{t}=G_z(0)+G_y(0) \qquad \text{in } \mathrm{L}^1(\Omega)$$

This proves differentiability of H at 0 with $H'(0)=G_z(0)+G_y(0)$ in $\mathrm{L}^1(\Omega)$, and the same reasoning with only notational changes gives differentiability of H at points t, $|t|\ll 1$, with $H'(t)=G_z(t)+G_y(t)$ in $\mathrm{L}^1(\Omega)$. □

(2) **If all variations** $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ of u in a subset $\mathcal{X}$ of $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ **in direction $\boldsymbol{\varphi}$ satisfy the requirements of the proposition**, we can read off that $\boldsymbol{\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{F}[u_t]}$ **depends only on $\boldsymbol{u}$ and $\boldsymbol{\varphi}$** and not on the variation $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ as a whole. In this case (and if there is at least one such variation $(u_t)_{t\in(-\varepsilon,\varepsilon)}$), we **continue to use the** (then well-defined) **notation for the first variation**

$$\delta\mathcal{F}[u;\varphi]:=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{F}[u_t]\,.$$

If, for $\mathcal{A}\subset\mathcal{X}$, the requirements of the proposition are satisfied only for all $\mathcal{A}$-admissible variations $(u_t)_{t\in(-\varepsilon,\varepsilon)}$ of u in direction φ and there is one such variation at least, we also write

$$\delta_{\mathcal{A}}\mathcal{F}[u;\varphi]:=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{F}[u_t]$$

(which will exist only for $u\in\mathcal{A}$ and specific choices of φ).

Here, in principle both $\mathcal{X}$ and $\mathcal{A}$ are arbitrary subsets of $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and thus there is no true difference between $\delta\mathcal{F}[u;\varphi]$ and $\delta_{\mathcal{A}}\mathcal{F}[u;\varphi]$. However, when using these notations later on, $\mathcal{X}$ will rather be a function class needed for a correct technical implementation (smooth functions or functions in a Sobolev space, for instance), while the subset $\mathcal{A}$ of $\mathcal{X}$

is specified by a constraint and represents a principal feature of the underlying variational problem. In this connection, the notation $\delta_{\mathcal{A}}\mathcal{F}[u;\varphi]$ will indeed be useful to indicate that only $\mathcal{A}$-admissible variations are under consideration.

(3) The necessary criterion for minimum points in single-variable calculus immediately implies that a **necessary criterion for a minimizer u of $\mathcal{F}$ in $\mathcal{A}$** is

$$\delta_{\mathcal{A}}\mathcal{F}[u;\,\cdot\,] \equiv 0$$

(which is just meant to indicate $\delta_{\mathcal{A}}\mathcal{F}[u;\varphi]=0$ whenever $\delta_{\mathcal{A}}\mathcal{F}[u;\,\cdot\,]$ exists).

In the sequel we will determine a more concrete form of the necessary criterion for variational problems with either 'isoperimetric' constraints or holonomic constraints. We start with the isoperimetric case, in which the admissible class $\mathcal{A}$ is defined by finitely many **integral constraints**:

Theorem (necessary criterion for minimizers subject to isoperimetric constraints). *Consider $m \in \mathbb{N}$, $c_1,c_2,\ldots,c_m \in \mathbb{R}$, an open $\Omega \subset \mathbb{R}^n$, $\mathcal{M}^n\otimes\mathcal{B}(\mathbb{R}^N)\otimes\mathcal{B}(\mathbb{R}^{N\times n})$-measurable integrands $F,G_1,G_2,\ldots,G_m\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\overline{\mathbb{R}}$, and define the integral functionals*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x \qquad\qquad \mathcal{G}_i[w] := \int_\Omega G_i(\,\cdot\,,w,\mathrm{D}w)\,\mathrm{d}x \qquad \textit{for } i=1,2,\ldots,m$$

(whenever they exist in $\overline{\mathbb{R}}$). Furthermore, endow $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ with $\mathscr{D}$-convergence[5]*, and suppose, for $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $F(\,\cdot\,,u,\mathrm{D}u)\in\mathrm{L}^1(\Omega)$, that the first variations $\delta\mathcal{F}[u;\varphi]$ and $\delta\mathcal{G}_i[u+\sum_{j=0}^m t_j\psi_j;\varphi]$, $i=1,2,\ldots,m$, taken in $u+\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$, exist, are linear in φ, and continuous in $(t_0,t_1,t_2,\ldots,t_m)$ for all $\psi_0,\psi_1,\psi_2,\ldots,\psi_m,\varphi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ and $|(t_0,t_1,t_2,\ldots,t_m)|\le\varepsilon$ with some $\varepsilon=\varepsilon(\psi_0,\psi_1,\psi_2,\ldots,\psi_m,\varphi)>0$.* ***If u*** *with $\mathcal{G}_1[u]=c_1$, $\mathcal{G}_2[u]=c_2$, $\ldots$, $\mathcal{G}_m[u]=c_m$* ***minimizes $\mathcal{F}$ in the constrained class***

$$\mathcal{A} := \{w\in u+\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)\,:\,\mathcal{G}_1[w]=c_1\,,\,\mathcal{G}_2[w]=c_2\,,\,\ldots\,,\,\mathcal{G}_m[w]=c_m\}\,,$$

the linear functionals $\delta\mathcal{F}[u;\,\cdot\,]$, $\delta\mathcal{G}_1[u;\,\cdot\,]$, $\delta\mathcal{G}_2[u;\,\cdot\,]$, $\ldots$, $\delta\mathcal{G}_m[u;\,\cdot\,]$ on $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ are linearly dependent*, that is, there exist* ***Lagrange multipliers*** *$\lambda_1,\lambda_2,\ldots,\lambda_m\in\mathbb{R}$ such that*

$$\delta\mathcal{F}[u;\varphi]=\sum_{i=1}^m\lambda_i\delta\mathcal{G}_i[u;\varphi] \qquad\qquad \textit{for all } \varphi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)\,.$$

Clearly, the theorem also applies if u minimizes $\mathcal{F}$ in an admissible class which contains the above $\mathcal{A}$. In particular, the theorem applies to problems with an arbitrary type of boundary conditions (and also to problems with no boundary condition at all).

[5] Here, $\mathscr{D}$-convergence of a sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ to $w\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ means uniform convergence $\lim_{k\to\infty}\partial^\alpha w_k=\partial^\alpha w$ on Ω for all $\alpha\in\mathbb{N}_0^n$ with $\bigcup_{k=1}^\infty\mathrm{spt}(w_k)\Subset\Omega$. This convergence results from a topology on $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$, where the sets $\{v\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)\,:\,\sup_\Omega|\partial^\alpha v-\partial^\alpha w|<\varepsilon \text{ for all } \alpha\in\mathbb{N}_0^n\,,\,|\alpha|\le\ell \text{ and } \mathrm{spt}(v)\subset K\}$ with $\ell\in\mathbb{N}$, $\varepsilon>0$, and compact $K\subset\Omega$ are basis open neighborhoods of $w\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$. On $u+\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ we use the u-shifted version of this topology.

Remarks (on the necessary criterion subject to isoperimetric constraints)**.**

(1) We emphasize that the **technical assumptions on** the existence, linearity, and continuity in a parameter of **the first variations can be ensured via the preceding proposition**. The requirements needed for the proposition, in turn, can be obtained from growth conditions on the integrands F, G_1, G_2, $\ldots$, G_m (where the usage of the quite restrictive $\mathscr{D}$-convergence in theorem means that we allow few variations of u and thus need the requirements of the proposition only for comparably few 'smooth' variations).

(2) Whenever the first-variation formula applies, the conclusion of the theorem can be rephrased by saying that u weakly solves the equation

$$\Big(\mathrm{E}_{\mathcal{F}} - \sum_{i=1}^{m} \lambda_i \mathrm{E}_{\mathcal{G}_i}\Big) u \equiv 0 \qquad \text{on } \Omega$$

with the Euler-Lagrange operators $\mathrm{E}_{\mathcal{F}}$ and $\mathrm{E}_{\mathcal{G}_i}$ of $\mathcal{F}$ and $\mathcal{G}$, respectively. Clearly, this equation is nothing but the Euler-Lagrange equation of the integral functional $\mathcal{F}-\sum_{i=1}^{m} \lambda_i \mathcal{G}_i$.

(3) The constraint $\int_\Omega w \,\mathrm{d}x = c$ (for scalar w in case $N = 1$) of the actual non-parametric isoperimetric problem is contained in the theorem as the special case $m = 1$, $G_1(x, y, z) = y$. However, since this constraint is in fact linear, a much simpler treatment is possible and has already been discussed in the exercises.

(4) Still in the scalar case $N = 1$, the theorem shows that minimizers $u \in \mathrm{W}_0^{1,2}(\Omega)$ of the Dirichlet integral $\mathcal{E}_2$ in the constrained class

$$\Big\{ w \in \mathrm{W}_0^{1,2}(\Omega) \,:\, \int_\Omega w^2 \,\mathrm{d}x = 1 \Big\}$$

are necessarily weak solution to the Helmholtz equation

$$-\Delta u = \lambda u \qquad \text{on } \Omega\,.$$

In other words, this basic minimization problem produces solutions to the **eigenvalue problem for $-\boldsymbol{\Delta}$ with zero Dirichlet boundary values** on Ω.

(5) The theorem incorporates very general integral constraints with possibly non-linear dependence of the integrands $G_i(x, y, z)$ on z. We remark that the existence theory for minimizers does not apply in comparable generality, but rather remains restricted (apart from very specific cases) to integral constraints without z-dependence or with linear z-dependence only.

Proof of the theorem. We can assume that the functionals $\delta\mathcal{G}_1[u;\,\cdot\,]$, $\delta\mathcal{G}_2[u;\,\cdot\,]$, $\ldots$, $\delta\mathcal{G}_m[u;\,\cdot\,]$ are linearly independent (since otherwise there is nothing to prove). With some linear algebra it then follows that there exist[6] $\psi_1, \psi_2, \ldots, \psi_m \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ such that

$$\delta\mathcal{G}_i[u;\psi_j] = \delta_{ij} \qquad \text{for } i, j = 1, 2, \ldots, m\,.$$

[6]Indeed, the above claim on the existence of $\psi_1, \psi_2, \ldots, \psi_m$ can be verified as follows. Assume first that $\bigcap_{i\in\{1,2,\ldots,m\}\setminus\{j\}} \ker(\delta\mathcal{G}_i[u;\,\cdot\,]) \subset \ker(\delta\mathcal{G}_j[u;\,\cdot\,])$ for some $j \in \{1, 2, \ldots, m\}$. Then, $K := \bigcap_{i=1}^{m} \ker(\delta\mathcal{G}_i[u;\,\cdot\,])$ can effectively be represented as the intersection of only $(m-1)$ kernels of co-dimension 1 in $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ and thus has a co-dimension $\ell \le m-1$ in $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$. We consider the m functionals induced by $\delta\mathcal{G}_1[u;\,\cdot\,]$, $\delta\mathcal{G}_2[u;\,\cdot\,]$, $\ldots$, $\delta\mathcal{G}_m[u;\,\cdot\,]$ on the ℓ-dimensional space $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)/K$. These m functionals are necessarily linearly dependent (since there are at most $\ell \le m-1$ linearly independent linear functionals on the ℓ-dimensional space), and then it

Now we fix an arbitrary $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$, and we observe that

$$\Gamma_i(t_0,t_1,t_2,\ldots,t_m) := \mathcal{G}_i\bigg[u+t_0\varphi+\sum_{j=1}^m t_j\psi_j\bigg]$$

for $i=1,2,\ldots,m$ defines an $\mathbb{R}^m$-valued C^1 function $\Gamma=(\Gamma_1,\Gamma_2,\ldots,\Gamma_m)$ on a neighborhood of 0 in $\mathbb{R}^{m+1}$ (where the C^1 property results from the assumption that the partial derivatives $\frac{\partial\Gamma_i}{\partial t_k}(t_0,t_1,t_2,\ldots,t_m)=\partial\mathcal{G}_i\big[u+\sum_{j=0}^m t_j\psi_j\,;\psi_k\big]$ with understanding $\psi_0:=\varphi$ exist and are continuous in $(t_0,t_1,t_2,\ldots,t_m)$ with $|(t_0,t_1,t_2,\ldots,t_m)|\ll 1$). We record $\Gamma(0,0,0,\ldots,0)=(c_1,c_2,\ldots,c_m)$ (since u satisfies the constraints) and $\frac{\partial\Gamma}{\partial(t_1,t_2,\ldots,t_m)}(0,0,0,\ldots,0)=\mathrm{I}_{m\times m}$ (by the above choice of ψ_j). At this point we decisively apply the implicit function theorem to conclude that $\Gamma^{-1}(c_1,c_2,\ldots,c_m)$ is near 0 the graph of C^1 function. In particular, we obtain a number $\delta>0$ and a C^1 function $\tau=(\tau_1,\tau_2,\ldots,\tau_m)\colon(-\delta,\delta)\to\mathbb{R}^m$ with $\tau(0)=0$ such that $\Gamma(t,\tau(t))=(c_1,c_2,\ldots,c_m)$ for all $t\in(-\delta,\delta)$ or equivalently

$$\mathcal{G}_i\bigg[u+t\varphi+\sum_{j=1}^m\tau_j(t)\psi_j\bigg]=c_i \qquad\text{for } i=1,2,\ldots,m \text{ and all } t\in(-\delta,\delta)\,.$$

At this stage we have constructed the $\mathcal{A}$-admissible variation $\big(u+t\varphi+\sum_{j=1}^m\tau_j(t)\psi_j\big)_{t\in(-\delta,\delta)}$ of u in $u+\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ in direction $\varphi+\sum_{j=1}^m\tau_j'(0)\psi_j$. Since the functions in $\mathcal{A}$ satisfy the constraints, since $\delta\mathcal{G}_i$ is linear in the direction (here by assumption), and since we have $\delta\mathcal{G}_i[u;\psi_j]=\delta_{ij}$, we infer

$$0=\delta_{\mathcal{A}}\mathcal{G}_i\bigg[u;\varphi+\sum_{j=1}^m\tau_j'(0)\psi_j\bigg]=\delta\mathcal{G}_i[u;\varphi]+\sum_{j=1}^m\tau_j'(0)\,\delta\mathcal{G}_i[u;\psi_j]=\delta\mathcal{G}_i[u;\varphi]+\tau_i'(0)$$

for $i=1,2,\ldots,m$. Thus, we have determined $\tau_j'(0)=-\delta\mathcal{G}_j[u;\varphi]$ for $j=1,2,\ldots,m$. Finally, using the necessary criterion for the minimality of u, we conclude

$$0=\delta_{\mathcal{A}}\mathcal{F}\bigg[u;\varphi+\sum_{j=1}^m\tau_j'(0)\psi_j\bigg]=\delta\mathcal{F}[u;\varphi]+\sum_{j=1}^m\tau_j'(0)\,\delta\mathcal{F}[u;\psi_j]=\delta\mathcal{F}[u;\varphi]-\sum_{j=1}^m\lambda_j\,\delta\mathcal{G}_j[u;\varphi]$$

with Lagrange multipliers $\lambda_j:=\delta\mathcal{F}[u;\psi_j]$ (which do not depend on the arbitrary $\varphi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ but only on the initially fixed choices of ψ_j). We have thus proved the claim. □

Next we turn to admissible classes (of $\mathbb{R}^N$-valued functions on open $\Omega\subset\mathbb{R}^n$) defined by finitely many **holonomic constraints**

$$g_1(w)\equiv c_1\,,\qquad g_2(w)\equiv c_2\,,\qquad \ldots\,,\qquad g_m(w)\equiv c_m \qquad\qquad \text{a.e. on } \Omega$$

follows that $\delta\mathcal{G}_1[u;\,\cdot\,]$, $\delta\mathcal{G}_2[u;\,\cdot\,]$, $\ldots$, $\delta\mathcal{G}_m[u;\,\cdot\,]$ themselves are also linearly dependent. Thus, the above assumption must have been wrong, and indeed we have $\bigcap_{i\in\{1,2,\ldots,m\}\setminus\{j\}}\ker(\delta\mathcal{G}_i[u;\,\cdot\,])\not\subset\ker(\delta\mathcal{G}_j[u;\,\cdot\,])$ for $j=1,2,\ldots,m$. This means that there exist $\psi_j\in\bigcap_{i\in\{1,2,\ldots,m\}\setminus\{j\}}\ker(\delta\mathcal{G}_i[u;\,\cdot\,])\setminus\ker(\delta\mathcal{G}_j[u;\,\cdot\,])$ for $j=1,2,\ldots,m$, and these ψ_j consequently satisfy $\delta\mathcal{G}_i[u;\psi_j]=0$ for $j\neq i$ and $\delta\mathcal{G}_i[u;\psi_j]\neq0$ for $j=i$ in $\{1,2,\ldots,m\}$. Normalizing the ψ_j by multiplication with suitable non-zero scalar factors and using the assumed linearity of $\delta\mathcal{G}_i[u;\,\cdot\,]$, we end up with the claimed property $\delta\mathcal{G}_i[u;\psi_j]=\delta_{ij}$.

with $c_1, c_2, \ldots, c_m \in \mathbb{R}$ and $g = (g_1, g_2, \ldots, g_m) \colon \mathbb{R}^N \to \mathbb{R}^m$. One typically assumes that the constraints are independent in the sense that $g_1, g_2, \ldots, g_m$ are C^1 on $\mathbb{R}^N$ with $\nabla g_1(x), \nabla g_2(x), \ldots, \nabla g_m(x)$ linearly independent in $\mathbb{R}^N$ for every $x \in \bigcap_{i=1}^m \{g_i = c_i\}$, or equivalently that g is C^1 on $\mathbb{R}^N$ with $\operatorname{rank}(\mathrm{D}g) \equiv m$ on $\bigcap_{i=1}^m \{g_i = c_i\}$ (clearly possible in case $m \le N$ only). In fact, $\bigcap_{i=1}^m \{g_i = c_i\}$ is then a submanifold and thus holonomic constraints of the described type are a special case of a **manifold constraint**

$$w \in \mathcal{M} \qquad \text{a.e. on } \Omega$$

with a general $(N{-}m)$-dimensional C^1-submanifold $\mathcal{M}$ in $\mathbb{R}^N$. In fact, in order to obtain a necessary criterion for problems with this type of constraints, we need and work with a certain **technical retraction property**. In order to set this clear and single out the core variational statement, we introduce:

Definition (strong $\mathbf{C^2}$-neighborhood retracts). *We say that a subset $\mathcal{M}$ of $\mathbb{R}^N$ is a strong C^2-neighborhood retract in $\mathbb{R}^N$ if there exist and open set $\mathcal{U} \subset \mathbb{R}^N$ and a function $R \in \mathrm{C}^2(\mathcal{U}, \mathbb{R}^N)$ with $\mathrm{D}R$, D^2R bounded on $\mathcal{U}$ such that we have*

$$\operatorname{dist}(\mathcal{M}, \mathbb{R}^N \setminus \mathcal{U}) > 0\,, \qquad R(\mathcal{U}) \subset \mathcal{M}\,, \qquad R|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}\,.$$

Remarks (on strong C^2-neighborhood retracts)**.**

(1) A strong C^2-neighborhood retract $\mathcal{M}$ in $\mathbb{R}^N$ is always closed in $\mathbb{R}^N$, since, for $\mathcal{U}$ and R as in the definition, the third, first, and second condition (applied in this order) yield $\overline{\mathcal{M}} = R(\overline{\mathcal{M}}) \subset R(U) \subset \mathcal{M}$.

(2) Most importantly, **every closed (i.e. compact without-boundary) $\mathbf{C^2}$-submanifold $\boldsymbol{\mathcal{M}}$ in $\mathbb{R}^{\boldsymbol{N}}$ is a strong $\mathbf{C^2}$-neighborhood retract in $\mathbb{R}^{\boldsymbol{N}}$.** If $\mathcal{M}$ is even C^3, this is comparably straightforward to check by choosing the required mapping R as the nearest-point projection onto $\mathcal{M}$ on a suitably small neighborhood $\mathcal{U}$ of $\mathcal{M}$ in $\mathbb{R}^N$ (where the C^3 assumption is needed to have a C^2 normal vector and then a C^2 projection). In the general C^2 case a more delicate construction of R is based on the idea to represent $\mathcal{M}$ locally as a C^2 graph and 'glue together' the C^2 graph projections. A detailed account on this construction — even though not at all relevant for the main purposes of this chapter — follows:

We start by fixing the terminology that a C^2 retraction from an open set $\mathcal{U}$ in $\mathbb{R}^N$ to an arbitrary set K in $\mathbb{R}^N$ is a map $R \in \mathrm{C}^2(\mathcal{U}, \mathbb{R}^N)$ such that $R(x) \in K$ for all $x \in \mathcal{U}$ and $R(x) = x$ for all $x \in K$.

Now, if $\mathcal{M}$ is an $(N{-}m)$-dimensional without-boundary C^2-submanifold in $\mathbb{R}^N$, for every fixed $x \in \mathcal{M}$, there exist $f \in \mathrm{C}^2(P, \mathbb{R}^m)$, defined on an open cube $P \subset \mathbb{R}^{N-m}$ and with values in an open cube $Q \subset \mathbb{R}^m$, and a rotation $T \in \mathcal{O}(\mathbb{R}^N)$ such that $x \in T(P{\times}Q)$ and $\mathcal{M}$ has, locally near x, the rotated-graph representation

$$\mathcal{M} \cap T(P{\times}Q) = T(\mathrm{Graph}(f))\,.$$

Using this representation, one easily obtains a C^2 retraction R from the open neighborhood $\mathcal{U} \coloneqq T(P{\times}Q)$ of x to $\mathcal{M} \cap \mathcal{U}$: It suffices to simply set $R(T(p,q)) \coloneqq T(p, f(p))$ for $(p,q) \in P{\times}Q$. Moreover, for later convenience we record that $\mathcal{U}$ is convex by construction.

If $\mathcal{M}$ is additionally compact, the definition of compactness then yields a finite cover $\mathcal{M} \subset \bigcup_{i=1}^k \mathcal{U}_i$, $k \in \mathbb{N}$, with convex open sets $\mathcal{U}_i$, each of which comes with a C^2 retraction from $\mathcal{U}_i$ to $\mathcal{M} \cap \mathcal{U}_i$, for $i = 1, 2, \ldots, k$. With the help of the subsequent lemma, it is then possible to glue or patch these retractions together in an iterative way: In a first step the lemma, applied with $\mathcal{U}_1, \mathcal{U}_2$, and the remainder neighborhood $\mathcal{V} = \bigcup_{i=3}^k \mathcal{U}_i$, allows to replace $\mathcal{U}_1$ and $\mathcal{U}_2$ with a new open set $\mathcal{U}_{12}$, which again comes with a C^2 retraction from $\mathcal{U}_{12}$ to $\mathcal{M} \cap \mathcal{U}_{12}$, such that still $\mathcal{M} \subset \mathcal{U}_{12} \cup \bigcup_{i=3}^\infty \mathcal{U}_i$. In a second step, the resulting $\mathcal{U}_{12}$ and the original (and thus still convex) $\mathcal{U}_3$ are then replaced with a new open set $\mathcal{U}_{123}$ (again with corresponding retraction) such that $\mathcal{M} \subset \mathcal{U}_{123} \cup \bigcup_{i=4}^\infty \mathcal{U}_i$, and so on. Ultimately, one ends up with a single open neighborhood $U_{123\ldots k}$ of $\mathcal{M}$ and with a C^2 retraction from $\mathcal{U}$ to $\mathcal{M}$. Since $\mathcal{M}$ is compact, the requirements from the definition are met on a possibly smaller neighborhood, and $\mathcal{M}$ — which, as we recall, is assumed only C^2 itself — turns out to be a strong C^2-neighborhood retract as claimed.

At this point, to complete the argument, it is enough to establish the following lemma:

Lemma (gluing glemma for retractions). *Consider a compact set $\mathcal{M}$ and open sets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}$ in $\mathbb{R}^N$ with $\mathcal{M} \subset \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{V}$, where $\mathcal{U}_2$ is convex. Assume that there exist C^2 retractions R_1 from $\mathcal{U}_1$ to $\mathcal{M} \cap \mathcal{U}_1$ and R_2 from U_2 to $\mathcal{M} \cap \mathcal{U}_2$. Then there also exist an open set $\mathcal{U}$ in $\mathbb{R}^N$ with $\mathcal{M} \subset \mathcal{U} \cup \mathcal{V}$ and a C^2 retraction from $\mathcal{U}$ to $\mathcal{M} \cap \mathcal{U}$.*

In order to approach the proof of the lemma we introduce, for open $\mathcal{U} \subset \mathbb{R}^N$, arbitrary $\mathcal{M} \subset \mathbb{R}^N$, and $\varepsilon > 0$, the notations $\mathcal{U}^\varepsilon := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \mathbb{R}^N \setminus U) > \varepsilon\} \subset \mathcal{U}$ and $\mathcal{N}_\varepsilon(\mathcal{M}) := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \mathcal{M}) < \varepsilon\} \supset \mathcal{M}$.

Proof of the lemma. Using the definition of compactness (for the open cover of $\mathcal{M}$ which consists of all $\mathcal{U}_1^{2\varepsilon}$, $\varepsilon > 0$, all $\mathcal{U}_2^\varepsilon$, $\varepsilon > 0$, and $\mathcal{V}$), we can fix some $\varepsilon > 0$ such that $\mathcal{M} \subset \mathcal{U}_1^{2\varepsilon} \cup \mathcal{U}_2^\varepsilon \cup \mathcal{V}$. We then observe that $R_1 \in C^2(\mathcal{U}_1, \mathbb{R}^N)$ is in particular Lipschitz continuous on $\mathcal{U}_1^{\varepsilon/2}$ and write L for its optimal Lipschitz constant there. Moreover, we set $\delta := \varepsilon/(L{+}2)$, and we next verify the auxiliary claim

$$R_1(x) \in \mathcal{U}_2 \qquad \text{for all } x \in \mathcal{U}_1^\varepsilon \cap \mathcal{U}_2^\varepsilon \cap \mathcal{N}_\delta(\mathcal{M})\,.$$

Indeed, for such x there exists $y \in \mathcal{M}$ with $|y{-}x| < \delta$, and in view of $x \in \mathcal{U}_1^\varepsilon$ and $\delta \le \varepsilon/2$ we infer $y \in \mathcal{U}_1^{\varepsilon/2}$. Via the retraction property and the Lipschitz continuity of R_1 we get $|R_1(x){-}x| \le |R_1(x){-}R_1(y)|{+}|R_1(y){-}x| \le (L{+}1)|y{-}x| < (L{+}1)\delta < \varepsilon$. Thus, taking into account $x \in \mathcal{U}_2^\varepsilon$, we conclude $R_1(x) \in \mathcal{U}_2$ as claimed.

We now introduce the open set $\mathcal{U}_* := \mathcal{U}_1^{2\varepsilon} \cup (\mathcal{U}_2^\varepsilon \cap \mathcal{N}_\delta(\mathcal{M}))$ and record that $\mathcal{M} \subset \mathcal{U}_* \cup \mathcal{V}$ holds by the initial choice of ε. We also choose a cut-off function $\eta \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^N)$ with $0 \le \eta \le 1$ on $\mathbb{R}^N$, $\eta \equiv 1$ on $U_1^{2\varepsilon}$, and $\operatorname{spt}\eta \subset \mathcal{U}_1^\varepsilon$, which in particular allows us to understand $\eta R_1 \in C^2_{\mathrm{cpt}}(\mathbb{R}^N, \mathbb{R}^N)$ as globally defined (clearly with $\eta R_1 \equiv 0$ outside $\mathcal{U}_1^\varepsilon$). At this stage, we finally define $R \in C^2(\mathcal{U}_*, \mathbb{R}^N)$ by setting

$$R(x) := \begin{cases} R_2\big(\eta R_1(x){+}(1{-}\eta(x))x\big) & \text{if } x \in \mathcal{U}_2^\varepsilon \cap \mathcal{N}_\delta(\mathcal{M}) \\ R_1(x) & \text{if } x \in \mathcal{U}_1^{2\varepsilon} \end{cases}.$$

Indeed, for R to be well-defined, we need to be sure in the first case $x \in \mathcal{U}_2^\varepsilon \cap \mathcal{N}_\delta(\mathcal{M})$ that $\eta R_1(x){+}(1{-}\eta(x))x \in \mathcal{U}_2$. In the subcase $x \notin \mathcal{U}_1^2\varepsilon$ this is obvious, since we get $\eta(x) = 0$ and thus $\eta R_1(x){+}(1{-}\eta(x))x = x \in \mathcal{U}_2$. In the other subcase $x \in \mathcal{U}_1^\varepsilon$, however, the auxiliary claim established above yields $R_1(x) \in \mathcal{U}_2$, and then the assumed convexity of $\mathcal{U}_2$ ensures that with x and $R_1(x)$ also the convex combination $\eta R_1(x){+}(1{-}\eta(x))x$ is again in $\mathcal{U}_2$. Moreover, in the overlap case $x \in \mathcal{U}_1^{2\varepsilon} \cap \mathcal{U}_2^\varepsilon \cap \mathcal{N}_\delta(\mathcal{M})$, we have $\eta(x) = 1$ and, still by the auxiliary claim, $R_1(x) \in \mathcal{U}_2$. Therefore, with $R_1(x) \in \mathcal{M} \cap \mathcal{U}_2$ and $R_2\big(\eta R_1(x){+}(1{-}\eta(x))x\big) = R_2(R_1(x)) = R_1(x)$ we obtain the consistency of the definition in this case. In view of these arguments and since $\mathcal{U}_2^\varepsilon \cap \mathcal{N}_\delta(\mathcal{M})$ and $\mathcal{U}_1^{2\varepsilon}$ are open sets with union $\mathcal{U}_*$, it is now fully verified that R is well-defined and C^2 on $\mathcal{U}_*$. Moreover, it follows straightforwardly from the retraction properties of R_1 and R_2 that $R(x) \in \mathcal{M}$ for all $x \in \mathcal{U}_*$ and $R(x) = x$ for all $x \in \mathcal{M} \cap \mathcal{U}_*$. But indeed, since we merely get $R(x) \in \mathcal{M}$ and not precisely $R(x) \in \mathcal{M} \cap \mathcal{U}_*$, this does not yet mean that R is a C^2 retraction from $\mathcal{U}_*$ to $\mathcal{M} \cap \mathcal{U}_*$. However, introducing the open set $U := R^{-1}(U_*) = R^{-1}(\mathcal{M} \cap \mathcal{U}_*) \subset \mathcal{U}_*$, we have $\mathcal{M} \cap \mathcal{U} = \mathcal{M} \cap \mathcal{U}_*$ and thus $\mathcal{M} \subset \mathcal{U} \cup \mathcal{V}$, and we can be sure that R is a C^2 retraction from $\mathcal{U}$ to $\mathcal{M} \cap \mathcal{U}$. ☐

(3) Unlike manifolds strong C^2-neighborhood retracts may consist of connected components of different dimensions. In such cases, the notion allows to conveniently unify the following arguments (even though such cases are rare in applications and a separate treatment of the different components is also possible).

Theorem (necessary criterion for minimizers subject to a manifold constraint). *Consider an open $\Omega \subset \mathbb{R}^n$, an $\mathcal{M}^n \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^{N\times n})$-measurable $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \overline{\mathbb{R}}$, and set*

$$\mathcal{F}[w] := \int_\Omega F(\,\cdot\,, w, \mathrm{D}w)\,\mathrm{d}x$$

(whenever this exists in $\overline{\mathbb{R}}$). Moreover, consider a strong C^2-neighborhood retract $\mathcal{M}$ in $\mathbb{R}^N$ with a corresponding retraction R as in the definition, fix $p \in [1,\infty)$, and endow $\mathcal{V}^p := \mathrm{W}^{1,p}_{\mathrm{cpt}}(\Omega, \mathbb{R}^N) \cap$

$\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$ *with* $\mathrm{W}^{1,p}_{\mathrm{cpt}}$*-convergence*[7]*. Then, for* $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *with* $F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^1(\Omega)$*, suppose that* $\delta\mathcal{F}[u;\varphi]$*, taken in* $u{+}\mathcal{V}^p$*, exists and is given by the first-variation formula for all* $\varphi \in \mathcal{V}^p$*.* ***If u*** *with* $u \in \mathcal{M}$ *a.e. on* Ω ***minimizes*** $\boldsymbol{\mathcal{F}}$ ***in the constrained class***

$$\mathcal{A} := \{w \in u{+}\mathcal{V}^p \,:\, w \in \mathcal{M} \text{ a.e. on } \Omega\}\,,$$

***then u solves the* variational equality**

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot \mathrm{D}(\mathrm{D}R(u)\psi) + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot \mathrm{D}R(u)\psi\big]\,\mathrm{d}x = 0 \qquad \text{for all } \psi \in \mathcal{V}^p\,. \tag{vE}$$

Once more the assumptions on the existence of the first variation and the validity of the first-variation formula can be obtained from the earlier proposition.

Proof. Consider $\mathcal{U}$, R as in the definition of strong C^2-neighborhood retracts, u as in the theorem, and an arbitrary $\psi \in \mathcal{V}^p$. For simplicity of notation assume $\|\psi\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^N)} \le 1$ (which is no restriction, since the claimed variational equality is linear in ψ). Then, for $|t| < \delta := \mathrm{dist}(\mathcal{M},\mathbb{R}^N\setminus\mathcal{U})$ we have $u{+}t\psi \in \mathcal{U}$ a.e. on Ω, and thus

$$u_t := R(u{+}t\psi)$$

is well-defined. In view of $u \in \mathcal{M}$ a.e. on Ω and $R|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$, we have $u_0 = u$. Furthermore, we now show that

$$(u_t)_{t\in(-\delta,\delta)} \text{ is a variation of } u \text{ in } u{+}\mathcal{V}^p \text{ with } \frac{\mathrm{d}}{\mathrm{d}t}u_t = \mathrm{D}R(u{+}t\psi)\psi \text{ for } t \in (-\delta,\delta)\,.$$

We first observe that $u_t{-}u = R(u{+}t\psi){-}R(u)$ is continuous in $t \in (-\delta,\delta)$ as $\mathrm{L}^\infty(\Omega,\mathbb{R}^N)$-valued curve (since R is Lipschitz) and that $\partial_i u_t = \mathrm{D}R(u{+}t\psi)(\partial_i u{+}t\partial_i\psi)$ (derivative computed with the first chain rule) is continuous in $t \in (-\delta,\delta)$ as $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$-valued curve for every $i \in \{1,2,\ldots,n\}$ (by a reasoning with the dominated convergence theorem). Since u_t equals u outside the compact support of ψ, this means that u_t is continuous in $t \in (-\delta,\delta)$ as curve in $u{+}\mathcal{V}^p$.

To check $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}u_t = \mathrm{D}R(u)\psi$ in $u{+}\mathcal{V}^p$ we abbreviate $\omega_t(x) := \sup_{\mathrm{B}_{|t|}(u(x))}|\mathrm{D}R{-}\mathrm{D}R(u(x))|$ and estimate $|\frac{u_t-u}{t} - \mathrm{D}R(u)\psi| \le \omega_t$ a.e. on Ω with the definition of u_t and a standard derivative estimate. Now dominated convergence yields $\lim_{t\to0}\omega_t = 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and, as ψ is compactly supported, we can conclude $\lim_{t\to0}\frac{u_t-u}{t} = \mathrm{D}R(u)\psi$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$. In a similar way, with the abbreviation $\omega^2_t(x) := \sup_{\mathrm{B}_{|t|}(u(x))}|\mathrm{D}^2R{-}\mathrm{D}^2R(u(x))|$ and the estimate $|\frac{\partial_iu_t-\partial_iu}{t} - \partial_i(\mathrm{D}R(u)\psi)| \le \omega^2_t|\partial_iu|{+}\omega_t|\partial_i\psi|$ one finds $\lim_{t\to0}\frac{\partial_iu_t-\partial_iu}{t} = \partial_i(\mathrm{D}R(u)\psi)$ in $\mathrm{L}^p(\Omega,\mathbb{R}^N)$ also for the derivatives. This shows $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}u_t = \mathrm{D}R(u)\psi$ in $u{+}\mathcal{V}^p$, and for general $t \in (-\delta,\delta)$ one can check $\frac{\mathrm{d}}{\mathrm{d}t}u_t = \mathrm{D}R(u{+}t\psi)\psi$ in an analogous way.

Finally, by further applications of the dominated convergence theorem, we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}u_t = \mathrm{D}R(u{+}t\psi)\psi$ and $\partial_i\frac{\mathrm{d}}{\mathrm{d}t}u_t = \partial_i(\mathrm{D}R(u{+}t\psi)\psi) = \mathrm{D}^2R(u{+}t\psi)(\partial_iu{+}t\partial_i\psi,\psi){+}\mathrm{D}R(u{+}t\psi)\partial_i\psi$ are still continuous in $t \in (-\delta,\delta)$ as $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$-valued curves. Therefore, u_t is finally C^1 in $t \in (-\delta,\delta)$ as curve in $u{+}\mathcal{V}^p$ and is by definition a variation of u in $u{+}\mathcal{V}^p$ in direction $\mathrm{D}R(u)\psi \in \mathcal{V}^p$.

[7] By $\mathrm{W}^{1,p}_{\mathrm{cpt}}$-convergence of a sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,p}_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ to a limit $w \in \mathrm{W}^{1,p}_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ we mean L^p-convergence $\lim_{k\to\infty}w_k = w$ and $\lim_{k\to\infty}\mathrm{D}w_k = \mathrm{D}w$ on Ω with $\bigcup_{k=1}^\infty \mathrm{spt}(w_k) \Subset \Omega$. In the same way as described earlier for $\mathscr{D}$-convergence, $\mathrm{W}^{1,p}_{\mathrm{cpt}}$-convergence comes with a topology and is also used in a shifted version on $u{+}\mathrm{W}^{1,p}_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$.

After these technical details, the essential observation is now that the variation $(u_t)_{t\in(-\delta,\delta)}$ is indeed $\mathcal{A}$-admissible (since we required $R(\mathcal{U}) \subset \mathcal{M}$). Therefore, the direction $\mathrm{D}R(u)\psi$ of the variation is admissible in the basic necessary criterion for $\mathcal{A}$-constrained minimizers. From this criterion and the assumed validity of the first-variation formula, we then obtain

$$0 = \delta_{\mathcal{A}}\mathcal{F}[u;\mathrm{D}R(u)\psi] = \int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot \mathrm{D}(\mathrm{D}R(u)\psi) + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot \mathrm{D}R(u)\psi\big]\,\mathrm{d}x$$

as claimed in (vE). □

Remarks (on the necessary criterion for $\mathcal{M}$-constrained minimizers)**.** Using the notations of the definition and the theorem, the following comments are in order:

(1) From the basic requirements on R we obtain $R\circ R = R$ on $\mathcal{U}$, thus $\mathrm{D}R(R(y))\mathrm{D}R(y) = \mathrm{D}R(y)$ for $y \in \mathcal{U}$ and in fact

$$\mathrm{D}R(y)^2 = \mathrm{D}R(y) \qquad \text{for } y \in \mathcal{M}\,.$$

Thus, **$\mathrm{D}R(y)$ with $y \in \mathcal{M}$ is in fact a linear projection** from $\mathbb{R}^N$ to the vector subspace $\mathrm{T}_y\mathcal{M} := \operatorname{range}\mathrm{D}R(y)$ of $\mathbb{R}^N$, the **(generalized) tangent space** to $\mathcal{M}$ at $y \in \mathcal{M}$.

(2) For arbitrary $\varphi\colon \Omega \to \mathbb{R}^N$, we have the equivalence

$$\varphi = \mathrm{D}R(u)\psi \text{ for some } \psi \in \mathcal{V}^p \iff \varphi \in \mathcal{V}^p \text{ with } \varphi \in \mathrm{T}_u\mathcal{M} \text{ a.e. on } \Omega\,.$$

Here, ‘ $\Longrightarrow$ ’ results from $\mathrm{D}R(u) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times N}) \cap \mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times N})$, a product rule, and the definition of $\mathcal{V}^p$, while for ‘ $\Longleftarrow$ ’ one simply uses that $\varphi \in \mathrm{T}_u\mathcal{M}$ gives $\varphi = \mathrm{D}R(u)\varphi$ by the preceding Remark (1).

This equivalence identifies the test functions in the variational equality of the theorem as ‘tangential to $\mathcal{M}$ at the values of u’, and the variational equality can be equivalently rewritten in these terms as

$$\int_\Omega \big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\cdot \mathrm{D}\varphi + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)\cdot\varphi\big]\,\mathrm{d}x = 0$$
$$\text{for all } \varphi \in \mathcal{V}^p \text{ with } \varphi \in \mathrm{T}_u\mathcal{M} \text{ a.e. on } \Omega\,.$$

(3) In regular cases (e.g. if $\nabla_z F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^{p'}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$, $\nabla_y F(\,\cdot\,,u,\mathrm{D}u) \in \mathrm{L}^{p'}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and $\operatorname{div}\big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\big] \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$), integration by parts transforms the variational equality (vE) into

$$0 = \int_\Omega \mathrm{E}_{\mathcal{F}}u\cdot \mathrm{D}R(u)\psi\,\mathrm{d}x = \int_\Omega \mathrm{D}R(u)^*\mathrm{E}_{\mathcal{F}}u\cdot\psi\,\mathrm{d}x \qquad \text{for all } \psi \in \mathcal{V}^p\,,$$

where $\mathrm{E}_{\mathcal{F}}$, given by $\mathrm{E}_{\mathcal{F}}u = -\operatorname{div}\big[\nabla_z F(\,\cdot\,,u,\mathrm{D}u)\big] + \nabla_y F(\,\cdot\,,u,\mathrm{D}u)$, is the Euler-Lagrange operator of $\mathcal{F}$ and $\mathrm{D}R(y)^*$ is the adjoint linear map or transpose matrix of $\mathrm{D}R(y)$. By the fundamental lemma of the calculus of variations, one then finds that the variational equality is also equivalent to the pointwise equation

$$\mathrm{D}R(u)^*\mathrm{E}_{\mathcal{F}}u \equiv 0 \qquad \text{a.e. on } \Omega\,,$$

and in view of the elementary identity $\ker(\mathrm{D}R(y)^*) = (\mathrm{range}\,\mathrm{D}R(y))^\perp = (\mathrm{T}_y\mathcal{M})^\perp$ for $y \in \mathcal{M}$, it can finally be recast in form of the pointwise **perpendicularity relation**

$$\boxed{\mathrm{E}_\mathcal{F}u \perp \mathrm{T}_y\mathcal{M} \qquad\qquad \text{a.e. on } \Omega\,,}$$

Sometimes this is further shortened to $(\mathrm{E}_\mathcal{F}u)^{\tan} \equiv 0$ a.e. on Ω, where $(\mathrm{E}_\mathcal{F}u)^{\tan}$ stands for the component of $\mathrm{E}_\mathcal{F}u$ tangent to $\mathcal{M}$ at the values of u.

In less regular cases, the variation equality (vE) should be seen as a weak formulation of the perpendicularity relation.

(4) We now come back to the initially discussed case that $\mathcal{M}$ is (at least locally) given in the form $\mathcal{M} = \bigcup_{i=1}^m \{g_i = c_i\}$ with $c_1, c_2, \ldots, c_m \in \mathbb{R}$ and $g = (g_1, g_2, \ldots, g_m) \in \mathrm{C}^2(\mathbb{R}^N, \mathbb{R}^m)$ such that $\mathrm{rank}(\mathrm{D}g) \equiv m$ on $\mathbb{R}^N$. In this case one obtains the tangent spaces as $(\mathrm{T}_y\mathcal{M})^\perp = \mathrm{Span}\{\nabla g_i(y) : i = 1, 2, \ldots, m\}$ for $y \in \mathcal{M}$. Therefore, the perpendicularity relation is then equivalent to the requirement that

$$\mathrm{E}_\mathcal{F}u = \sum_{i=1}^m \lambda_i \nabla g_i(u) \qquad\qquad \text{holds a.e. on } \Omega$$

with certain functions $\lambda_i \colon \Omega \to \mathbb{R}$ as Lagrange multipliers. Solving for the $\mathbb{R}^N$-valued u and the m scalar Lagrange multiplier functions at the same time is reasonable, since the above $\mathbb{R}^N$-valued equation is complemented with the m scalar constraints $g_i(u) = c_i$.

(5) A version of the theorem remains valid if the C^2 regularity of R is weakened to $\mathrm{W}^{2,\infty}_{\mathrm{loc}}$ regularity (with $\mathrm{D}R$, D^2R still bounded).

(6) Holonomic constraints $g_i(\,\cdot\,, w) \equiv c_i$ a.e. on Ω and manifold constraints $w(x) \in \mathcal{M}_x$ for a.e. $x \in \Omega$ with x-dependence can be covered in a similar way under the technical assumption that there exist suitable retractions $R(x, \cdot\,)$ from a neighborhood $\mathcal{U}$ to $\mathcal{M}_x$ with D_yR, D_y^2R, $\mathrm{D}_x\mathrm{D}_yR$ bounded. One then obtains the variational equality with the help of admissible variations $u_t = R(\,\cdot\,, u{+}t\psi)$ of u in direction $\varphi = \mathrm{D}_yR(\,\cdot\,, u)\psi$.

Example (harmonic mappings into spheres). As a basic example consider the **minimization**[8] **of the Dirichlet integral among unit-sphere-valued mappings**, that is, take $\mathcal{F} = \mathcal{E}_2$ and $\mathcal{M} = \mathrm{S}^{N-1} = \{y \in \mathbb{R}^N : |y| = 1\}$. Then a suitable retraction R is given by $R(y) := \frac{y}{|y|}$ for $|y| > \delta$ with arbitrary $\delta \in (0,1)$ (which is only needed to define the neighborhood $\mathcal{U} = \{y \in \mathbb{R}^N : |y| > \delta\}$). The derivative $\mathrm{D}R(y) = \frac{\mathrm{I}_{N\times N}}{|y|} - \frac{y\otimes y}{|y|^3} \overset{y\in \mathrm{S}^{N-1}}{=} \mathrm{I}_{N\times N} - y\otimes y$ of R at $y \in \mathrm{S}^{N-1}$ gives the linear projection onto $\{y\}^\perp = \mathrm{T}_y\mathrm{S}^{N-1}$. Thus, for $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $|u| \equiv 1$ a.e. on Ω, we can record:

- The variational equality (vE) here reads

$$\int_\Omega \mathrm{D}u \cdot \mathrm{D}(\psi{-}(\psi\cdot u)u)\,\mathrm{d}x = 0 \qquad\qquad \text{for all } \psi \in \mathrm{W}^{1,2}_{\mathrm{cpt}}(\Omega, \mathbb{R}^N) \cap \mathrm{L}^\infty(\Omega, \mathbb{R}^N)\,.$$

or equivalently

$$\int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,2}_{\mathrm{cpt}}(\Omega, \mathbb{R}^N) \cap \mathrm{L}^\infty(\Omega, \mathbb{R}^N) \text{ with } \varphi \perp u \text{ a.e. on } \Omega\,.$$

[8] As it is the case already in the unconstrained case, the minimization is a non-trivial problem with non-constant solutions only if boundary conditions are imposed. However, such conditions do not take effect on the necessary criterion under consideration. Thus, it is not necessary to discuss or specify them at this point.

- In case $\Delta u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, the variational equality is equivalent to the perpendicularity relation $\Delta u \perp \mathrm{T}_u \mathrm{S}^{N-1} = \{u\}^\perp$ a.e. on Ω, which here means in fact that Δu is a.e. parallel to u, that is $\Delta u = \lambda u$ a.e. on Ω with a Lagrange multiplier function $\lambda \colon \Omega \to \mathbb{R}$. In the given case, one can in fact eliminate λ from the equation, since $|u| \equiv 1$ implies $0 = \Delta(|u|^2) = 2u \cdot \Delta u + 2|\mathrm{D}u|^2$. Combining this with $\Delta u = \lambda u$ and using $|u| \equiv 1$ again, we find $\lambda = -|\mathrm{D}u|^2$ a.e. on Ω. Therefore, the necessary criterion for minimizers of $\mathcal{E}_2$ with S^{N-1}-manifold constraint is in fact (a weak formulation of) the **PDE (system) for harmonic maps into spheres**

$$\Delta u = -|\mathrm{D}u|^2 u \qquad\qquad \text{a.e. on } \Omega\,.$$

This PDE system exhibits a quadratic first-order non-linearity in $\mathrm{D}u$, which makes its theory quite a bit more difficult and subtle than the theory of linear elliptic systems.

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